# Formalizing the slow-roll approximation in inflation

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The meaning of the inflationary slow-roll approximation is formalized. Comparisons are made between an approach based on the Hamilton-Jacobi equations, governing the evolution of the Hubble parameter, and the usual scenario based on the evolution of the potential energy density. The vital role of the inflationary attractor solution is emphasized, and some of its properties described. We propose a new measure of inflation, based upon contraction of the comoving Hubble length as opposed to the usual e-foldings of physical expansion, and derive relevant formulas. We introduce an infinite hierarchy of slow-roll parameters, and show that only a finite number of them are required to produce results to a given order. The extension of the slow-roll approximation into an analytic slowroll *expansion*, converging on the exact solution, is provided. Its role in calculations of inflationary dynamics is discussed. We explore rational approximants as a method of extending the range of convergence of the slow-roll expansion up to, and beyond, the end of inflation.

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#### I. INTRODUCTION

Inflationary universe models are based upon the possibility of slow evolution of some scalar field  $\phi$  in a potential  $V(\phi)$  [1, 2]. Although some exact solutions of this problem exist, most detailed studies of inflation have been made using numerical integration, or by employing an approximation scheme. The "slow-roll approximation" [3-5], which neglects the most slowly changing terms in the equations of motion, is the most widely used. Although this approximation works well in many cases, we know that it must eventually fail if inflation is to end. Moreover, even weak violations of it can result in significant deviations from the standard predications for observables such as the spectrum of density perturbations or the density of gravitational waves in the Universe [6, 2]. As observational data sharpen, it is important to derive a suite of predictions for the observables that are as accurate as possible, and which cover all possible inflationary models.

In the literature, one finds two different versions of the slow-roll approximation. The first [3, 5] places restrictions on the form of the potential, and requires the evolution of the scalar field to have reached its asymptotic form. This approach is most appropriate when studying inflation in a specific potential. We shall call it the potential slow-roll approximation (PSRA). The other form of the approximation places conditions on the evolution of the Hubble parameter during inflation [7]. We call this the Hubble slow-roll approximation (HSRA). It has distinct advantages over the PSRA, possessing a clearer geometrical interpretation and more convenient analytic properties. These make it best suited for general studies, where the potential is not specified.

In this paper, we clarify the meaning of the different slow-roll approximations that exist in the literature, which often describe a variety of slightly different approximation schemes applied to different variables at different orders. By formalizing the slow-roll approximation in detail, we will show how to use it as the basis of a *slow-roll* expansion—a sequence of analytic approximations which converge to the exact solution of the equations of motion for an inflationary universe. Such a technique relies strongly on the notion of the inflationary attractor, whose properties we describe. The use of Padé and Canterbury approximants [8,9] allows us to further improve the range and rate of convergence of this slow-roll expansion.

## II. EQUATIONS OF MOTION AND THEIR SOLUTION

We shall deal with the equations of motion in two different forms, both appropriate for a homogeneous scalar field  $\phi$ , evolving in a potential  $V(\phi)$ . We assume that enough inflation has occurred to render the densities of all other types of matter negligible, and to establish homogeneity in a patch at least as big as the horizon. The most familiar form of the equations, in a zero-curvature Friedmann universe, is

$$H^{2} = \frac{8\pi}{3m_{\rm Pl}^{2}} \left[\frac{1}{2}\dot{\phi}^{2} + V(\phi)\right],\tag{1}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'\,,\tag{2}$$

where  $H \equiv \dot{a}/a$  is the Hubble parameter, a is the scale factor (synchronous rather than conformal),  $m_{\rm Pl}$  the Planck mass, overdots indicate derivatives with respect to cosmic time t, and primes indicate derivatives with respect to the scalar field  $\phi$ .

