Chirality transitions in gravitational fields

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The chirality transitions induced by gravitational fields on Dirac particles are studied within the framework of field theory in curved spaces. To have these transitions both a non-null mass for the particle and an angular momentum for the source of the gravitational field are necessary. The results of this analysis are applied to some simple examples, and an upper bound for the corresponding amplitude is estimated.

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I. INTRODUCTION

A well-known phenomenon is the precession of gyroscopes in the vicinity of a gravitational field source. It suggests the possibility of an analogous effect for spinors in such regions, and in particular of left-right neutrino transitions. This paper studies the effect of gravitational fields on the propagation of Dirac particles within the field theory in a curved space-time framework, and focuses on the induced chirality transitions. This approach can be considered a semiclassical approximation to quantum gravity, a more fundamental theory not yet developed in a consistent way. Nevertheless, for our problem the effect in most cases is very small, and the quantum gravity corrections are completely negligible. Some papers have investigated these transitions, but have forgotten significant contributions and thus given misleading and contradictory results [1].

Our starting point is the Dirac equation in a curved space [2, 3]:

$$[i\gamma^\mu(x)(\partial_\mu + \Gamma_\mu) - m]\ \Psi = 0\,, \tag{1}$$

where γ^a are the Dirac matrices in the Minkowski space, $\Gamma_\mu = -\frac{i}{8}e^\nu_a e_{\nu b;\mu}[\gamma^a,\gamma^b]$ is the spin connection, and e^μ_a are the tetrads. The metric tensor $g_{\mu\nu}$ and the tetrads are related by

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu\,, \tag{2}$$

with η_{ab} the Minkowski metric. These equations do not have a unique solution, but as the different solutions are simply related by local Lorentz transformations, they are physically equivalent. The matrices $\gamma^\mu = \gamma^a e^\mu_a$ and the covariant derivatives $D_\mu = \partial_\mu + \Gamma_\mu$ satisfy

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\,,\quad [D_\mu, D_\nu] = \frac{i}{8}\left[\gamma^\alpha,\gamma^\beta\right] R_{\alpha\beta\mu\nu}\,, \tag{3}$$

where $R_{\alpha\beta\mu\nu}$ is the Riemann tensor. With these definitions we have $D_\mu\gamma^\nu = 0$.

For most astrophysical systems the gravitational field is weak. Therefore, we will restrict our analysis to linear perturbations of the flat space-time metric:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}\,, \tag{4}$$

and accordingly the tetrads will be

$$e^a{}_\mu = \delta^a{}_\mu + \omega^a{}_\mu\,, \tag{5}$$

where $h \ll 1$ and $\omega^a_\mu \ll 1$. In the following we will adopt $\omega_{i0} = 0$, by using the freedom given by local Lorentz transformations.

II. THE HAMILTONIAN

The Hamiltonian is

$$i\partial_0 = H = \frac{1}{g^{00}}\gamma^a e^0_a[m - i\gamma^b e^i_b(\partial_i + \Gamma_i)] - i\Gamma_0\,, \tag{6}$$

which is Hermitian under the positive definite scalar product for the solutions of the Dirac equation in curved space-time:

$$(\Psi_1,\Psi_2) = i\int_\Sigma d\Sigma_\mu\ \bar\Psi_1\gamma^\mu\Psi_2\sqrt{-g}\,, \tag{7}$$

where Σ is a spacelike hypersurface. To recover the most usual expression for the scalar product we can perform a transformation to absorb the extra factor $\sqrt{-g}$ of the measure in the wave function. In the linear approximation it reads

$$\Psi \longrightarrow \left(1 - \frac{h^{ii}}{4}\right)\Psi\,, \tag{8}$$

and thus the Hamiltonian becomes

$$H = \gamma^0 m(1 - \tfrac{1}{2}h^{00}) + \boldsymbol{\alpha}\cdot\mathbf{p} - \tfrac{1}{4}\{h^{00},\boldsymbol{\alpha}\cdot\mathbf{p}\} - \tfrac{1}{4}\{h^{ij},\alpha^j p^i\} + \tfrac{1}{2}\{\mathbf{h},\mathbf{p}\} + \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{h}\cdot\mathbf{s}\,, \tag{9}$$

where $s^i = \frac{i}{8}\epsilon^{ijk}[\gamma^j,\gamma^k]$.

