Gravitational instantons and moduli spaces in topological two-form gravity

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A topological version of four-dimensional (Euclidean) Einstein gravity which we propose regards anti-self-dual two-forms and an anti-self-dual part of the frame connections as fundamental fields. The theory describes the moduli spaces of conformally self-dual Einstein manifolds for a cosmological constant $\Lambda \neq 0$ case and an Einstein-Kählerian manifold with the vanishing real first Chern class for $\Lambda = 0$. In the $\Lambda \neq 0$ case, we evaluate the index of the elliptic complex associated with the moduli space and calculate the partition function. We also clarify the moduli space and its dimension for $\Lambda = 0$ which are related to Plebansky's heavenly equations.

I. INTRODUCTION

Recently a number of noteworthy connections have been revealed between a class of field theories called topological quantum field theories on one hand and the mathematical advances in the topology and geometry of low dimensional manifolds on the other. The study of these relations has been introduced by Schwarz [1] and Witten [2]. Topological quantum field theories are constructed by fields, symmetries, and equations.

One concept that lies in the topological quantum field theory is the realization of the moduli spaces. The moduli space is defined as the equivalent set of the solutions of the fields for the equations associated with the symmetries of topological quantum theories. These theories can be described by the moduli spaces and are characterized by their topological and geometrical invariants which depend only on moduli parameters. There may be various topological quantum field theories which describe the same moduli space. The prim interest of these theories is these invariants, which are computable by standard techniques in quantum field theories.

Some gravitational versions of topological quantum field theories are also given by Witten [3,4]. The two-dimensional gravity models are of importance and promise new insight into string theory [4]. He conjectured that a certain series of critical points in the matrix model approach (i.e., the dynamical simplicial decomposition of Riemannian surfaces) is equivalent to two-dimensional topological gravity. In fact, Kontsevich used the intersection theory [5] to support the conjecture. This result is important to know the nonperturbative effect of string theory.

Since the work of Witten, there have been several attempts to construct four-dimensional topological gravity theories over different kinds of gravitational moduli

spaces [6–11]. For example, the moduli space of the conformally self-dual gravitational instantons was investigated in detail by Perry and Teo [6].

In previous papers [12,13] we proposed a four-dimensional topological gravity model. This model contains two types of topological field theories: Witten-type topological field theory in the cosmological constant $\Lambda \neq 0$ case and Schwarz-type topological field theory in the $\Lambda = 0$ case. They are obtained by modifying a chiral formulation of Einstein gravity developed by Capovilla *et al.* [14]. In these theories three quaternionic Kähler forms and the anti-self-dual part of the frame connections of the principal bundle $P_{SO(4)}$ are used as fundamental fields. The advantages of using these fields are that the treatment analogous to that of Yang-Mills field is possible and that the moduli space can be easily defined in terms of them efficiently.

In the $\Lambda \neq 0$ case, the moduli space is the set of the equivalence class of the fields defining the Einstein conformally self-dual Riemannian manifolds. This moduli space is up to orientation, identical with the one considered by Torre [7]. In his paper the dimension of the moduli space is found to be zero when the cosmological constant is positive, and the result is true also in our case. In the $\Lambda = 0$ case, the moduli spaces are those of Einstein-Kählerian manifolds with vanishing real first Chern class.

The purpose of our attempt is to investigate the four-dimensional gravitational instantons and derive the topological invariants such as the partition function and observables. We explore the relation between the simplicial decomposition of four-dimensional manifolds and the four-dimensional topological gravity.

We can regard the $\Lambda \neq 0$ case as a simple example of a gravitational analogue of the Donaldson theory and expect that we can calculate some topological invariants

such as the partition function in four dimensions.

On the contrary, the $\Lambda = 0$ case is a BF-type[1] topological gravity model. The partition function of the Abelian BF theory is represented by the Ray-Singer torsion [16]. Thus it is interesting to confirm whether or not our partition function in the $\Lambda = 0$ can be related to the Ray-Singer torsion.

Another aspect of the $\Lambda = 0$ case is that it provides the self-dual equations of Riemannian curvature two-form. There have been discovered various kinds of noncompact gravitational instantons [i.e., asymptotically locally Euclidean (ALE) [17] or asymptotically locally flat (ALF) [18]] which satisfy these equations. In this paper we will treat the compact manifolds only. In the near future we will extend our investigation to the noncompact case.

The plan of the paper is as follows. In Sec. II, we present a classical action, fields content, and equations of motion, and define the moduli spaces in our theory. We explain each case separately to avoid the confusion. In Sec. III, we formulate the Becchi-Rouet-Stora-Tyutin (BRST) transformations of our model. In Sec. IV, we mention the dimension of the moduli space and zero modes which appear in the partition function in the $\Lambda \neq 0$ case. In Sec. V, the gauge fixing conditions are introduced and the partition function is derived in the $\Lambda \neq 0$ case. In Sec. VI, we explain the dimension of the moduli space in the $\Lambda = 0$ case. Section VII is devoted to discussion.

II. TOPOLOGICAL TWO-FORM GRAVITY

We adopted the following action suggested by Capovilla et al. [14] and Horowitz [19] for our topological gravity model on a four-dimensional manifold M_4 [12,13]:

$$S_{\text{TG}} = \frac{1}{\alpha} \int_{M_4} [\Sigma^k \wedge F_k - \frac{\Lambda}{24} \Sigma^k \wedge \Sigma_k]$$
$$(k = 1, 2, 3), \quad (1)$$

where α is a dimensionless parameter and Λ is a cosmological constant (as we will see later that it will appear in Einstein equation $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\mu, \nu = \{0, \ldots, 3\}$).

We start with fundamental fields, a trio of su(2) valued two-forms Σ^k and a su(2) valued one-form $\omega^k = \omega^k_\mu dx^\mu$. F^k denotes the su(2) valued two-form with $F^k = F^k_{\mu\nu} dx^\mu \wedge dx^\nu \equiv d\omega^k + (\omega \times \omega)^k = d\omega^k + \epsilon^{ijk} \omega^i \wedge \omega^j$ [ϵ^{ijk} is the structure constant of SU(2)]. Varying the action with respect to each of fields Σ^k and ω^k we obtain the equations of motion

$$\Lambda \neq 0: \quad F^k - \frac{\Lambda}{12}\Sigma^k = 0, \quad D\Sigma^k = 0, \quad (2a)$$

$$\Lambda = 0: \quad F^k = 0, \quad D\Sigma^k = 0, \quad (2b)$$

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[1] Non-Abelian or Abelian theories whose Lagrangian forms are $B \wedge F$ will be called BF-type models, where B will be an $(n-2)$-form and $F = F_A$ is the curvature two-form of some connection A.

where $D\Sigma^k = d\Sigma^k + 2(\omega \times \Sigma)^k$.

In this paper we take $\alpha \to 0$ limit in Eq. (1) to make the contribution from Eq. (2a) or (2b) dominant in our theory. We are interested in the moduli spaces which are defined by Eq. (2a) or (2b) and the gauge fixing conditions which we will explain later. This treatment is similar to that of the large k limit of the Chern-Simons theory [15].

For $\Lambda \neq 0$, one of equations of motion $D\Sigma^k = 0$ can be derived by $F^k - (\Lambda/12)\Sigma^k = 0$ and Bianchi identity $DF^k = 0$. Eliminating Σ^k from the action by using Eq. (2a) we obtain the effective action proportional to the second Chern number $\int F^k \wedge F_k$, which is the classical action of the topological Yang-Mills theory (TYMT) (the Donaldson theory) for the SU(2) gauge group [20]. Thus the theory reduces to a Witten-type topological gravity model on-shell. On the other hand, for the $\Lambda = 0$ case, the action describes a Schwarz-type (BF-type) topological field theory [16,19].

We suppose the following conditions for the action.

Postulate 1a for M_4 with $\Lambda \neq 0$. M_4 is a four-dimensional oriented Riemannian manifold.

Postulate 1b for M_4 with $\Lambda = 0$. M_4 is a four-dimensional oriented Riemannian manifold and has an almost complex structure with its real first Chern class $c_1(M_4)_R = 0$.

Postulate 2 for the field ω^k. We consider the principal bundle $P_{\text{SO}(4)}$ of oriented orthonormal frames over M_4 with the structure group SO(4). This bundles is associated by the tangent bundle with a metric $\tilde{g}_{\mu\nu} = \tilde{e}^a_\mu \tilde{e}^b_\nu \delta_{ab}$, where $a, b = \{0, \ldots, 3\}$. $\tilde{e}^a = \tilde{e}^a_\mu dx^\mu$ is a vierbein [a section of $\text{End}(TM_4) = T^*M_4 \otimes TM_4$ with the assumption of $\det(\tilde{e}) \neq 0$]. We suppose that the field ω^k_μ denotes an anti-self-dual part of the frame connections (a connection of $P_{\text{SU}(2)}$ which comes from $P_{\text{SO}(4)} \sim P_{\text{SU}(2)} \oplus P_{\text{SU}(2)}$). ω^k is related to the anti-self-dual part of the so(4) valued one-form connection $^{(-)}\omega^{ab}$ via η^k_{ab}:

$$^{(-)}\omega^{ab}_\mu(\tilde{e}) = \eta^{ab}_k \omega^k_\mu(\tilde{e}), \quad (3)$$

where η^k_{ab} is an anti-self-dual constant called 't Hooft's η symbols [21] $\eta^{ab}_k = \epsilon_k{}^{ab0} + \frac{1}{2}\epsilon_{kij}\epsilon^{ijab}$ with $i, j, k = \{1, 2, 3\}$. Some useful properties of η^{ab}_k are given in Appendix A. A point to notice is that M_4 is an oriented Kählerian manifold with $c_1(M_4)_R = 0$ from postulate 1b. Thus at least the reduction of the structure group SO(4)$\to$U(2) is possible for the $\Lambda = 0$ case (see Ref. [21] and Fig. 1).