One can derive a very useful alternative form of these equations by using the scalar field as a time variable [10, 4, 11]. This requires that  $\dot{\phi}$  not change sign during inflation. Without loss of generality, we can choose  $\dot{\phi} > 0$  throughout (this will determine some signs in later equations). Differentiating Eq. (1) with respect to t and using Eq. (2) gives

$$2\dot{H} = -\frac{8\pi}{m_{\rm Pl}^2} \dot{\phi}^2 \,. \tag{3}$$

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We may divide each side by  $\dot{\phi}$  to eliminate the time dependence in the Friedmann equation, obtaining

$$\left[H'(\phi)\right]^2 - \frac{12\pi}{m_{\rm Pl}^2} H^2(\phi) = -\frac{32\pi^2}{m_{\rm Pl}^4} V(\phi) , \qquad (4)$$

$$\dot{\phi} = -\frac{m_{\rm Pl}^2}{4\pi} H'(\phi) ,$$
 (5)

and Eq. (3) implies  $\dot{H} \leq 0$ . This new set of equations, the Hamilton-Jacobi equations, is normally more convenient than Eqs. (1) and (2). They were used by Salopek and Bond [4] to establish several important results to which we refer later.

These equations allow one to generate an endless collection of exact inflationary solutions via the following procedure [12, 11]: Choose a form of  $H(\phi)$ , and use Eq. (4) to find the potential for which the exact solution applies; now use Eq. (5) to find  $\phi$ , which allows the  $\phi$  dependences to be converted into time dependences, to get H(t); if desired, a further integration gives a(t)(though this last step is seldom required). For example, this procedure gives a very easy derivation of the exact solution describing "intermediate" inflation, which corresponds to the choice  $H(\phi) \propto \phi^{-\alpha}$  with  $\alpha$  a positive constant and  $\phi > 0$ . This exact solution was derived using the Hamilton-Jacobi equations by Muslimov [10] (and independently by Barrow [13] using a different technique). Other fully integrated exact solutions have been found by Barrow [14, 15].

It is normally impossible to make analytic progress by first choosing a potential  $V(\phi)$ , because Eq. (4) is unpleasantly nonlinear. The simplest exception is the exponential potential, known to drive power-law inflation, for which the Hamilton-Jacobi formalism was used by Salopek and Bond [4] to find (in parametric form) the general isotropic solution.

#### A. Potential slow-roll approximation

When provided with a potential  $V(\phi)$  from which to construct an inflationary model, the slow-roll approximation is normally advertised as requiring the smallness of two parameters (both functions of  $\phi$ ), defined by [5]

$$\epsilon_V(\phi) = \frac{m_{\rm Pl}^2}{16\pi} \left(\frac{V'(\phi)}{V(\phi)}\right)^2,\tag{6}$$

$$\eta_V(\phi) = \frac{m_{\rm Pl}^2}{8\pi} \frac{V''(\phi)}{V(\phi)} \,. \tag{7}$$

Henceforth, we refer to them as potential-slow-roll (PSR) parameters.<sup>1</sup> Their smallness is used to justify the neglect of the kinetic term in the Friedmann equation, Eq. (1), and the acceleration term in the scalar wave

equation, Eq. (2). Unfortunately, the smallness of the PSR parameters is a necessary consistency condition, but *not* a sufficient one to guarantee that those terms can be neglected. The PSR parameters only restrict the form of the potential, not the properties of dynamic solutions. The solutions are more general because they possess a freely specifiable parameter, the value of  $\dot{\phi}$ , which governs the size of the kinetic term. The kinetic term could, therefore, be as large as one wants, regardless of the smallness, or otherwise, of these PSR parameters.

In general, this PSR formalism requires a further "assumption," that the scalar field evolves to approach an asymptotic attractor solution, determined by

$$\dot{\phi} \simeq -\frac{V'}{3H}$$
 (8)

The word "assumption" is placed in quotes here because, in general, one is able to test whether Eq. (8) is approached for a wide range of initial conditions. The inflationary attractor is of vital importance in the application of the slow-roll approximation, and we discuss its properties in Sec. II C.

## B. Hubble slow-roll approximation

If  $H(\phi)$  is taken as the primary quantity, then there is a better choice of slow-roll parameters. We define the Hubble slow-roll (HSR) parameters  $\epsilon_H$  and  $\eta_H$  by [7]

$$\epsilon_H(\phi) = \frac{m_{\rm Pl}^2}{4\pi} \left(\frac{H'(\phi)}{H(\phi)}\right)^2 , \qquad (9)$$

$$\eta_H(\phi) = \frac{m_{\rm Pl}^2}{4\pi} \frac{H''(\phi)}{H(\phi)} \,. \tag{10}$$

These possess an extremely useful set of properties which make them superior choices to  $\epsilon_V$  and  $\eta_V$  as descriptors of inflation.