Before undertaking our discussion of the chirality transitions by the gravitational field, we first review the physical meaning of this Hamiltonian. The first three terms have a straightforward significance. If we consider that h^{00} is a slowly varying function, they can be written as $(1-\frac{h^{00}}{2})[\gamma^0 m + \alpha_i p_i]$. This is the flat space-time Dirac

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Hamiltonian affected by the factor $(1 - \frac{h^{00}}{2})$. In general h^{00} is positive definite and leads to a red shift of the energy levels. This effect has a geometrical origin independent of the spin of the particles, and has been verified for photons from white dwarfs [4].

To facilitate the interpretation of the remaining terms we will perform a further but not very restrictive simplification, namely, the post-Newtonian approximation [3], where

$$\begin{aligned} h^{00} &= -2\phi\,, \\ h^{ij} &= -2\delta^{ij}\phi\,, \\ h^{i0} &= h^i\,, \end{aligned} \tag{10}$$

with

$$\phi = \frac{1}{2}G\int dy^3 \frac{T^{00}(y,t)}{\mid x-y\mid}\,, \quad h^i = -4G\int dy^3 \frac{T^{0i}(y,t)}{\mid x-y\mid}\,. \tag{11}$$

$T^{\mu\nu}$ is the energy-momentum tensor, and G is the gravitational constant. The source of the potential ϕ is the mass distribution, and the vector potential h^i is originated by the angular momentum of this mass distribution. As illustrative examples the value of ϕ is 2×10^{-6} on the surface of the Sun, 6×10^{-10} on earth, and 6×10^{-7} is the contribution due to the galactic center on the solar system. With respect to h^i, on the surface of the Earth it is of the order of 10^{-16}, and on a neutron star of 10^{-6}. Using the post-Newtonian approximation, the Hamiltonian becomes

$$H = (1+\phi)(\gamma^0 m + \alpha^i p^i) + \frac{1}{2}\{h^i, p^i\} + \frac{1}{2}\epsilon_{ijk}\nabla^i h^j s^k\,. \tag{12}$$

From this the velocity operator is easily obtained:

$$\dot{r}^i = [r^i, H] = (1+2\phi)\alpha^i + h^i\,. \tag{13}$$

It has two contributions from the gravitational field. The one due to ϕ mainly alters the module of the velocity, but the other implies a velocity drift, the same for particles and antiparticles, due to the angular momentum of the gravitational field source.

To obtain a further insight into the physical meaning of this Hamiltonian we can perform a Foldy-Wouthuysen transformation [5], to make the low velocity limit explicit. This transformation leads to

$$\begin{aligned} H_{\rm FW} = &\frac{1+3\phi}{2m}(\mathbf{p}+m\mathbf{h})^2 + m\phi \\ &+\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{h}\cdot\mathbf{s} - \frac{3}{2m}(\mathbf{s}\times\boldsymbol{\nabla}\phi + i\boldsymbol{\nabla}\phi)\cdot\mathbf{p}\,. \end{aligned} \tag{14}$$

The first two terms contain the red shift of the kinetic energy and of the mass, where the latter is interpreted as the gravitational potential energy. This has been verified by neutron interference in the gravitational field of the earth [6]. The remaining terms are analogous to the corresponding ones for a Dirac particle in an electromagnetic field, provided the identification

$$m\mathbf{h} = -e\mathbf{A}\,, \quad m\phi = e\varphi\,. \tag{15}$$

In this way we can recognize a Bohm-Aharanov gravitational effect and a coupling of $\mathbf{h}$ to the orbital angular momentum of the particle. The latter reduces to the Sagnac effect [7] when we consider a rotating reference frame. The third term is a coupling of $\mathbf{h}$ to the spin of the particle, but with a gyrogravitational factor equal to 1. Finally, the fourth term corresponds to a spin-orbit coupling.

On the other hand we can also consider the ultrarelativistic limit, appropriate to describe neutrinos, generated by a Cini-Toushek transformation [8]:

$$\begin{aligned} H_{\rm CT} = &\boldsymbol{\alpha}\cdot\mathbf{p} + \{\phi, \boldsymbol{\alpha}\cdot\mathbf{p}\} + \frac{1}{2}\{\mathbf{h},\mathbf{p}\} + \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{h}\cdot\mathbf{s} \\ &+m\gamma^0\left\{\frac{i}{4}[h_{i,j}+h_{j,i}]\left[\frac{\alpha^i p^j}{p^2} - 2\frac{p^i p^j}{p^4}\boldsymbol{\alpha}.\mathbf{p}\right] + \nabla\phi\times\mathbf{s}\cdot\frac{\mathbf{p}}{p^2} + \frac{i}{2}\phi\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{p^2}\right\}\,. \end{aligned} \tag{16}$$

In this Hamiltonian it is clear that the chirality violation is due to the term that contains the mass m and the deviations of the metric from $\eta^{\mu\nu}$, and does not commute with γ_5. So if m is null or the space-time is flat this equation factorizes in two independent equations by means of the projection operators $\frac{1}{2}(1\pm\gamma_5)$.