Furthermore we assume the parallelizability of $\tilde{e}^a_\nu$ with the Levi-Civita connection $\Gamma^\sigma_{\mu\nu}$ and the frame connection ω^{ab}_μ defined by $\tilde{e}^a_\mu$. This is a sufficient condition for the metricity of $\tilde{g}_{\mu\nu}$ ($\nabla_\tau \tilde{g}_{\mu\nu} = \partial_\tau \tilde{g}_{\mu\nu} - \Gamma^\sigma_{\mu\tau}\tilde{g}_{\sigma\nu} - \Gamma^\sigma_{\nu\tau}\tilde{g}_{\sigma\mu} = 0$):

$$\nabla_\nu(\tilde{e})\tilde{e}^\mu_a = \partial_\nu \tilde{e}^\mu_a + \Gamma^\mu_{\sigma\nu}(\tilde{e})\tilde{e}^\sigma_a - \omega^b{}_{a\nu}(\tilde{e})\tilde{e}^\mu_b = 0. \quad (4)$$

From this equation, the relation between Riemannian tensor and the curvature tensor F^k is given by

$$^{(-)}R_{\mu\nu}{}^{\rho\tau}(\tilde{g}(\tilde{e})) = 4F_{\mu\nu}{}^{ab}(\tilde{e})\tilde{e}^\rho_a \tilde{e}^\tau_b$$
$$= 4F_{\mu\nu}{}^k(\tilde{e})\eta^{ab}_k \tilde{e}^\rho_A \tilde{e}^\tau_b, \quad (5)$$

FIG. 1. Hol(g) on M_4 with torsionless connections: (1) almost complex $\{M_4, J\}$, GL(2, C) $\supseteq$ Hol(g). (2) Kählerian $\{M_4, g, J\}$, U(2) $\supseteq$Hol(g), e.g., CP^2 with Kähler metrics. (3) Riemannian $\{M_4, g\}$, O(4) $\supseteq$ Hol(g), e.g., S^4 with Riemannian metrics. (4) Ricci-flat Kählerian $\{M_4, g, J\}$, SU(2) $\supseteq$ Hol$_0$(g), e.g., $K3/Z_2$ with Ricci-flat Kähler metrics. (5) hyper-Kählerian $\{M_4, g, J^1, J^2, J^3\}$, SP(1) $\cong$ SU(2) $\supseteq$ Hol(g), e.g., $K3$ with Calabi-Yau metrics, T^4 with flat metrics.

where $R_{\mu\nu\rho\tau} = {}^{(+)}R_{\mu\nu\rho\tau} + {}^{(-)}R_{\mu\nu\rho\tau}$. It is well known that the Riemann curvature tensors over M_4 are written in block diagonal form of 6×6 matrix [24]:

$$R = \begin{pmatrix} {}^{(+)}R_{\mu\nu\rho\tau} \\ {}^{(-)}R_{\mu\nu\rho\tau} \end{pmatrix}$$

$$= \begin{pmatrix} {}^{(+)}W_{\mu\nu\rho\tau} + {}^{(+)}S_{\mu\nu\rho\tau}, & K_{\mu\nu\rho\tau} \\ {}^{t}K_{\mu\nu\rho\tau}, & {}^{(-)}W_{\mu\nu\rho\tau} + {}^{(-)}S_{\mu\nu\rho\tau} \end{pmatrix}.$$

${}^{(+)}W_{\mu\nu\rho\tau}$ is the self-dual part of the Weyl tensor and ${}^{(-)}W_{\mu\nu\rho\tau}$ is its anti-self-dual part. ${}^{(\pm)}S_{\mu\nu}{}^{\rho\tau} \propto (\delta_{[\mu}^{[\rho}\delta_{\nu]}^{\tau]} \pm \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\tau})R$ and $K_{\mu\nu}{}^{\rho\tau} \propto \delta_{[\mu}^{[\rho}\Phi_{\nu]}^{\tau]}$, where $\Phi_{\mu}{}^{\tau} = R_{\mu}{}^{\tau} - \frac{1}{4}\delta_{\mu}^{\tau}R$ is the trace-free part of the Ricci tensor and R is the scalar curvature.

Postulate 3 for the Σ^k field. $\{\Sigma^k\}$ are a trio of su(2) valued two-forms. We suppose that the index k of Σ^k and ω^k denotes the anti-self-dual part of so(4) index. Namely, su(2) is a Lie algebra of SU(2)$_L$ which comes from Spin(4) = SU(2)$_L$ × SU(2)$_R$ [the double covering group of SO(4)].

Our stance for this model is that the fundamental variables are not $\tilde{e}^a$ but ω^k and Σ^k. We seek the solutions of them which satisfy the above postulates, equations of motion, and the gauge-fixing conditions. These conditions specify the manifolds concerning our model and the property of metrics or almost complex structures on them.

We now turn to the gauge fixing conditions which we set to restrict 5 degrees of freedom of Σ^k. In this topological model, there exists a symmetry generated by a parameter one-form θ^k in addition to the SU(2)$_L$ [with a su(2) valued zero-form v^k] and diffeomorphism (with a vector field ξ^μ) symmetries:

$$\delta\omega_\tau^k = D_\tau v^k + (\mathcal{L}_\xi \omega^k)_\tau + \frac{\Lambda}{12}\theta_\tau^k,$$

$$\delta\Sigma_{\mu\nu}^k = 2(\Sigma_{\mu\nu} \times v)^k + (\mathcal{L}_\xi \Sigma^k)_{\mu\nu} + D_{[\mu}\theta_{\nu]}^k. \tag{6}$$

The θ^k symmetry is regarded as a "restricted" topological symmetry which preserves the equations of motion (2a) or (2b). With the appearance of the θ^k symmetry, the theory turns out to be on-shell reducible in the sense that the transformation laws (6) are invariant under

$$\delta v^k = -\frac{\Lambda}{12}\epsilon^k + \rho^\sigma \omega_\sigma^k,$$

$$\delta\theta_\mu^k = D_\mu \epsilon^k + 2\rho^\nu \Sigma_{\nu\mu}^k, \tag{7}$$

$$\delta\xi^\mu = -\rho^\mu,$$

as long as the equations of motion are satisfied. The transformations with parameters ϵ^k and ρ^μ correspond to redundant SU(2) and redundant diffeomorphism, respectively.

Our strategy to construct a topological quantum field theory is to consider the following five equations as gauge fixing conditions for the θ^k symmetry (except for the redundant part of the symmetry).

Postulate 4 for the gauge fixing conditions for the θ^k symmetry:

$$\text{TF} \Sigma^i \wedge \Sigma^j \equiv \Sigma^{(i} \wedge \Sigma^{j)} - \frac{1}{3}\delta_{ij}\Sigma^k \wedge \Sigma_k = 0. \tag{8}$$

These constraints were imposed in the original two-form Einstein gravity [14] and are necessary and sufficient conditions that Σ^k comes from a vierbein $e^a = e_\mu^a dx^\mu$:

$$\Sigma^k(e) = -\eta_{ab}^k e^a \wedge e^b. \tag{9}$$

(We should remark that e^a is independent of $\tilde{e}^a$ in this stage.) $\{\Sigma^k(e)\}$ have 13 degrees of freedom. The Riemannian metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ can be expressed in terms of $\Sigma^k(e)$:

$$g^{1/2}g_{\mu\nu} = -\frac{1}{12}\epsilon^{\alpha\beta\gamma\delta}\Sigma_{\mu\alpha k}(\Sigma_{\beta\gamma} \times \Sigma_{\delta\nu})^k,$$

$$g \equiv \det(g_{\mu\nu}). \tag{10}$$

Such a two-form $\Sigma^k(e)$ is anti-self-dual with respect to world indices μ, ν by the Hodge dual operation $^*g(\Sigma(e))$ which is defined via Eq. (10). [But it does not necessarily mean that $\Sigma^k(e)$ belongs to su(2) valued anti-self-dual two-forms part only because these eigenspaces of the dual operation vary as the deformations of Σ^k and $g_{\mu\nu}(\Sigma)$.]

The set of Eqs. (2a) or (2b) and (8) arose before as an ansatz within the framework of two-form Einstein gravity with the cosmological constant [14,23]. We consider them to be *gravitational instanton equations*. The degrees of the freedom of the fundamental fields are completely fixed by the above conditions (see Table I).