(i) We have exactly

$$\epsilon_H = 3 \frac{\dot{\phi}^2/2}{V + \dot{\phi}^2/2} \quad \left( = -\frac{d\ln H}{d\ln a} \right) , \qquad (11)$$

$$\eta_H = -3 \frac{\ddot{\phi}}{3H\dot{\phi}} \quad \left( = -\frac{d\ln\dot{\phi}}{d\ln a} = -\frac{d\ln H'}{d\ln a} \right) . \tag{12}$$

(ii)  $\epsilon_H \ll 1$  is the condition for neglecting the first term of Eq. (4) [the kinetic term in Eq. (1)].

(iii)  $|\eta_H| \ll 1$  is the condition for neglecting the derivative of the first term of Eq. (4) [the acceleration term in Eq. (2)]. As a consequence, all the necessary dynamical information is encoded in the HSR parameters. They do not need to be supplemented by any assumptions about the inflationary attractor, Eq. (8).

(iv) The condition for inflation to occur is precisely

$$\ddot{a} > 0 \Longleftrightarrow \epsilon_H < 1. \tag{13}$$

There is an algebraic expression relating  $\epsilon_V$  to  $\epsilon_H$  and  $\eta_H$  [using Eq. (4)]:

$$\epsilon_V = \epsilon_H \left(\frac{3 - \eta_H}{3 - \epsilon_H}\right)^2 \,. \tag{14}$$

<sup>&</sup>lt;sup>1</sup>To preview what is to come, these parameters are sufficient to obtain results to first order in slow roll. However, the general slow-roll expansion requires an infinite hierarchy of parameters which will be defined in Sec. IV.

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The true end point of inflation, gauged by the HSR parameters, occurs *exactly* at  $\epsilon_H = 1$ . When using the PSR parameters, this condition is approximate; inflation ending at  $\epsilon_V = 1$  is only a first-order result.

For  $\eta_V$ , the relation to the HSR parameters is differential rather than algebraic:

$$\eta_V = \sqrt{\frac{m_{\rm Pl}^2 \epsilon_H}{4\pi}} \frac{\eta'_H}{3 - \epsilon_H} + \left(\frac{3 - \eta_H}{3 - \epsilon_H}\right) (\epsilon_H + \eta_H) , \qquad (15)$$

although a more compact representation in terms of higher-order parameters will be presented in Sec. IV. This will show that the first term in Eq. (15) is of higher order in slow roll, so that to lowest order, one has  $\eta_V = \eta_H + \epsilon_H$ . Note that  $\eta_H$  and  $\eta_V$  are not the same to first order in slow roll, since  $H^2 \propto V$ . We could have defined  $\eta_H$  to coincide with  $\eta_V$  in slow roll, by defining  $\bar{\eta}_H = \eta_H - \epsilon_H$ , but we prefer to regard the definitions in Eqs. (9) and (10) as fundamental.

The definitions can be used to derive two useful relations between parameters of the same type:

$$\eta_H = \epsilon_H - \sqrt{\frac{m_{\rm Pl}^2}{16\pi}} \, \frac{\epsilon'_H}{\sqrt{\epsilon_H}} \,, \tag{16}$$

$$\eta_V = 2\epsilon_V - \sqrt{\frac{m_{\rm Pl}^2}{16\pi}} \frac{\epsilon'_V}{\sqrt{\epsilon_V}} \,. \tag{17}$$

Note that although, as functions, the parameters of a given type, either HSR or PSR, are related, their values at a given  $\phi$  are independent of one another. One immediately sees the different "normalization" of the  $\eta$  from Eqs. (16) and (17).

It is important to stress that although we have derived self-consistent exact expressions relating the PSR to the HSR parameters, we cannot invert these expressions without first assuming that the evolution has reached the attractor, Eq. (8). As already mentioned, the attractor constraint is part of the structure of the HSRA, but is absent from the PSRA. So, while the HSRA implies the PSRA, the converse does not hold without assuming the attractor constraint.

# C. Inflationary attractor

Already in this section we have seen how important the notion of the inflationary attractor is. The behavior of this attractor was established by Salopek and Bond [4], and we now discuss its properties.

