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Remark on black hole entropy in string theory

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We extend the string-theoretic calculation of black hole entropy, first performed by Susskind and Uglum, away from the infinite mass limit. It is shown that the result agrees with that obtained from the classical action of string theory, using the Noether charge method developed by Wald. Also shown in the process is the equivalence of two general techniques for finding black hole entropies—the Noether charge method and the method of conical singularities.

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I. BLACK HOLE ENTROPY FROM STRING THEORY

One of the things we hope to get from a quantum theory of gravity is a microscopic understanding of black hole thermodynamics. In particular we would like to see a microscopic structure associated with the horizon, the number of whose states is counted by the usual black hole entropy. Recently, it has been argued by Susskind [1], and Susskind and Uglum [2], that string theory may be able to provide this.

In particular, it was observed in [1] that the string partition function contains contributions which describe strings stuck onto the horizon at their two end points; this stringy "hair" then seems like a natural candidate for microscopic structure. Then in [2], the authors made this idea more precise by calculating the genus zero contribution to the partition function, in the infinite mass limit, and reproducing the expected result of $\frac{1}{4}$ per unit area. Unfortunately these calculations are fraught with peril, as they require elements of off-shell string theory; we just have to use the best available ansantz and hope. Perhaps the reasonable nature of the results adds to its credibility.

In this paper, we extend the computation of [2] away from the limit of infinite mass. Since we are computing the classical (genus zero) part of the partition function, we expect that the answer should be the same as that obtained from the classical string action, where the latter can be found using the Noether charge technique developed by Wald [3].

The ansatz used in [2], which we will also use here, is that of Tseytlin [4]. He argued that the string partition function and the string action should be closely related; indeed, he stated

$$I = \frac{\partial}{\partial t} Z_R \ , \tag{1.1}$$

where Z_R is the renormalized genus zero σ model partition function, t is the renormalization parameter, and I is the classical string action, alternately derivable from conformal invariance, or from scattering amplitudes. (We use I since S will denote entropy.) Z_R still contains the (renormalized) Mobius infinity, and taking $\partial/\partial t$ is the prescription Tseytlin found for removing it. Then the right-hand side of (1.1) should be identified with the genus zero contribution to the generating functional W (the quantity usually defined in field theory by $W = -\ln Z$), and we get $W = I$. From W one gets the Helmholtz free energy by $W = \beta F$.

Then F determines the thermodynamics. In particular, the entropy is given by the standard formula

$$S = \beta^2 \frac{\partial}{\partial \beta} F \ , \tag{1.2}$$

where β is the inverse temperature, proportional to the periodicity of the regular Euclidean continuation of the black hole spacetime. The derivatives $\partial/\partial\beta$ in (1.2) direct us to vary this periodicity, which creates a conical singularity in the spacetime. We can then rewrite (1.2) using ϵ, the angular excess, instead of β, getting

$$S = (2\pi + \epsilon)^2 \frac{\partial}{\partial \epsilon} \frac{1}{2\pi + \epsilon} I \bigg|_{\epsilon=0} \ . \tag{1.3}$$

So the computation boils down to computing the first variation of the Euclidean action under the introduction of a conical singularity; see [5,2] for more on this idea. The computation will be carried out in the next section, where its equivalence to the Noether charge method will be demonstrated.

II. COMPUTING THE ENTROPY

First we review Wald's Noether charge method (for further details see [3,6,7]). The starting point is a covariant Lagrangian L, written as a d form, where d is the number of dimensions. Then one computes the variation of L under a diffeomorphism generated by an arbitrary vector field ζ. This can always be written schematically as

$$\delta L = E^i \delta\psi_i + d\theta(\delta\psi_i) \ , \tag{2.1}$$

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where ψ_i are the fields, $\delta\psi_i$ are their variations, E^i are the equations of motion, and θ is some $d-1$ form depending on the $\delta\psi_i$. The condition of covariance of L is

$$\delta L = d(\zeta \cdot L) , \tag{2.2}$$

where $\zeta \cdot L$ means ζ contracted onto the first index of L. Then for on-shell fields ($E = 0$), the last two equations imply

$$d(\theta - \zeta \cdot L) = 0 , \tag{2.3}$$

so that $J \equiv \theta - \zeta \cdot L$ is a closed form; J is just the dual of the expected conserved current. Ordinarily one would not expect J to be exact as well, but here the fact that it is closed for *all* ζ allows one to prove exactness [8]. So one has

$$J = dQ \tag{2.4}$$

for some $d-2$ form Q, which depends on ζ. The final step is to specialize ζ to be the horizon Killing field for the black hole, normalized to give unit surface gravity; then the entropy is identified as

$$S = 2\pi \int_H Q , \tag{2.5}$$

where H is the horizon $d-2$ surface.

Now for the method of conical singularities. We start with a somewhat formal computation, and later sketch how to make it rigorous. For simplicity, we specialize to the spherically symmetric case, and we choose coordinates such that the Euclidean metric is

$$ds^2 = dr^2 + f(r)^2 d\phi^2 + g(r) d\Omega^2 , \tag{2.6}$$

where $f(r) \sim r^2$ as $r \sim 0$, and $d\Omega^2$ represents the other $d-2$ coordinates, all angular, which play no role in the computation. The Euclidean time coordinate is ϕ, which has a 2π period for regularity at $r = 0$.