We also assume the parallelizability of e_ν^a with the Levi-Civita connection and the frame connection defined by e_ν^a as before. The equation $\nabla(e)_{[\mu}\nabla(e)_{\nu]}\Sigma_{\sigma\tau}^k = 0$, which comes from Eq. (9) and this parallelizability yield the following relation between the curvature tensor $F^k(e)$ and Riemannian curvature tensor:

$${}^{(-)}R_{\mu\nu\rho\tau}(e) = 4F_{\mu\nu}{}^k(e)\Sigma_{\rho\tau k}(e). \tag{11}$$

Now we will explain what kinds of Riemmanian tensors are derived from the solutions of $\{\Sigma^k\}$ which satisfy Eqs.

TABLE I. A dimension counting of fundamental variables.

$\Lambda \neq 0$ case	Degrees	$\Lambda = 0$ case	Degrees
Σ^k	$3 \times 6 = 18$	Σ^k	$3 \times 6 = 18$
ω^k	$3 \times 4 = 12$	ω^k	$1 \times 4 = 4$
Total	30	Total	22
{diffeo. $\times$ SU(2)}		{diffeo. $\times$ U(1)}	
gauge fix. condi.	$4 + 3 = 7$	gauge fix. condi.	$4 + 1 = 5$
{rest. top. / red.}		{rest. top. / red.}	
gauge fix. condi.	5	gauge fix. condi.	5
Eq. of the motion		Eqs. of motion	
$F^k = (\Lambda/12)\Sigma^k$	18	$\{F^k = 0\}/\{DF^k = 0\}/$	
		$\{D^2 F^k = 0\}$,	$6 - 4 + 1 = 3$
		$\{D\Sigma^k = 0\}/\{D^2\Sigma^k = 0\}$	$3 \times (4 - 1) = 9$
Total	30	Total	22

(8), (11), and (2a) for $\Lambda \neq 0$ [or Eqs. (8), (11), and (2b) for $\Lambda = 0$] and show the definitions of each moduli space.

(a) $\Lambda \neq 0$ case. Using the property of Riemannian manifold such as torsion tensor $T^a \propto D(e)e^a = 0$ and Eq. (9) we obtain $D(e)\Sigma^k(e) = 0$. Comparing this with $D(\tilde{e})\Sigma^k(e) = 0$,

$$\omega^k(\tilde{e}) = \omega^k(e), \quad F^k_{\mu\nu}(\tilde{e}) = F^k_{\mu\nu}(e). \tag{12}$$

Thus we obtain that

$$^{\mathrm{TF}}F^i(e) \wedge F^j(e) = ^{\mathrm{TF}}F^i(\tilde{e}) \wedge F^j(\tilde{e}) = 0. \tag{13}$$

which leads to $\Sigma^k(e) = \Sigma^k(\tilde{e})$. (Note that $\Sigma^k(e) = \Sigma^k(\tilde{e})$ is not a sufficient condition for $e^a = \tilde{e}^a$ [14].

By substituting $F^k(e) = F^k(\tilde{e}) = (\Lambda/12)\Sigma^k(e)$ into Eq. (11),

$$^{(-)}R_{\mu\nu\rho\tau}(\Sigma(e)) = ^{(-)}S_{\mu\nu\rho\tau}(\Sigma(e)). \tag{14}$$

Namely, M_4 becomes a conformally self-dual Einstein manifold:

$$\Lambda \neq 0: \quad R_{\mu\nu}(e) = \Lambda g_{\mu\nu}(e) \quad \text{and} \quad ^{(-)}W_{\mu\nu\rho\tau}(e) = 0. \tag{15}$$

The moduli space in this case is defined by ω^k only:

$$\mathcal{M}(\omega) = \{\omega^k | \omega^k \in \mathrm{su}(2) \otimes \wedge^1, \; ^{\mathrm{TF}}F^i \wedge F^j = 0\}/\{\mathrm{SU}(2) \times \mathrm{diffeo.}\}. \tag{16}$$

It corresponds to the moduli space of the conformally self-dual Einstein metrics because these metrics are represented by ω^k via Eqs. (2a) and (10). If we consider only compact Einstein conformally self-dual Riemannian manifolds with $R > 0$ ($\Lambda > 0$), then M_4 is either isometric to S^4, or to $\mathbb{CP}^2$, with their standard metrics from the theorem given by Hitchin [24]. So the solution $\Sigma^k(e)$ determines the standard metric on S^4 or the Fubini-Study metric on $\mathbb{CP}^2$ for $\Lambda > 0$.

(b) $\Lambda = 0$ case. From Eqs. (5) and (2b) the Riemannian tensor defined by $\tilde{e}^a$ is self-dual:

$$^{(-)}R_{\mu\nu\rho\tau}(\tilde{e}) = 0 \tag{17}$$

so (M_4, g) is a Ricci-flat Kählerian manifolds. The following theorem gives the characterization of Ricci-flat Kählerian manifolds.

Theorem (Hitchin [24]). Let M_4 be a compact connected oriented Riemannian manifold. If M_4 is Ricci-flat and $^{(+)}W_{\mu\nu\tau\rho} = 0$, then (M_4, g) is one of the following four cases: (1) (M_4, g) is flat, i.e., is covered by a flat four-torus; (2) (M_4, g) is a Kähler-Einstein $K3$-surface ($\pi_1 = 1$); (3) (M_4, g) is a Kähler-Einstein Enriques sur-

face ($\pi_1 = Z_2$); (4) (M_4, g) is the quotient of a Kähler-Einstein Enriques surface by a free antiholomorphic isometric involution ($\pi_1 = Z_2 \times Z_2$).

[We should better take the opposite orientation of M_4 and replace η^k_{ab} by self-dual notation $\bar{\eta}^k_{ab}$ for the $\Lambda = 0$ case so that $^{(+)}R_{\mu\nu\rho\tau}(e) = 0$ and Einstein-Kähler forms $\{\Sigma\}$ belong to self-dual (1,1) form.]

The relation between the vanishing covariant derivative with a Levi-Civita connection and the holonomy group $\mathrm{Hol}(g)$ asserts that $\mathrm{Hol}(g) \subseteq \mathrm{U}(2)$ for Kählerian manifolds with the complex dimension two. These compact Kählerian manifolds with $c_1(M_4)_R = 0$ are exactly the compact complex manifolds admitting a Kähler metric with zero Ricci form (or equivalently the compact complex manifolds with restricted holonomy group contained in the special unitary group). We now investigate the properties of the metrics or the complex structures defined by $\Sigma^k(e)$ on these manifolds. We divide these manifolds into two groups.

Case b-1 (when the canonical bundle K is trivial). M_4 is a $K3$-surface or a four-torus T^4. On these two manifolds, the following reductions of $P_{\mathrm{U}(2)}$ are possible due to the fact that the canonical bundles K over them are triv-

ial. Actually, the restricted holonomy group $\mathrm{Hol}_0(g)$ reduces to the identity exactly when a metric is flat for T^4. Although T^4 is not simply connected, $\mathrm{Hol}(g) = \mathrm{Hol}_0(g)$ happens. Therefore the reduction $P_{\mathrm{U}(2)} \to P_I$ is possible and all frame connections can be gauged away when metric is flat [24]. A $K3$-surface is, by definition, a compact simply connected complex surface with trivial canonical bundle K and $b_1 = 0$. For a Calabi-Yau metric (a Kähler-Einstein metric) $\mathrm{Hol}(g) = \mathrm{Hol}_0(g) \subseteq \mathrm{SU}(2)_R \cong \mathrm{SP}(1)$. Thus $\mathrm{SU}(2)_L$ connections $\{\omega^k(\tilde{e})\}$ can be gauged away.

In these cases $D(\tilde{e})\Sigma^k(e) = 0$ reduces to $d\Sigma^k(e) = 0$ for all k. From $d\Sigma^k(e) = 0$ and $D(e)\Sigma^k(e) = 0$ we obtain $\omega^k(e) = 0$ and the Ricci-flat Kähler metric:

$$\Lambda = 0 : \quad R_{\mu\nu}(e) = 0, \quad {}^{(+)}W_{\mu\nu\tau\sigma}(e) = 0, \quad \omega^k(e) = 0.$$
(18)

$\{\Sigma^k(e)\}$ define a Calabi-Yau metric on a $K3$-surface or a flat metric on T^4. The gravitational instanton equations

for a $K3$-surface with Calabi-Yau metrics and a four-torus with flat metrics reduce to

$$d\Sigma^k = 0, \quad {}^{\mathrm{TF}}\Sigma^i \wedge \Sigma^j = 0.$$
(19)

These equations give the Ricci-flat condition and restrict $\{\Sigma^k\}$ to be a trio of closed Einstein-Kähler forms [8,14]. In Ref. [24] Plebanski used these equations to derive his "heavenly equations." A manifold which satisfies Eq. (19) is called hyper-Kählerian. On hyper-Kählerian manifolds, a trio of Kähler forms $\{\Sigma^k(e)\}$ is represented by

$$\Sigma^k(e) = -\eta^k_{ab}e^a \wedge e^b \propto g_{\alpha\bar{\beta}}J^{k\bar{\beta}}_{\bar{\gamma}}dz^\alpha \wedge d\bar{z}^{\bar{\gamma}},$$
(20)

where $\{J^k\}$ are a trio of the g-orthogonal complex structures which satisfy the quaternionic relations and $g_{\alpha\bar{\beta}}$ is an Hermite symmetric metric. z and $\bar{z}$ denote complex local coordinates on these manifolds.