Suppose  $H_0(\phi)$  is any solution to the full equation of motion, Eq. (4), either inflationary or noninflationary. Consider, first, a linear perturbation  $\delta H(\phi)$ . We shall also assume, and discuss further below, that the perturbation does not reverse the sign of  $\dot{\phi}$ . It therefore obeys the linearized equation

$$H_0^\prime \delta H^\prime \simeq \frac{12\pi}{m_{\rm Pl}^2} H_0 \delta H \,, \tag{18}$$

which has the general solution

$$\delta H(\phi) = \delta H(\phi_i) \exp\left(\frac{12\pi}{m_{\rm Pl}^2} \int_{\phi_i}^{\phi} \frac{H_0}{H'_0} d\phi\right) \,. \tag{19}$$

Since  $H'_0$  and  $d\phi$  have, by construction, opposing signs, the integrand within the exponential term is negative definite, and hence all linear perturbations die away.

If  $H_0$  is inflationary, the behavior is particularly dramatic because the condition for inflation bounds the integrand away from zero. Consequently one obtains

$$\delta H(\phi) < \delta H(\phi_i) \exp\left(-\frac{6\sqrt{\pi}}{m_{\rm Pl}} |\phi - \phi_i|\right) \,. \tag{20}$$

That is, if there is an inflationary solution, all linear perturbations approach it *at least* exponentially fast as the scalar field rolls.

Another way of writing the solution for the perturbation, regardless of whether or not  $H_0$  is inflationary, is in terms of the amount of expansion which occurs, by using the number of *e*-foldings N as defined in the following section [Eq. (22)]. We get the precise result [4]

$$\delta H(\phi) = \delta H(\phi_0) \exp\left(-3|N_i - N|\right) \,. \tag{21}$$

For nonlinear perturbations, the problem is more complex; though all the solutions are easily seen to approach each other, we have not shown that they do so exponentially quickly. The most awkward case is where a perturbation actually reverses the sign of  $\dot{\phi}$ , as the Hamilton-Jacobi equations are singular when that happens. Nevertheless, as long as the perturbation is insufficient to knock the scalar field over a maximum in the potential, the perturbed solution will inevitably reverse and subsequently pass through the initial value  $\phi_i$  again; then it can be treated as a perturbation with the same sign of  $\dot{\phi}$ as the original solution.

The picture that emerges is therefore as follows. Provided the potential is able to support inflation, the inflationary solutions all rapidly approach one another, with exponential rapidity once in the linear regime. Even when inflation ends, the universe continues to expand and therefore the solutions continue to approach one another. Consequently, even the exit from inflation is independent of initial conditions. Note that there is no concept of a single "attractor solution"; all solutions are attractors for one another and converge asymptotically. As we shall see, this is a vital requirement if a slow-roll expansion is to make any sense.

The situation where the inflationary attractor does not apply is therefore soon after inflation begins. Normally this is in the distant past and of no concern. An exception, recently noted [16], is hybrid inflation. There, the slow-roll parameters rise above unity, halting inflation, and then fall back below unity, reaching very small values. Nevertheless, it is easy to show that, except for a tiny window in parameter space, inflation fails to restart despite the smallness of the PSR parameters, because there is insufficient time for the solution to approach the inflationary attractor. Another similar situation will be discussed in Sec. VI.

# FORMALIZING THE SLOW-ROLL APPROXIMATION IN INFLATION

## **III. BETTER MEASURE OF INFLATION**

An important quantity for making inflationary predictions is the amount of inflation that has taken place. Inflation is commonly characterized by the number of efoldings of physical expansion that occur, as given by the natural logarithm of the ratio of the final scale factor to the initial one. This can be expressed *exactly* as

$$N \equiv \ln \frac{a_f}{a_i} = -\sqrt{\frac{4\pi}{m_{\rm Pl}^2}} \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{\epsilon_H(\phi)}} \, d\phi \,. \tag{22}$$

If one is working in the PSRA then, provided the attractor solution Eq. (8) is attained, this may be approximated by

$$N(\phi_i, \phi_f) \simeq -\sqrt{\frac{4\pi}{m_{\rm Pl}^2}} \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{\epsilon_V(\phi)}} d\phi.$$
 (23)