III. CHIRALITY TRANSITIONS

According to the discussion in the last section, in a flat space $[H, \gamma_0\gamma_5] = 0$, and therefore for two states with energies E_1 and E_2 we have

$$(E_1 + E_2)\langle E_1 \mid \gamma^0\gamma^5 \mid E_2\rangle = 0\,. \tag{17}$$

But the equation of motion of γ_5 is

$$\frac{d}{dt}\langle\gamma^5\rangle = i\langle[H,\gamma^5]\rangle = 2im\langle\gamma^0\gamma^5\rangle\,, \tag{18}$$

and so even in the case of $m \neq 0$ there are no chirality transitions. The situation changes when there is a gravitational field. In such a case we can write the Hamiltonian as $H = H_1 + H_2$, with

$$\begin{aligned} H_1 &= \gamma^0 m(1+\phi) + \boldsymbol{\alpha}\cdot\mathbf{p} + \{\phi, \boldsymbol{\alpha}\cdot\mathbf{p}\} \\ H_2 &= \frac{1}{2}\{\mathbf{h},\mathbf{p}\} + \frac{1}{2}\boldsymbol{\nabla}\times\mathbf{h}\cdot\mathbf{s}\,, \end{aligned} \tag{19}$$

such that

$$\{H_1, \gamma^0, \ \gamma^5\} = 0, \ \ [H_2, \gamma^0\gamma^5] = 0\,, \tag{20}$$

and hence the equation of motion for $\langle\gamma_5\rangle$ becomes:

$$\frac{d}{dt}\langle\gamma^5\rangle = 2im\langle\gamma^0\gamma^5(1+\phi)\rangle \ . \tag{21}$$

To explicitly see the effect of **h** on the evolution of $\langle\gamma_5\rangle$, provided that the state considered has an energy E, we can rewrite Eq. (21) as

$$\frac{d}{dt}\langle\gamma^5\rangle \simeq i\frac{m}{E}\langle\{H,\gamma^0\gamma^5(1+\phi)\}\rangle$$
$$= i\frac{m}{E}\langle\gamma_0\gamma_5(2H_2 - i\alpha_i\phi_{,i})\rangle \ . \tag{22}$$

The right member depends on H_2. Iterating this procedure once more we can highlight the role of **h** in the evolution of $\langle\gamma^5\rangle$:

$$\frac{d}{dt}\langle\gamma^5\rangle \simeq -\frac{m}{2E^2}\left\langle\gamma^5\gamma^0[(h_{i,j}+h_{j,i})\alpha^i p^j + \frac{i}{2}\sigma^{ij}\phi_{,j}p^i]\right\rangle . \tag{23}$$

The expression between square brackets is just the chirality-violating term that appears in the Cini-Toushek Hamiltonian. To compute the change of the chirality amplitude we can decompose the state in components of definite chirality:

$$\Psi = a\Psi_+ + b\Psi_-, \quad a^2+b^2 = 1 . \tag{24}$$

From here

$$\frac{d}{dt}\langle\gamma^5\rangle = 2a\frac{da}{dt} - 2b\frac{db}{dt} = 4a\frac{da}{dt} , \tag{25}$$

and using Eq. (23) we have

$$\frac{da}{dt} = -\frac{m}{4E^2}\sqrt{1-a^2}\left\langle\Psi_+\left|\gamma^5\gamma^0\left[(h_{i,j}+h_{j,i})\alpha^i p^j + \frac{i}{2}\sigma^{ij}\phi_{,j}p^i\right]\right|\Psi_-\right\rangle . \tag{26}$$

The chirality transition depends both on the symmetric part of $h_{i,j}$ and on the gradient of ϕ. The first contribution comes from the angular momentum of the gravitational field source, and the second one from its mass. In particular for an ultrarelativistic particle $|\mathbf{p}| \sim E$, and thus in this case the amplitude is proportional to $\frac{m}{E}$. This result is consistent with the one obtained considering the interaction between fermionic and scalar fields in quantum gravity at the tree level [9].

In general we can have a chirality transition even when the angular momentum is null, except when additional considerations imply that the gradient dependent term contribution is negligible, as in the cases we discuss in the following sections.