The moduli space is the equivalent class of a trio of the Einstein-Kähler forms (the hyper-Kähler forms) $\{\Sigma^k(e)\}$:

$$\mathcal{M}(\Sigma) = \{\Sigma^k | \Sigma^k \in \mathrm{SU}(2) \otimes \wedge^2, \ {}^{\mathrm{TF}}\Sigma^i \wedge \Sigma^j = 0, \ d\Sigma^k = 0\}/\{\text{diffeo.}\}.$$
(21)

Case b-2 (when the canonical bundle K is not trivial). M_4 is $K3/Z_2$, $K3/Z_2 \times Z_2$, or T^4/Γ where Γ is some discrete group. In these cases the reductions of $P_{\mathrm{U}(2)} \to P_{\mathrm{SU}(2)_R}$ are not possible because the canonical bundles are not trivial. From Eqs. (2b), (8), and (11), the Riemannian self-dual tensors are also derived:

$$\Lambda = 0 : \quad R_{\mu\nu}(e) = 0, \quad {}^{(+)}W_{\mu\nu\tau\sigma}(e) = 0.$$
(22)

The Hitchin's theorem states that $\{\Sigma^k(e)\}$ form Einstein-Kähler metrics on these manifolds. In these cases $\mathrm{Hol}_0(g) \subset \mathrm{SU}(2)_R$ but $\mathrm{Hol}_0(g) \neq \mathrm{Hol}(g)$ is held. They are called as the locally hyper-Kählerian Kählerian manifolds and some information from ω^k and Σ^k will be needed to describe the moduli spaces.

III. BRST SYMMETRY

In this section we will explain the BRST symmetry of the model in the $\Lambda \neq 0$ case. Our action is invariant under the usual gravitational transformations, the restricted topological transformations. These transformations are invariant under the redundant transformations of them. We shall denote the BRST versions of the gravitational transf. $\delta^G_{\tilde{c}}$, the restricted topological transf. δ^S, and the redundant transf. $\delta^G_{\tilde{\gamma}}$, respectively.

(For $\Lambda = 0$ this model belongs to the BF-type model so more careful investigations into the symmetries is necessary [26].) We introduce the following notation for the BRST symmetry:

diffeo. × SU(2)	→	BRST		
		ghost	antighost	N-L field
(ξ^ν, v^k)	→	$\tilde{c}^k \equiv (c^\nu, c^k)$	$\tilde{b}^k \equiv (b_\nu dx^\nu, b^k)$	$\tilde{\pi}^k \equiv (\pi_\nu dx^\nu, \pi^k)$
red. diffeo. × red. SU(2)	→	BRST		
		ghost	antighost	N-L field
(ρ^ν, ϵ^k)	→	$\tilde{\gamma}^k \equiv (\gamma^\nu, \gamma^k)$	$\tilde{\beta}^k \equiv (\beta_\nu dx^\nu, \beta^k)$	$\tilde{\tau}^k \equiv (\tau_\nu dx^\nu, \tau^k)$
rest. topological sym.	→	BRST		
		ghost	antighost	N-L field
θ^i	→	ϕ^i	χ^{ij}	π^{ij}

The on-shell BRST transformations of this model are given by

$$
\begin{aligned}
&(1) && \delta_B \omega_\mu^i = D_\mu c^i + (\mathcal{L}_c \omega^i)_\mu + \tfrac{\Lambda}{12}\phi_\mu^i \equiv \delta_{\hat{\varepsilon}}^G \omega_\mu^i + \delta^S \omega_\mu^i, \\
&(2) && \delta_B \Sigma_{\mu\nu}^i = 2(\Sigma_{\mu\nu} \times c)^i + (\mathcal{L}_c \Sigma^i)_{[\mu\nu]} + D_{[\mu}\phi_{\nu]}^i \equiv \delta_{\hat{\varepsilon}}^G \Sigma_{\mu\nu}^i + \delta^S \Sigma_{\mu\nu}^i, \\
&(3) && \delta_B c^i = -(c \times c)^i + \mathcal{L}_c c^i - \tfrac{\Lambda}{12}\gamma^i + \gamma^\sigma \omega_\sigma^i \equiv -(c \times c)^i + \mathcal{L}_c c^i + \hat{\gamma}^i, \\
&(4) && \delta_B b^i = -2(b \times c)^i + \mathcal{L}_c b^i + \pi^i, \\
&(5) && \delta_B \pi^i = 2(\pi \times c)^i + \mathcal{L}_c \pi^i + \tfrac{\Lambda}{6}(b \times \gamma)^i + \gamma^\sigma D_\sigma b^i, \\
&(6) && \delta_B c^\mu = c^\sigma \partial_\rho c^\mu - \gamma^\mu, \\
&(7) && \delta_B b_\mu = (\mathcal{L}_c b)_\mu + \pi_\mu, \\
&(8) && \delta_B \pi_\mu = (\mathcal{L}_c \pi)_\mu + (\mathcal{L}_\gamma b)_\mu, \\
&(9) && \delta_B \gamma^i = 2(\gamma \times c)^i + \mathcal{L}_c \gamma^i + \gamma^\sigma \phi_\sigma^i \Rightarrow \delta_B \hat{\gamma}^i = 2(\hat{\gamma} \times c)^i + \mathcal{L}_c \hat{\gamma}^i, \\
&(10) && \delta_B \beta^i = 2(\beta \times c)^i + \mathcal{L}_c \beta^i + \tau^i, \\
&(11) && \delta_B \tau^i = -2(\tau \times c)^i + \mathcal{L}_c \tau^i + \tfrac{\Lambda}{6}(\beta^i \times \gamma)^i + \gamma^\sigma D_\sigma \beta^i, \\
&(12) && \delta_B \gamma^\mu = \mathcal{L}_c \gamma^\mu, \\
&(13) && \delta_B \beta_\mu = (\mathcal{L}_c \beta)_\mu + \tau_\mu, \\
&(14) && \delta_B \tau_\mu = (\mathcal{L}_c \tau)_\mu + (\mathcal{L}_\gamma \beta)_\mu, \\
&(15) && \delta_B \phi_\mu^i = -2(\phi_\mu \times c)^i + (\mathcal{L}_c \phi^i)_\mu + D_\mu \gamma^i + 2\gamma^\sigma \Sigma_{\sigma\mu}^i \equiv \delta_{\hat{\varepsilon}}^G \phi_\mu^i + \delta_{\hat{\gamma}}^G \omega_\mu^i + \tfrac{24}{\Lambda}\gamma^\rho \left(F_{\mu\rho}^i - \tfrac{\Lambda}{12}\Sigma_{\mu\rho}^i \right), \\
&(16) && \delta_B \chi^{ij} = 2(\chi \times c)^{ij} + \mathcal{L}_c \chi^{ij} + \tfrac{\Lambda}{6}(\chi \times \gamma)^{ij} + \gamma^\sigma D_\sigma \chi^{ij}, \\
&(17) && \delta_B \pi^{ij} = \chi^{ij} - 2(\pi \times c)^{ij} + (\mathcal{L}_c \pi^{ij}),
\end{aligned}
$$

$$(23)$$

where $\delta_{\hat{\gamma}}^G \omega_\mu^i$ denote the redundant transformations and are given by the replacement of the parameters $\hat{c} \to \hat{\gamma}$. The characteristic feature of our BRST symmetries is the presence of the restricted topological symmetry of the fundamental fields: $\delta^S \omega_\mu^i = (\Lambda/12)\phi_\mu^i$, $\delta^S \Sigma_{\mu\nu}^i = D_{[\mu}\phi_{\nu]}^i$.

As already pointed out, this action comes to a Witten-type action for $\Lambda \neq 0$ under $\alpha \to 0$ limit by removing Σ^k using the equation of motion. The restricted topological symmetry can be interpreted as the supersymmetry for a Witten-type model. The supersymmetric pair $(\delta \omega_\mu^k, \phi_\mu^k)$ is important because it forms a basis of the tangent space of the moduli space while the other pair $(\delta \Sigma_{\mu\nu}^k, D_{[\mu}\phi_{\nu]}^k)$ is the auxiliary one and can be removed by using the equation of motion. The symmetries in the $\Lambda \neq 0$ case are interpretable as

$$\{\text{SU(2)} \times \text{diffeo.} \times \text{super sym.}\}/\{\text{redundant SU(2)} \times \text{redundant diffeo.}\}. \tag{24}$$

TABLE II. Fields and their ghost assignment (-1 for form means a "vector").