These formulas are needed to make the connection between horizon-crossing times in calculations of the production of scalar and tensor perturbations. A comoving scale k crosses outside the Hubble radius at a time which is N(k) e-foldings from the end of inflation, where

$$N(k) = 62 - \ln \frac{k}{a_0 H_0} - \ln \frac{10^{16} \text{GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{\text{end}}^{1/4}} - \frac{1}{3} \ln \frac{V_{\text{end}}^{1/4}}{\rho_{\text{reh}}^{1/4}}.$$
(24)

The subscript zero indicates present values, the subscript k specifies the value when the wave number k crosses the Hubble radius during inflation (k = aH), the subscript "end" specifies the value at the end of inflation, and  $\rho_{\rm reh}$  is the energy density of the Universe after reheating to the standard hot big bang evolution. This calculation assumes that instantaneous transitions occur between regimes, and that during reheating the Universe behaves as if matter dominated. Ordinarily, it is taken as a perfectly good approximation that the comoving scale presently equal to the Hubble radius crossed outside the Hubble radius 60 *e*-foldings from the end of inflation, with all other scales relevant to large-scale structure studies following within the next few *e*-foldings.

Something like 70 e-foldings is normally advertised as the minimum for inflation to solve the various cosmological conundrums, such as the flatness and horizon problems. However, it is well known that this is an approximation (for example, in the standard big bang model, the Universe expands by more than a factor  $e^{70}$  between the Planck time and the present, without solving the flatness or horizon problems), based on the assumption of a constant Hubble parameter. A better measure of inflation is the reduction in the size of the comoving Hubble length 1/aH. First of all, the condition for inflation ( $\ddot{a} > 0$ ) is equivalent to  $d[(aH)^{-1}]/dt < 0$ . Second, it is the reduction of 1/aH, not that of 1/a, which solves the flatness and horizon problems. And finally, for the generation of perturbations, it is the relation of the comoving wave number k to aH that is important. We therefore define

$$\bar{N} \equiv \ln \frac{(aH)_f}{(aH)_i} \,. \tag{25}$$

It can be shown that

$$\bar{N}(\phi_i, \phi_f) = -\sqrt{\frac{4\pi}{m_{\rm Pl}^2}} \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{\epsilon_H(\phi)}} \left[1 - \epsilon_H(\phi)\right] d\phi \,.$$
(26)

Since *H* always decreases,  $\bar{N} \leq N$  by definition, with the difference indicating the extra amount of expansion required by the decrease of *H* during inflation. In the extreme slow-roll limit (HSR) *N* and  $\bar{N}$  coincide. This tells us that the true condition for sufficient inflation should be that  $\bar{N}$  (not *N*) exceeds 70.

Again, using the potential, we can only write down an approximate relation. At lowest order, N and  $\overline{N}$  coincide, while to next order one obtains

$$\bar{N}(\phi_i, \phi_f) \simeq -\sqrt{\frac{4\pi}{m_{\rm Pl}^2}} \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{\epsilon_V(\phi)}} \left(1 - \frac{1}{3}\epsilon_V(\phi) - \frac{1}{3}\eta_V(\phi)\right) d\phi \,.$$

$$(27)$$

Equation (26) gives  $\bar{N}$  during an arbitrary inflationary epoch (not just quasi-de Sitter). It also holds if inflation is interrupted for a period, while the dynamics are still dominated by the scalar field. This requires  $\epsilon_H > 1$ , causing the integrand in Eq. (26) to change sign over a range of  $\phi$ . This is unlikely, but it has recently been noted [17] that it arises in a variant of hybrid inflation with a quartic potential, where the potential that drives inflation is of the form  $V \propto (1 + \lambda \phi^4)$ . For some values of  $\lambda$ , the potential temporarily steepens sufficiently to suspend inflation, while 1/aH increases. This must be compensated by extra inflationary expansion later on.

To make use of the new expressions Eqs. (26) and (27), we need an  $\overline{N}(k)$  relation to replace Eq. (24). This is simply

$$ar{N}(k) = 62 - \ln rac{k}{a_0 H_0} - \ln rac{10^{16} {
m GeV}}{V_{
m end}^{1/4}} - rac{1}{3} \ln rac{V_{
m end}^{1/4}}{
ho_{
m reh}^{1/4}} \,.$$
 (28)

Although this strongly resembles Eq. (24), it is in fact slightly simpler; the difference between  $V_k$  and  $V_{end}$  is now part of the definition of  $\overline{N}$ .