IV. PARTICLES IN A ROTATING FRAME

The metric is a flat space one, seen from a rotating frame with velocity ω:

$$\phi = \frac{1}{2}[\omega^2 r^2 - \boldsymbol{\omega}\cdot\mathbf{r})^2] \quad \mathbf{h} = \mathbf{r}\times\boldsymbol{\omega} . \tag{27}$$

The corresponding Hamiltonian is

$$H = m\gamma^0 + \boldsymbol{\alpha}\cdot\mathbf{p} - \boldsymbol{\omega}\cdot\mathbf{L} - \boldsymbol{\omega}\cdot\mathbf{s} + [m\gamma^0\phi + \{\phi,\boldsymbol{\alpha}\cdot\mathbf{p}\}] , \tag{28}$$

where the last term, which depends on the scalar potential, is of second order in ω. For this metric $h_{i,j}$ is antisymmetric, and thus the transition amplitude is due only to ϕ. At first order in ω this potential is null and so is the transition amplitude. This result can be expressed in an alternative and more intuitive way by considering that at this order the coupling terms $\boldsymbol{\omega}\cdot\mathbf{s}$ and $\boldsymbol{\omega}\cdot\mathbf{L}$ generate the same rotation on **s** and **p**, and so the helicity remains invariant.

V. THE ULTRARELATIVISTIC PARTICLES

In this section we will analyze ultrarelativistic particles, which can be the case of neutrinos supposing their mass is non-null. They are very important from the astrophysical point of view; for example, their change of chirality could give way to sterile particles, which could be significant in processes such as the cooling of a supernova or the baryogenesis in the early Universe.

For neutrinos we know with a very high precision that they are left-handed particles with $\frac{m}{E} \ll 1$ and thus $a \ll 1$. The amplitude we are considering is a mean value between states of definite helicity, and the interaction with the gravitational field changes the initial momentum and spin of the neutrino. If we take into account that we are considering a weak field approximation, we can use a semiclassical argument to state an upper bound for the term that contains $\nabla\phi$ in the chirality transition amplitude, as we discuss in the following.

To be more specific we will refer to spherically symmetric mass distributions, with total mass M and radius r_0. In these cases the scalar potential outside the distribution is $\phi = -\frac{M}{r}$ and $\nabla\phi$ has the direction of **r**. Our expressions arise from a weak field approximation for the gravitational interaction and thus, to be consistent with it, we must restrict their application to the range $r \gg M$. Hence we will consider only particles with a large impact parameter ρ ($\rho \gg M$). To estimate an upper bound for the gradient-dependent term it is sufficient to consider the change of **p**. In this case we have a small scattering angle $\theta \sim \left(\frac{m}{E}\right)^2\frac{M}{\rho} \ll \left(\frac{m}{E}\right)^2$, and the transversal variation of **p** is at most of the same order. But this implies that $\langle +|\nabla\phi\cdot(\mathbf{p}\times\mathbf{s})|-\rangle \sim (\frac{m}{E})^2$, which is not significant compared with the contribution depending on **h**. Therefore Eq. (26) reduces to

$$a = -\frac{m}{4E^2}\int dt\langle\Psi_+ \mid \gamma^5\gamma^0[\nabla(\boldsymbol{\alpha}\cdot\mathbf{h})\cdot\mathbf{p}$$
$$+(\boldsymbol{\alpha}\cdot\nabla)\mathbf{h}\cdot\mathbf{p}] \mid \Psi_-\rangle \ . \tag{29}$$

In such a way the consistency with the weak field approximation constrains us to a region where the dependence of the amplitude of transition on ϕ is negligible. Thus, the necessary conditions to have a chirality transition at this order for large impact parameter neutrinos reduce to $m \neq 0$ and $h_{i,j} + h_{j,i} \neq 0$. The latter implies in particular that the gravitational field of a nonrotating mass, which has $\mathbf{h} = 0$, cannot induce a transition between states of definite chirality for small scattering angles.