Field	Content	Fermion/boson	Ghost number	Form	Representation
$\delta \omega^i = \delta \omega_{\mu\nu}^i dx^\mu \wedge dx^\nu$		B	0	1	$\Omega^{2,0} \otimes \Lambda^1$
$\delta \Sigma^i = \delta \Sigma_{\mu\nu}^i dx^\mu \wedge dx^\nu$		B	0	2	$\Omega^{2,0} \otimes \Lambda^2$
diffeo.$\to$ BRST					
c^ν	Ghost	F	1	-1	$TM_4 \simeq \Lambda^1$
$b = b_\nu dx^\nu$	Antighost	F	-1	1	$T^* M_4 \simeq \Lambda^1$
$\pi = \pi_\nu dx^\nu$	N-L field	B	0	1	$T^* M_4 \simeq \Lambda^1$
SU(2)$\to$ BRST					
c^i	Ghost	F	1	0	$\Omega^{2,0} \otimes \Lambda^0$
b^i	Antighost	F	-1	0	$\Omega^{2,0} \otimes \Lambda^0$
π^i	N-L field	B	0	0	$\Omega^{2,0} \otimes \Lambda^0$
red. diffeo.$\to$ BRST					
γ^ν	Ghost	B	2	-1	$TM_4 \simeq \Lambda^1$
$\beta = \beta_\nu dx^\nu$	Antighost	B	-2	1	$T^* M_4 \simeq \Lambda^1$
$\tau = \tau_\nu dx^\nu$	N-L field	F	-1	1	$T^* M_4 \simeq \Lambda^1$
red. SU(2)$\to$ BRST					
γ^i	Ghost	B	2	0	$\Omega^{2,0} \otimes \Lambda^0$
β^i	Antighost	B	-2	0	$\Omega^{2,0} \otimes \Lambda^0$
τ^i	N-L field	F	-1	0	$\Omega^{2,0} \otimes \Lambda^0$
SUSY$\to$ BRST					
$\phi^i = \phi_\nu^i dx^\nu$	Ghost	F	1	1	$\Omega^{2,0} \otimes \Lambda^1$
χ^{ij}	Antighost	F	-1	0	$\Omega^{4,0} \times \Lambda^0$
π^{ij}	N-L field	B	0	0	$\Omega^{4,0} \times \Lambda^0$

The transformations of these fields also end in those of the ordinary Witten-type theory even for off-shell except ϕ_μ^k and $\Sigma_{\mu\nu}^k$ by redefining the redundant SU(2) ghost as $\hat{\gamma}^i \equiv -(\Lambda/12)\gamma^i + \gamma^\mu\omega_\mu^i$. The BRST symmetry of ϕ^i agrees with Witten-type theory on shell. Thus this model coincides with the Witten-type topological gravity model given by Torre [7] up to the secondary Chern number which is our classical action after eliminating Σ^k under $\alpha \to 0$ limit.

From now on we will replace the Lie derivative $\mathcal{L}_c$ ($\mathcal{L}_\gamma$) with the modified one $\tilde{\mathcal{L}}_c\omega_\mu^k \equiv (\mathcal{L}_c\omega^k)_\mu - D(c^\mu\omega_\mu^k)$ $[\tilde{\mathcal{L}}_\gamma\omega_\mu^k \equiv (\mathcal{L}_\gamma\omega^k)_\mu - D(\gamma^\mu\omega_\mu^k)]$ so that $\delta_B\omega_\mu^k$ ($\delta_B\phi_\mu^k$) still remains in $P_{\mathrm{SU}(2)} \times \mathrm{Adsu}(2) \otimes \Lambda^1$:

$$(\tilde{\mathcal{L}}_c\omega^i)_\mu = 2c^\tau F_{[\tau\mu]},$$

(25)

$$\tilde{\mathcal{L}}_c\Sigma_{\mu\nu}^i = 2D_{[\mu}c^\sigma\Sigma_{\sigma\nu]}^i + 3c^\sigma D_{[\mu}\Sigma_{\sigma\nu]}^i.$$

Before we proceed, it will be useful to introduce general

spin bundles $\Omega^{m,n}$, the space of fields with spin (m, n) of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ [27]. Let us denote by $\Omega^{1,0}$, $\Omega^{0,1}$, the two complex vector bundles on M_4 associated with the defining two-dimensional representations of the two factors. These will only exist globally if M_4 is a spin manifold, i.e., the second Stiefel-Whitney class $w_2(M_4) = 0$. Let us denote by $\Omega^{m,n} \equiv S^m\Omega^{1,0} \otimes S^n\Omega^{0,1}$ the tensor product of the mth symmetric power bundle of $\Omega^{1,0}$ and the nth symmetric power bundle of $\Omega^{0,1}$. For example, the space of $P_{\mathrm{SU}(2)} \times \mathrm{Adsu}(2)$ valued one-form $\delta\omega^k$ is $\Omega^{2,0} \otimes \Lambda^1 \simeq \Omega^{2,0} \otimes \Omega^{1,1}$ while the space of (ξ^μ, v^k) is equivalent to $\Lambda^1 \oplus \Omega^{2,0} \simeq \Omega^{1,1} \oplus \Omega^{2,0}$ (see Table II).

IV. ZERO MODES IN THE $\Lambda \neq 0$ CASE

To know the number of zero modes in the quantum action S_q we consider the moduli space $\mathcal{M}$ defined by our instanton equations Eq. (16) for conformally self-dual Einstein manifolds:

$$\mathcal{M}(\omega) = \{\omega_0^k | \omega^k \in \mathrm{su}(2) \otimes \Omega^{1,1}, \;^{\mathrm{TF}}F^i \wedge F^j = 0\}/\{\mathrm{SU}(2) \times \mathrm{diffeo.}\}.$$

(26)

Given a solution (Σ_0^k, ω_0^k) of the instanton equations, the tangent space $T(\mathcal{M})$ of $\mathcal{M}$ is the space of infinitesimal deformations $\delta\omega^k$ which satisfy linearized instanton equations modulo deformations generated by SU(2) [the subgroup of SO(4)] transformation and diffeomorphism:

$$T(\mathcal{M}(\omega)) = \{\delta\omega^k | \delta\omega^k \in \Omega^{2,0} \otimes \Omega^{1,1}, D_1\delta\omega^k = 0\}/\{\mathrm{SU}(2) \times \mathrm{diffeo.}\},$$

(27)

where

$$D_1\delta\omega^k \equiv \;^{\mathrm{TF}}F_0^i \wedge D\delta\omega^j = 0.$$

(28)

This linearized instanton equation is derived by substituting equation $D\delta\omega^k - (\Lambda/12)\delta\Sigma^k = 0$ into $^{\mathrm{TF}}\Sigma_0^i \wedge \delta\Sigma^j = 0$.

We define the following sequence of mappings on a compact conformally self-dual Einstein manifold in terms of the spin bundles:

$$0 \xrightarrow{D_{-1}} C^\infty(\Omega^{1,1} \oplus \Omega^{2,0})$$
$$\xrightarrow{D_0} C^\infty(\Omega^{2,0} \otimes \Omega^{1,1})$$
$$\xrightarrow{D_1} C^\infty(\Omega^{4,0}) \xrightarrow{D_2} 0,$$

(29)

where the symbol sequence is

$$0 \to \overbrace{\Omega^{1,1} \oplus \Omega^{2,0}}^{V_0} \to \overbrace{\Omega^{2,0} \otimes \Omega^{1,1}}^{V_1} \to \overbrace{\Omega^{4,0}}^{V_2} \to 0.$$

In the above sequence D_{-1} and D_2 are identically zero operators. The operator D_0 is defined by

$$D_0(\xi^\mu, v^k) \equiv \tilde{\mathcal{L}}_\xi\omega^k + Dv^k.$$

(30)

We can easily check the ellipticity of the deformation

complex. Defining the inner product in each space V_i, we can introduce the adjoint operators D_0^* and D_1^* for D_0 and D_1, respectively, and the Laplacians Δ_i; $\Delta_0 = D_0^*D_0$, $\Delta_1 = D_0D_0^* + D_1^*D_1$, $\Delta_2 = D_1D_1^*$. We may then define the cohomology group on each V_i:

$$H^i \equiv \mathrm{Ker}D_i/\mathrm{Im}D_{i-1}.$$

(31)

It is easy to show that H^i is equivalent to the kernel of Δ_i, the harmonic subspace of V_i. These cohomology groups are finite dimensional. We call the dimension of H^i h^i. H^1 is exactly identical with the tangent space of $\mathcal{M}(\omega)$ in Eq. (27), the dimension of which we need to know. On the space V_0, H^0 is equal to $\mathrm{Ker}D_0$ because the image of D_{-1} is trivial. In the $\Lambda \neq 0$ case, Torre found that $\mathrm{Ker}D_0$ is equivalent to the space of the Killing vectors [7]. The kernel of D_2 is the whole of the space V_2. Hence H^2 is the subspace of V_2 orthogonal to the mapping D_1, or equivalently it is the kernel of D_1^*. The index of the elliptic complex is defined as the alternating sum:

$$\mathrm{index} \equiv h^0 - h^1 + h^2.$$

(32)

By applying the Atiyah-Singer index theorem [28] to the elliptic complex we obtain

$$
\begin{aligned}
\text{index} &= \int_{M_4} \frac{\mathrm{ch}(\Omega^{2,0} \oplus \Omega^{1,1} \ominus \Omega^{2,0} \otimes \Omega^{1,1} \oplus \Omega^{4,0})\mathrm{td}(TM_4 \otimes C)}{\mathrm{e}(TM_4)} \\
&= \int_{M_4} \frac{\mathrm{ch}(\Omega^{2,0} \ominus \Omega^{3,1} \oplus \Omega^{4,0})\mathrm{td}(TM_4 \otimes C)}{\mathrm{e}(TM_4)} \\
&= 5\chi - 7\tau ,
\end{aligned}
\tag{33}
$$

where ch, e, and td are the Chern character, Euler class, and Todd class of the various vector bundles involved. Therefore the alternating sum of h^i in Eq. (32) is determined by the Euler number χ and Hirzebruch signature τ. By changing $\tau \to |\tau|$, this index can also be adopted to manifolds with the opposite orientation.