To add a conical singularity with angular excess ϵ requires the metric variation

$$\delta g_{\phi\phi} = \frac{\epsilon}{\pi} f^2 . \tag{2.7}$$

In order to cast the computation in a form similar to the above, we look for a vector field ζ which generates this variation via the usual formula for diffeomorphisms, $\delta g_{ab} = \nabla_{(a}\zeta_{b)}$. This ζ cannot be smooth, since the metric variation in question does not result from a diffeomorphism. The vector field which does the trick is

$$\zeta^a = \frac{\epsilon}{2\pi} \phi \frac{\partial^a}{\partial\phi} , \tag{2.8}$$

which is smooth everywhere except on a cut at $\phi = 0$. Note that this is just $(\epsilon/2\pi)\phi$ times the horizon Killing vector used in the Noether charge computation (in particular, $\partial/\partial\phi$ is normalized for unit surface gravity).

From here the computation is almost the same as before. We need to compute $\delta I = \int \delta L$. We write δL in the form of (2.1) above. We use the on-shell condition $E = 0$, and integrate the $d\theta$ term onto the boundary B, which consists of both sides of the cut plus the asymptotic surface (see Fig. 1). Then

$$\begin{aligned} \delta I &= \int_B \theta \\ &= \int_B J + \frac{\epsilon}{2\pi} \int_B \left(\phi \frac{\partial}{\partial\phi} \cdot L \right) \\ &= \int_H Q|_0^{2\pi} + \epsilon \int_{\phi=0} \left(\frac{\partial}{\partial\phi} \cdot L \right) , \end{aligned} \tag{2.9}$$

where J and Q are as defined above. In the final step, we first used $J = dQ$ to integrate J onto the boundary of B, which we take to be $H|_0^{2\pi}$. Then we observed that $\phi(\partial/\partial\phi) \cdot L$ has no projection into the asymptotic part of B, so its only contribution comes from the cut.

But whereas above Q was evaluated for $\zeta = \partial/\partial\phi$, here we have $(\epsilon/2\pi)Q[\phi(\partial/\partial\phi)]$. It seems ϕ cannot be factored out, since $\nabla\phi$ terms may appear; but what saves us is that $\nabla\phi$ is smooth across the cut, so that all $\nabla\phi$ terms vanish from $Q|_0^{2\pi}$. So we can *can* factor out the $(\epsilon/2\pi)\phi$, giving

$$I + \delta I = \epsilon \int_H Q\left(\frac{\partial}{\partial\phi}\right) + (2\pi + \epsilon) \int_{\phi=0} \left(\frac{\partial}{\partial\phi} \cdot L \right) . \tag{2.10}$$

[Here we also used the $\partial/\partial\phi$ symmetry to write $I = 2\pi \int_{\phi=0} (\partial/\partial\phi) \cdot L$.] Finally we compute S by plugging into (1.3). Note that the second term is the classical contribution of the fields in the spacetime away from the horizon; we expect this to make no contribution to the entropy, and it does not, since it is proportional to $2\pi + \epsilon$. The remainder gives the same result obtained above: namely,

$$S = 2\pi \int_H Q , \tag{2.11}$$

with Q evaluated on $\partial/\partial\phi$.

Unfortunately, the above calculation is not rigorous, since for one thing, relevant quantities such as $\nabla_a\phi$ are

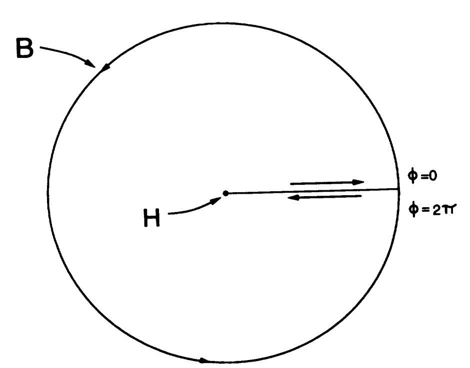

FIG. 1. r, ϕ cross section of the Euclidean black hole spacetime, showing the cut at $\phi = 0$ and the integration path B.

not defined at $r = 0$, and for another, the action I will typically diverge if a conical singularity is added (for example $\int R^2$ will diverge). Here we outline a more rigorous path to the same conclusion.

Starting from the Euclidean black hole spacetime, we cut out a small disk around $r = 0$. Then, in the remaining annulus (which extends to $r = \infty$), we choose again the metric generated by ζ^a [Eq. (2.8)]. This gives the annulus a conical geometry with angular excess ϵ. Then, we choose some smooth metric variation on the disk which matches smoothly onto that of the annulus. Then we calculate the variation in the action due to these metric variations, using (2.1).[1] Finally, we integrate $d\theta$ onto the boundaries, as in (2.9).

Now there are two extra boundary segments, the inner boundary of the annulus, and the outer boundary of the disk (see Fig. 2). But since the metric variation is smooth across the disk-annulus boundary, these extra contributions simply cancel each other. The remaining path is just B from Fig. 1, except that the cut only extends to the disk boundary, so in particular $r = 0$ is avoided. The final step is to take the limit as the disk shrinks to zero radius, recovering the result (2.10) above.

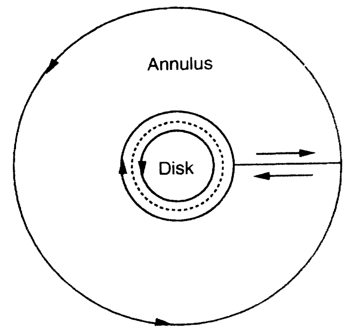

FIG. 2. The black hole spacetime with disk and annulus marked (the dotted line is the boundary). Also shown are the paths of the boundary integrations.

[1]Note that (2.1) holds for arbitrary variations, although it was used above only for variations resulting from a diffeomorphism.

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