A very different situation arises when we consider neutrinos propagating in the field of a rotating mass. In this case we have $\mathbf{h} = 2\mathbf{J} \times \frac{\mathbf{r}}{r^3}$, where $\mathbf{J}$ is the mass angular momentum. Now $h_{i,j}$ has a symmetric part and, from Eq. (29), the non-null chirality violation is given by

$$a = \frac{3}{2}\frac{mG}{E^2}\int dt \left\langle \Psi_+ \left| \frac{\gamma^5\gamma^0}{r^5}[(\mathbf{r}\cdot\boldsymbol{\alpha})(\mathbf{J}\times\mathbf{r}\cdot\mathbf{p}) + (\mathbf{J}\times\mathbf{r}\cdot\boldsymbol{\alpha})(\mathbf{r}\cdot\mathbf{p})] \right| \Psi_- \right\rangle . \tag{30}$$

To illustrate this result we can use two different situations, both considering a neutrino path tangential to the mass distribution on an equatorial point. If $\mathbf{p} \parallel \mathbf{J}$ and if we integrate on the trajectory from the radius of the mass distribution $r = r_0$ to $r = \infty$, we have, for the ratio between negative and positive chirality,

$$\Delta_{\parallel} = \frac{m\mathbf{J}}{Er_0^2} = \frac{2}{5}\frac{MG}{c^3}\left(\frac{mc^2}{E}\right)\omega . \tag{31}$$

This effect is odd; the contribution from $r = -\infty$ to $r = r_0$ exactly cancels the one from $r = r_0$ to $r = +\infty$. On the other hand, if $\mathbf{p} \perp \mathbf{J}$ we obtain

$$\Delta_{\perp} = \frac{1}{2}\Delta_{\parallel} , \tag{32}$$

but now the effect is even and both contributions add.

VI. FINAL REMARKS

The results obtained in the preceding section can be summarized as follows. Starting from the Dirac equation in a curved space-time, we determined that the chirality transition depends both on the mass of the particle and on the structure of the gravitational field, or more specifically on the symmetric part of $h_{i,j}$ for ultrarelativistic particles if we consider only the region where the weak field approximation is valid. This effect is in general non-null and characterized by an amplitude of order $\frac{m}{E}$, in contrast with the result given by the geometrical optics approximation. This last approach considers point particles moving according to geodesics, with their momentum and spin transported in a parallel way and thus with an invariant scalar product. In other words, in this framework we can say that the gravitation induces a local rotation of the inertial reference frames with respect to an asymptotic flat space, so that the action on a particle produces a simultaneous rotation of the momentum and the spin, $\boldsymbol{\omega}\cdot(\mathbf{s}+\mathbf{L})$.

To visualize why our results give a non-null transition amplitude we can develop $\mathbf{h}$ in the neighborhood of a point x_0:

$$\delta h_i = h_i - h_i|_{\mathbf{x}_0} + \frac{1}{2}[\nabla \times \mathbf{h}|_{\mathbf{x}_0} \times (\mathbf{x}-\mathbf{x}_0)]_i + \frac{1}{2}(h_{j,i}+h_{i,j})|_{\mathbf{x}_0}(x^j - x_0^j) . \tag{33}$$

Here we can see that the variation of $\mathbf{h}$ is given by two terms. The first one corresponds to a pure rotation around x_0. It does not produce any change of chirality and is the only one considered in the geometrical optics approximation. Instead of this our results keep track of the additional term proportional to $(h_{j,i} + h_{i,j})$, responsible for the chirality transitions.

Finally, and only with the purpose of estimating an upper bound, we can consider as a source of the gravitational field a black hole with maximum angular momentum (Kerr-Newman extreme metric) [4], although in this case the linear approximation is not necessarily appropriate. This system is characterized by $\frac{GJ}{r_0^2} = 1$, and we obtain

$$\Delta \le \frac{m}{E} . \tag{34}$$

Hence the effect is actually very small and significant transition probabilities can be observed only if there are enhancing mechanisms, as can be nonlinear effects due to very strong gravitational fields.

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[1] See, for example, Y. Q. Cai and G. Papini, Phys. Rev. Lett. **66**, 1259 (1991); J. Anandan, *ibid.* **68**, 3809 (1992); Y. Q. Cai and G. Papini, *ibid.* **68**, 3811 (1992); B. Mashhoon, *ibid.* **68**, 3812 (1992). In the first paper they study the gravity-spin coupling for the particular case of a rotating frame. But they only consider the coupling of the spin with the angular velocity of the frame, $\boldsymbol{\omega}\times\mathbf{s}$, and forget the coupling of the orbital angular momentum, $\boldsymbol{\omega}\times\mathbf{L}$, which exactly cancels the effect of the first one. In addition, the analogy pointed out in this paper with the electromagnetic interaction of an eventual magnetic moment is only valid in the nonrelativistic limit, because this interaction

has the form $\gamma^0 \mathbf{B} \cdot \mathbf{s}$ and explicitly violates chirality. On the other hand, Anandan's conclusions are correct in the limit of the geometrical optics, where the particles move along geodesics, which is not the case for the solutions of the Dirac equation. Our results have a different origin from the ones considered in these papers, as we state in our discussion.

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