If $\Lambda > 0$, h^1 and h^2 are found to be zero as shown by Torre [7]. The result such as $h^1 = 0$ for the $\Lambda > 0$ case agrees with the one of Perry and Teo. They showed that the dimension of the moduli space of conformally self-dual metrics is zero on S^4 or CP^2 by using the deformation complex for the metrics [6]. Therefore from Eqs. (32) and (33), the dimension h^0 is equal to the index:

$$
h^0 = 5\chi - 7\tau, \quad h^1 = h^2 = 0. \tag{34}
$$

The value of h^0, the dimension of the Killing vector space, agrees with that obtained by a different method in Ref. [23]. For S^4 with the standard metric, the dimension of the isometry is given by dim.$SO(5) = 10$, which coincides with $h^0 = 5\chi - 7\tau|_{\tau=0,\chi=2} = 10$. For CP^2 with the Fubini-Study metric, the dimension of the isometry is given by dim.$SU(3) = 8$ which agrees with $h^0 = 5\chi - 7\tau|_{\tau=1,\chi=3} = 8$. If $\Lambda < 0$, h^0 becomes zero [7], although h^1 and h^2 are not completely determined:

$$
h^0 = 0, \quad h^2 - h^1 = 5\chi - 7\tau. \tag{35}
$$

For conformally self-dual Einstein manifolds with $\Lambda < 0$, there are two known examples, which are hyperbolic surface/Γ and boundary domain/Γ where Γ is some discrete subgroup. The dimensions of their moduli spaces of conformally self-dual Einstein metrics are zero due to the Mostov's rigidity [29]. Thus in these cases the dimensions of the moduli spaces of the anti-self-dual frame connections are also zero.

V. THE PARTITION FUNCTION IN THE $\Lambda \neq 0$ CASE

For the purpose of the calculation of the partition function we first decompose ω^k and Σ^k fields as follows:

$$
\omega^k = \omega_0^k + \delta\omega^k, \quad \Sigma^k = \Sigma_0^k + \delta\Sigma^k, \tag{36}
$$

where ω_0 and Σ_0^k are the background solutions of conformally self-dual Einstein manifolds. $\delta\omega^k$ and $\delta\Sigma^k$ are quantum fluctuations (=infinitesimal deformations).

The BRST quantization of the Witten-type topological gravity model in the $\Lambda \neq 0$ case is straightforward. Twelve gauge fixing conditions for the supersymmetry and seven ones for $SU(2) \times$ diffeo. are imposed. The gauge fixing conditions for the supersymmetry consist of five gauge fixing conditions for the supersymmetry except for the redundancy and seven fixing conditions to remove the freedom of the redundant symmetries. We are fixing the gauge to be $D_0^* \delta\omega^k = 0$ for the diffeomorphism and $SU(2)$ and $D_0^* \phi^k = 0$ for the redundant diffeomorphism and redundant $SU(2)$:

$$
D_0^* \delta\omega^k \equiv \underset{\text{diffeo.}}{(\tilde{\mathcal{L}}_c^* \delta\omega^k,} \quad \underset{SU(2)}{D^* \delta\omega^k)} \quad = 0
\tag{37}
$$

$$
D_0^* \phi^k \equiv \underset{\text{red. diffeo.}}{(\tilde{\mathcal{L}}_\gamma^* \delta\omega^k,} \quad \underset{\text{red. } SU(2)}{D^* \psi^k)} \quad = 0
\tag{38}
$$

$$
(D_1^\delta \omega)^{ij} \equiv \underset{\text{super/\{red. diffeo.} \times \text{red. } SU(2)\}}{{}^{\mathrm{TF}}F^i \wedge D\delta\omega^j} \quad = 0
\tag{39}
$$

where the asterisk denotes the Hodge star dual operation and $\mathcal{O}^* \equiv -*\mathcal{O}*$ is the adjoint operator of $\mathcal{O}$. The operator $D_0^* : C^\infty(\Omega^{2,0} \otimes \Lambda^1) \to C^\infty(\Omega^{2,0} \otimes \Lambda^0)$ is the adjoint operator of $D_0 : \tilde{c}^k \to \delta\omega^k$,

$$
\begin{aligned}
D_0 \tilde{c}^k(\omega) &\equiv \tilde{\mathcal{L}}_c \omega^k + D_\tau c^k dx^\tau \\
&= 2c^\nu D_{[\nu}\omega_{\tau]}^k dx^\tau + D_\tau c^k dx^\tau,
\end{aligned}
\tag{40}
$$

where $D = d + (\omega \times \cdot)$. The elements of the image of $D_1 : C^\infty(\Omega^{2,0} \otimes \Lambda^1) \to C^\infty(\Omega^{4,0} \otimes \Lambda^4)$ are four-forms with symmetric trace-free $SU(2)$ indices, i.e., sections of $\Omega^{4,0} \otimes \Lambda^4 \simeq \Omega^{4,0} \otimes \Lambda^0$.

The gauge-fixed quantum action is given by

$$
\begin{aligned}
S_q = S_{\mathrm{TG}} + \int \delta_B[\chi_{ij}(D_1\delta\omega)^{ij} + \tilde{b}_k * D_0^* \delta\omega^k \\
+ \tilde{\beta}_k * D_0^* \phi^k].
\end{aligned}
\tag{41}
$$

When expanded out by using the properties of δ_B in Eq. (23), Eq. (41) reads

$$
\begin{aligned}
S_q = S_{\mathrm{TG}} + \int \pi_{ij}(D_1\delta\omega)^{ij} + \tilde{\pi}_k * D_0^* \delta\omega^k + \tilde{b}_k * D_0^* D_0 \tilde{c}^k(\omega) + \tilde{b}_k * D_0^* \phi^k \\
+ \int \chi_{ij}(D_1\phi)^{ij} + \tilde{\tau}_k * D_0^* \phi^k + \tilde{\beta}_k * D_0^* D_0 \tilde{\gamma}^k(\omega) + \tilde{\beta}_k * D_0^* D_0 \tilde{c}^k(\phi) + \text{other higher order terms.}
\end{aligned}
\tag{42}
$$

We are now ready to evaluate the partition function:

$$Z = \int \mathcal{D}X(-S_q), \qquad (43)$$

where $\mathcal{D}X$ represents the path integral over the fields such as $\delta\Sigma^k, \delta\omega^k$, ghosts, antighosts, N-L fields, etc. The Gaussian integrals over the commuting $\bar{\beta} - \bar{\gamma}$ set of fields in Eq. (42) yields the determinant $(\det\Delta_0)^{-1}$ which cancels with the $\det\Delta_0$ contribution coming from the anticommuting set of fields $\bar{b} - \bar{c}$ set. The term of $\bar{\beta}_k * D_0^* D_0 \bar{c}^k(\phi)$ is three-point interaction of ghosts so does not contribute to the partition function.

Consider now two terms $\bar{\tau}_k * D_0^* \phi^k + \chi_{ij}(D_1\phi)^{ij}$ (we absorb $\bar{b}_k * D_0^* \phi^k$ term into $\bar{\tau}_k * D_0^* \phi^k$ term). In calculating their determinants we use a differential operator $T = D_0^* \oplus D_1$ and its adjoint operator T^*:

$$T^*T; \Omega^{3,1} \oplus \Omega^{1,1} \underset{T^*}{\overset{T}{\rightleftharpoons}} \Omega^{0,0} \oplus \Omega^{2,0} \oplus \Omega^{4,0} . \qquad (44)$$

One could show $\det T = \det^{1/2}(T^*T) = (\det\Delta_1)^{1/2}$ by using matrix notations for T and T^*. The $\bar{\tau}_k - \delta\omega^k - \pi_{ij}$ system of commuting fields gives $(\det\Delta_1)^{-1/2}$ which cancels with the determinant $\bar{\tau}_k - \phi^k - \chi_{ij}$ system of anticommuting fields.