## IV. HIERARCHY OF SLOW-ROLL PARAMETERS

We now reconsider the intrinsic structure of the slowroll approximation. The HSR parameters ( $\epsilon_H$  and  $\eta_H$  of Sec. II B), which measure the first and second derivatives of the Hubble parameter, are all we require to obtain results to first order in the slow-roll expansion. However, to go beyond this, we require more derivatives, necessitating further slow-roll parameters. In general, there will be an infinite number of these, incorporating derivatives to all orders; we shall prove that each additional order in the expansion requires the introduction of one new parameter. The formal order of the expansion parameter depends on the number of derivatives it contains, two derivatives for each order.

In [18], where second-order results were derived, the extra parameter was defined as  $\xi_{CKLL}$  $(m_{\rm Pl}^2/4\pi)H'''/H'$ , which is of the same order as the others. This definition is rather unfortunate; we mean to be expanding about a flat potential, and this parameter is not guaranteed to tend to zero in that limit because of the derivative in the denominator. This definition has led to some confusion (for instance, in [19] where  $\xi$  was not treated as an expansion parameter at all). The moral of this is that the hierarchy of slow-roll parameters should be carefully defined, so as to tend to zero as the potential approaches flatness in arbitrary ways. Even with this restriction, there are different ways one could define the hierarchy. If a superscript (n) indicates the *n*th derivative with respect to  $\phi$ , the simplest definition would appear to be

$${}^{n}\tilde{\beta}H \equiv \frac{m_{\rm Pl}^2}{4\pi} \left(\frac{H^{(n)}}{H}\right)^{2/n} , \qquad (29)$$

which gives  ${}^{1}\tilde{\beta}H \equiv \epsilon_{H}$  and  ${}^{2}\tilde{\beta}H \equiv \eta_{H}$ . However, there is a superior alternative.

## A. HSR hierarchy

We shall work with the set of definitions

$${}^{n}\beta_{H} \equiv \left\{ \prod_{i=1}^{n} \left[ -\frac{d\ln H^{(i)}}{d\ln a} \right] \right\}^{\frac{1}{n}} , \qquad (30)$$

which can be expressed as

$${}^{n}\beta_{H} = \frac{m_{\rm Pl}^{2}}{4\pi} \left(\frac{\left(H'\right)^{n-1}H^{(n+1)}}{H^{n}}\right)^{\frac{1}{n}} . \tag{31}$$

These have elegant properties, which we shall use to recast the HSR approximation as an HSR *expansion*. Furthermore, as we demonstrate below, only a finite number of these parameters are needed to obtain results to any given order.

It is difficult to incorporate  $\epsilon_H$  into this scheme naturally — it must be defined separately. We use the form given in Eq. (9), but shall refer to  $\epsilon_H$  as  ${}^0\beta_H$  in later sections. The first parameter this definition yields is  ${}^1\beta_H \equiv \eta_H$ , in accordance with Eq. (10). The next four HSR expansion parameters are

$$\xi_{H} \equiv {}^{2}\beta_{H} = \frac{m_{\rm Pl}^{2}}{4\pi} \left(\frac{H'H''}{H^{2}}\right)^{\frac{1}{2}}, \qquad (32)$$

$$\sigma_{H} \equiv {}^{3}\beta_{H} = \frac{m_{\rm Pl}^{2}}{4\pi} \left(\frac{H'^{2}H'''}{H^{3}}\right)^{\frac{1}{3}}, \qquad (33)$$

$$\tau_H \equiv {}^4\beta_H = \frac{m_{\rm Pl}^2}{4\pi} \left(\frac{{H'}^3 H^{(5)}}{H^4}\right)^{\frac{1}{4}}, \qquad (34)$$

$$\zeta_H \equiv {}^5\beta_H = \frac{m_{\rm Pl}^2}{4\pi} \left(\frac{H'{}^4 H^{(6)}}{H^5}\right)^{\frac{1}{5}} . \tag{35}$$

The strength of these definitions is that each parame-

ter combines H and its derivatives, raised as a whole to some power 1/n, where  $n \in \mathbb{Z}^+$ . This has the interesting consequence that if we wish to convert the Taylor series of some function of  $H(\phi)$ , and its derivatives, into an expansion in slow-roll parameters, we are guaranteed that the powers of any specific parameter  ${}^n\beta_H$  will be integer multiples of n. Thus, the lowest-order term we can expect to find involving this parameter will be  $({}^n\beta_H)^n$ . Consequently, if one is interested in expanding this function up to order m, then at most the first m + 1 parameters can appear. Thus, despite the potentially infinite number of slow-roll parameters, we require only a finite selection of them at any order.