Since the moduli space has a vanishing dimension $h^1 = 0$ for the $\Lambda > 0$ case, it consists of an isolated point such as $\mathbb{CP}^2$ with the Fubini-Study metric or S^4 with the standard metric. We can write the partition function as

$$Z = \Sigma_{\text{instanton}}(\det\Delta_1)^{-1/2}(\det\Delta_0)^{-1}(\det\Delta_1)^{1/2}(\det\Delta_0)$$
$$= \Sigma_{\text{instanton}} \pm 1 = \pm 1 \qquad (45)$$

up to the secondary Chern class by projecting out h^0 zero modes.

We comment the BRST symmetry of the $\Lambda = 0$ case briefly. Substituting $\Lambda = 0$ in Eq. (23) the restricted topological symmetry of the fundamental fields are $\delta^S\omega_\mu^k = 0$, $\delta^S\Sigma_{\mu\nu}^k = D_{[\mu}\phi_{\nu]}^k$. The difference is apparent from the Witten-type topological model given by Kunitomo [8], which has the following transformation for the fundamental fields $\delta^S\omega_\mu^k = \hat{\phi}_\mu^k$, $\delta^s\Sigma_{\mu\nu}^k = \Psi_{[\mu\nu]}^k$. The fermionic ghost zero modes for ϕ_μ^k of our model are contained in the basis of the tangent space of the moduli space and further investigation is necessary to know the precise value of the dimension.

However on a $K3$-surface and T^4, the special situation occurs; ω^k can be gauged away for on-shell and the topological symmetry reduces to $\delta^S\Sigma^k = d\phi^k$. The number of the free parameters of this symmetry is not 12 but 9 due to the redundancy of $\{d^2\phi = 0/d^3\phi = 0\}$. There is no need to fix SU(2) and redundant SU(2) in this case. We only fix 5 degrees of the restricted topological symmetry and 4 degrees of diffeomorphism and 4 degrees of the redundant diffeomorphism [26].

VI. THE DIMENSION OF THE MODULI SPACE IN THE $\Lambda = 0$ CASE

We focus our attention on two cases: a four-torus with flat metrics and a $K3$-surface with Calabi-Yau metrics.

For a $K3$-surface with Calabi-Yau metrics and a four-torus with flat metrics, the moduli space is represented only by the deformations of a trio of Einstein-Kähler forms (hyper-Kähler forms) due to our prerequisite conditions for the P bundle on M_4 and the gauge fixing conditions.

Let $K(g)$ be the moduli space of Einstein-Kähler forms on M_4, $\epsilon(g)$ be the moduli space of Einstein metrics, and $C(g)$ be the moduli space of complex structures, respectively. They are the equivalent classes under the all diffeomorphism. At first we quote the result about the dimension of $K(g)$ briefly when M_4 is a Kählerian manifold with vanishing real first Chern class, which is given by Ref. [23]. Then we clarify the difference between $\mathcal{M}(\Sigma)$, i.e., the moduli space of hyper-Kähler forms defined by Eq. (21) and the moduli space $K(g)$.

When the real first Chern class is zero, the deformation of the Kähler class with a fixed complex structure induces a deformation of an Einstein metric from the Calabi-Yau theorem. The deformation of Einstein-Kähler forms $\{\Sigma\}$ consists of those of Einstein metrics $\{g\}$ and of complex structures $\{J\}$ and needs a careful examination of its degenerated part:

$$\delta\Sigma = \delta g \circ J + g \circ \delta J \sim h \circ J + g \circ I, \qquad (46)$$

where $I = (d/dt)J(t)|_{t=0}$ is the variation of complex structure $J(t)$ of J and $h = (d/dt)g(t)|_{t=0}$ is the variation of Kähler-Einstein metric $g(t)$ of g. We quote the results of the dimensions about $\epsilon(g)$, $C(g)$, and $K(g)$ in order.

A. Deformation of Einstein-Kähler metrics

If some infinitesimal Einstein deformations of Einstein-Kähler metric g are contravariant two-tensor h, then they are decomposed into its Hermitian part h_h and anti-Hermitian part $h_{ah}\{h\} = \{h_h\} \oplus \{h_{ah}\}$,

$$\{h_h\} = \{h; h(Ju, Jv) = h(u, v)\},$$
$$\{h_{ah}\} = \{h; h(Ju, Jv) = -h(u, v)\}, \qquad (47)$$

where $u, v \in TM_4$. It is easy to see that both $\{h_h\}$ and $\{h_{ah}\}$ are infinitesimal Einstein deformations. $\{h_h \circ J\}$ are shown to be the real (1,1) harmonic differential two-forms and orthogonal to the Kähler forms (which means the fixing of the scale factor). Therefore they form a space whose dimension is $\dim.H_R^{(1,1)}(M,J) - 1$.

B. Deformation of complex structures

According to the Kodaira-Spencer deformation theorem [31], the tangent space of the moduli space of complex structures is isomorphic to $2H_C^1(M, \Theta)$ in our case, where Θ is the sheaf of the germs of holomorphic vector fields. The deformation of complex structures is separated into two parts. One is antisymmetric complex deformation and the other is the symmetric one. The dimension of the antisymmetric one $\{I_{as}\}$ is given by $2\dim H_C^{(2,0)}(M, \mathbf{J})$ because antisymmetric complex

deformations are in one-to-one correspondence to $(2,0)$ or $(0,2)$ harmonic differential forms of $\{g \circ I_{as}\}$. Thus the dimension of the remainder $\{I_s\}$ is given by $2\dim H_C^1(M,\Theta) - 2\dim H_C^{(2,0)}(M,\mathbf{J})$.

C. Deformation of Einstein-Kähler forms

The degenerated part of the deformations of the Einstein-Kähler forms consists of the anti-Hermitian Einstein deformations $\{h_{\text{ah}} \circ J\}$, and the symmetric complex deformations $\{g \circ I_s\}$. The former counterbalances the latter; $g \circ I_s + h_{\text{ah}} \circ J = 0$ and their correspondence is shown to be bijective. The dimension of $\{I_s\}$ is the same as that of $\{h_{\text{ah}}\}$ and is given by $2\dim H_C^1(M,\Theta) - 2\dim H_C^{(2,0)}(M,\mathbf{J})$. Consequently infinitesimal deformations of the Einstein-Kähler form is represented by

$$\delta\Sigma = \underbrace{\underbrace{h_h \circ J}_{\dim_R H^{(1,1)}(M,\mathbf{J}) - 1} + (h_{\text{ah}} \circ J}_{\dim\epsilon(g)}$$

$$\underbrace{+g \circ I_s) + \underbrace{g \circ I_{\text{as}}}_{2\dim_C H^{(2,0)}(M,\mathbf{J})}}_{\dim C(g) = 2\dim_C H^1(M,\Theta)} \tag{48}$$

Finally, the dimensions of moduli spaces can be summarized for Einstein metrics, for complex structures and for Einstein-Kähler forms over the Kählerian manifolds with $c_1(M)_R = 0$:

$$\dim\epsilon(g) = \dim H_R^{(1,1)}(M,\mathbf{J}) - 1 + 2\dim H_C^1(M,\Theta)$$
$$-2\dim H_C^{(2,0)}(M,\mathbf{J}), \tag{49}$$

$$\dim C(g) = 2\dim H_C^1(M,\Theta), \tag{50}$$

$$\dim K(g) = \dim H_R^{(1,1)}(M,\mathbf{J}) - 1 + 2\dim H_C^1(M,\Theta). \tag{51}$$

One can show that the canonical line bundle K is trivial over a $K3$-surface or over T^4, and that there is a nowhere vanishing holomorphic two-form λ. The isomorphism of sheaves due to λ and the Dolbeaut theorem: $H_C^1(M,\Theta^1) \cong H_C^1(M,\Omega^1) \cong H^{(1,1)}(M,C)$, leads to the equations

$$\dim\epsilon(g) = 3\dim H^{(1,1)}(M,\mathbf{C}) - 1 - 2\dim H^{(2,0)}(M,\mathbf{C}), \tag{52}$$

$$\dim C(g) = 2\dim H^{(1,1))}(M,\mathbf{C}), \tag{53}$$

$$\dim K(g) = 3\dim H^{(1,1)}(M,\mathbf{C}) - 1. \tag{54}$$

The results for a $K3$-surface and T^4 are given by

$$K3; \begin{cases} \dim K(g) = 59, \\ \dim C(g) = 40, \\ \dim\epsilon(g) = 57, \end{cases}$$

by substituting $b^{1,1} = 20$ and $b^{2,0} = 1$ and

$$T^4 \begin{cases} \dim K(g) = 11, \\ \dim C(g) = 8, \\ \dim\epsilon(g) = 9, \end{cases}$$

by substituting $b^{1,1} = 4$ and $b^{2,0} = 1$ except for a scale factor.

The difference between $\mathcal{M}(\Sigma)$ and $K(g)$ is as follows: the moduli space of the Einstein-Kähler forms $K(g)$ is defined in terms of (g, J^1) or equivalently Σ^1 only. On the other hand, the definition of $\mathcal{M}(\Sigma)$ specifies a set of (g, J^1, J^2, J^3) or equivalently $(\Sigma^1, \Sigma^2, \Sigma^3)$, which takes into account the degrees of freedom how one can choose g and a trio of the g-orthogonal complex structures up to a scale factor.

Before we present the difference between $\dim\mathcal{M}(\Sigma)$ and $\dim K(g)$, let us show that the degrees of freedom of a trio of g-orthogonal complex structures which satisfy the quaternionic relations for a fixed g is 3 (see Appendix B).

For a fixed g,

$$\{g - \text{orthogonal quaternionic almost complex structures } J^1\} \cong S^2 \cong \text{Im}\mathbf{H}|_{x_1^2+x_2^2+x_3^2=1}, \tag{55}$$

where $\text{Im}\mathbf{H}$ represents the imaginary part of the field of the quaternion $\mathbf{H}$;

$$\text{Im}\mathbf{H} \equiv \{J^1 = x_1\tilde{J}^1 + x_2\tilde{J}^2 + x_3\tilde{J}^3|(\tilde{J}^1)^2 = (\tilde{J}^2)^2 = (\tilde{J}^3)^2 = -1, \tilde{J}^1\tilde{J}^2 = -\tilde{J}^2\tilde{J}^1 = \tilde{J}^3,$$
$$(x_1, x_2, x_3) \in R^3, \quad \tilde{J}^2\tilde{J}^3 = -\tilde{J}^3\tilde{J}^2 = \tilde{J}^1, \tilde{J}^3\tilde{J}^1 = -\tilde{J}^1\tilde{J}^3 = \tilde{J}^2\}. \tag{56}$$

The degrees of freedom how one can choose J^1 for a fixed g is given by $\dim S^2 = 2$. The region of J^2 which is orthogonal to J^1 for a fixed pair (g, J^1) is equivalent to S^1 over S^2. J^3 is automatically arranged after (g, J^1, J^2) are fixed.

From these facts, the difference between $\dim K(g)$ and $\dim \epsilon(g)$ is given by 2, which corresponds to $\dim S^2$ (the degrees of freedom of how to choose J^1 for a given g) and coincides with $2\dim H^{(2,0)}(M, \mathbf{C}) = 2$ (the degrees of freedom of how to choose $J^1 + \delta J^1$ for a fixed $g + \delta g$) up to a scale factor. The difference between $\dim K(g)$ and $\dim \mathcal{M}(\Sigma)$ is given by $\dim S^1 = 1$ up to a scale factor.

After all the dimension of our moduli space becomes

$$\dim \mathcal{M}(\Sigma) = 60 \quad \text{for } K3,$$
$$\dim \mathcal{M}(\Sigma) = 12 \quad \text{for } T^4, \tag{57}$$

up to a scale factor. In fact, we have confirmed the dimension of the moduli space $\mathcal{M}(\Sigma)$ by applying Atiyah-Singer Index theorem to the deformation complex (we will report this in the next paper [26]).

The moduli space $\mathcal{M}(\Sigma)$ has a bundle structure with the fiber (J^1, J^2, J^3) over the base manifold which is the moduli space of the Einstein metrics up to a scale factor:

$$\mathcal{M}(\Sigma) \qquad \dim \mathcal{M}(\Sigma) = \dim K(g) + 1 = \dim \epsilon(g) + 3$$
$$\uparrow$$
$$K(g) \qquad \dim K(g) = \dim \epsilon(g) + 2.$$
$$\uparrow$$
$$\epsilon(g)$$

VII. CONCLUSION

In this paper we have presented a topological version of two-form Einstein gravity in four dimensions. For a compact manifold in the $\Lambda \neq 0$ case, we have defined the elliptic complex associated with the moduli space of our theory. By applying the Atiyah-Singer index theorem in the $\Lambda \neq 0$ case we have evaluated the index of the elliptic complex and the partition function. In the $\Lambda = 0$ case we have clarified the dimension of the moduli space which is related to the Plebansky's equations for T^4 and a $K3$-surface.

It would be intriguing to study the $\Lambda = 0$ case, since the relation of four-dimensional (Riemann) self-dual gravity and two-dimensional conformal field theory has been investigated. In fact, Park showed that the former arises from a large N-limit of the two-dimensional sigma model with SU(N) Weiss-Zumino terms only [32]. Our topological model will be useful to understand the relation and to develop the self-dual gravity.

As another approach, it would also be interesting to extend the BF-type model in the $\Lambda = 0$ case to the super BF-type model [8]. Since the dimension of the moduli space is nonzero, there arise as many fermionic zero modes as the dimension, which make the partition function trivial. To avoid this we need some functional $\mathcal{O}$ which absorbs the zero modes. If one calculates the vacuum expectation value of the "observable" $\mathcal{O}$, then

it may provide nontrivial information such as a differential invariant to distinguish differential structures on these manifolds. Such a functional $\mathcal{O}$ is required to be BRST invariant to preserve the topological nature of the theory and may be obtained from the BRST descendant equations as in two-dimensional topological gravity [4].

The extension of the algebraic curves (one-dimensional compact complex manifolds) with Einstein metrics to four dimensions may be the algebraic surfaces (two-dimensional compact complex manifolds) with Einstein metrics. T^4 and a $K3$-surface belong to the algebraic surfaces. To construct the topological gravity models by taking another gauge fixing conditions, which describe these algebraic surfaces is worth pursuing.

APPENDIX A: PROPERTIES OF THE η SYMBOLS

We list some useful identities of the η_{ab}^k [21]:

$$\eta_{ab}^k = \epsilon_{kab} \quad \text{for } a, b = 1, 2, 3, \tag{A1}$$

$$\eta_{a0}^k = \eta_{0a}^k = \delta_a^k \quad \text{for } a = 0, 1, 2, 3,$$

(1) $\eta_{ab}^k = -\eta_{ba}^k$,

(2) $\eta_{ab}^k = -\dfrac{1}{2}\epsilon_{ab}{}^{cd}\eta_{cd}^k$, \qquad (A2)

(3) $\eta_{ab}^k\eta_{cd}^k = 2(\delta_a^{[c}\delta_b^{d]} - \tfrac{1}{2}\epsilon_{abcd}) \equiv 2P_{ab}{}^{cd}$,

(4) $\eta_{ab}^k\eta_c^{lb} = 2\delta_k^l\delta_c^a + \epsilon_{kln}\eta_{ac}^n$,

where ϵ_{klm} and ϵ_{abcd} denote the antisymmetric constant tensors.

From Eqs. (3) and (4) the identities of $\Sigma^k(e)$ are derived:

(5) $\Sigma_{\mu\nu}^k(e)\Sigma^{k\tau\rho}(e) = 2P_{\mu\nu}{}^{\rho\tau}$,

(6) $\Sigma_\mu^{k\nu}(e)\Sigma_{\rho\nu}^l(e) = -\delta_k^l\delta_\mu^\rho + \epsilon_{kln}\Sigma_\mu^{n\rho}(e)$.

APPENDIX B: SOME DEFINITIONS, THEOREMS, AND PROPOSITION

In this appendix we put some definitions, theorems, and proposition which we have used.

Theorem (Kobayashi and Nomizu [22]). Let $P(M, G)$ be a principal fiber bundle with a connection Γ, where M is connected and paracompact. Let u_0 be an arbitrary point P. Denote by $P(u_0)$ the set of points in P which can be joined to u_0 by a horizontal curve. Then (1) $P(u_0)$ is a reduced bundle with structure group $\mathrm{Hol}(g)$ and (2) the connection Γ is reducible to a connection in $P(u_0)$.

Theorem (Calabi and Yau [30]). Let M be a compact Kählerian manifold, Σ is its Kähler form, and any closed (real) two-form of type (1,1) belonging to $2\pi c_1(M)_R$ is the Ricci form of one and only one Kähler class Σ.

As an immediate consequence we get the following fact: the compact Kählerian manifolds with zero real first Chern class are exactly the compact complex manifolds admitting a Kähler metric with zero Ricci form (equivalently with restricted holonomy group contained in the special unitary group).

Definition 1 (Besse [24]). A $4n$-dimensional Riemannian manifold is called (a) hyper-Kählerian if its holonomy group is contained in $\mathrm{SP}(n)$ and (b) locally hyper-Kählerian if its restricted holonomy is contained in $\mathrm{SP}(n)$.

Definition 2 (Besse [24]). A $4n$-dimensional Riemannian manifold is called (a) quaternion-Kähler if its holonomy group is contained in $\mathrm{SP}(N) \times \mathrm{SP}(1)$ and (b) locally quaternion-Kähler if its restricted holonomy group is contained in $\mathrm{SP}(N) \times \mathrm{SP}(1)$.

Proposition 1 (Besse [24]). A Riemannian manifold (M, g) is hyper-Kählerian if and only if there exist on M two complex structures J^1 and J^2 such that (a) J^1 and J^2 are parallel (i.e., g is a Kähler metric for each) and (b) $J^1 J^2 = -J^2 J^1$.

Notice that J^3 is still a parallel complex structure on M and more generally, given (x_1, x_2, x_3) in R^3 with $x_1^2 + x_2^2 + x_3^2 = 1$, then the complex structure $J = x_1 J^1 + x_2 J^2 + x_3 J^3$ on M is still parallel. So there is a whole manifold (isomorphic to S^2) on parallel complex structure on M.

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