#### **B. PSR hierarchy**

In the spirit of the previous subsection, we introduce a hierarchy of PSR parameters. As stressed earlier, these are not as useful in calculations of inflationary dynamics as the HSR parameters. They only classify the flatness of the potential, and so encode no information about initial conditions. This necessitates adding the additional attractor constraint, Eq. (8).

As before, we retain the standard definitions of  $\epsilon_V$  and  $\eta_V$  from Eqs. (6) and (7), and encapsulate the definition of  $\eta_V$ , and all higher-order parameters, in the new set of quantities

$${}^{n}\beta_{V} \equiv \frac{m_{\rm Pl}^{2}}{8\pi} \left(\frac{d\ln V}{d\phi}\right) \left\{\prod_{i=1}^{n} \left[\frac{d\ln V^{(i)}}{d\phi}\right]\right\}^{\frac{1}{n}},\qquad(36)$$

which reduce to

$${}^{n}\beta_{V} = \frac{m_{\rm Pl}^{2}}{8\pi} \left(\frac{\left(V'\right)^{n-1}V^{(n+1)}}{V^{n}}\right)^{\frac{1}{n}} . \tag{37}$$

This allows construction of a set of PSR expansion parameters

$$\xi_{V} \equiv {}^{2}\beta_{V} = \frac{m_{\rm Pl}^{2}}{8\pi} \left(\frac{V'V'''}{V^{2}}\right)^{\frac{1}{2}}, \qquad (38)$$

$$\sigma_V \equiv {}^{3}\beta_V = \frac{m_{\rm Pl}^2}{8\pi} \left( \frac{(V')^2 V''''}{V^3} \right)^3 , \qquad (39)$$

$$\tau_V \equiv {}^4\beta_V = \frac{m_{\rm Pl}^2}{8\pi} \left(\frac{\left(V'\right)^3 V^{(5)}}{V^4}\right)^{\frac{1}{4}}, \qquad (40)$$

$$\zeta_V \equiv {}^5\beta_V = \frac{m_{\rm Pl}^2}{8\pi} \left( \frac{\left(V'\right)^4 V^{(6)}}{V^5} \right)^{\frac{1}{5}} . \tag{41}$$

In Eqs. (14)-(17), we presented some additional properties of  $\epsilon$  and  $\eta$ . As one may suspect, these can be written in terms of the higher-order parameters. An extensive collection is given in the Appendix; here, we note the exact relations

$$\epsilon_V = \epsilon_H \left(\frac{3 - \eta_H}{3 - \epsilon_H}\right)^2 \,, \tag{42}$$

$$\eta_V = \frac{3\epsilon_H + 3\eta_H - \eta_H^2 - \xi_H^2}{3 - \epsilon_H} \,. \tag{43}$$

The latter demonstrates that  $\eta_V = \epsilon_H + \eta_H$  in the lowestorder HSRA, as stated earlier.

## V. FROM SLOW-ROLL APPROXIMATION TO SLOW-ROLL EXPANSION

A procedure for generating analytic approximations to inflationary solutions, from a slow-roll approximation, has a broad spectrum of applications. More useful still is a slow-roll expansion, from which solutions could be generated analytically to any required order in the slowroll approximation. An indication of how to go about this was given by Salopek and Bond [4]. We now show how this may be achieved within the framework of the slow-roll parameters.

Let us first emphasize the importance of the attractor behavior for this procedure. The general isotropic solution for a given potential possesses one free parameter, corresponding to the freedom to specify H (or equivalently  $\phi$ ) at some initial time. However, the traditional slow-roll solution, and its order-by-order corrections that we shall describe, generates only a single solution. Unless an attractor exists and has been attained, there is no need for this single solution to represent in any way the true solution for that potential. However, if the attractor has indeed been reached, then any particular solution provides an excellent approximation to those arising from a wide range of initial conditions. This is particularly important when inflation approaches its end, and the slow-roll parameters become large, because one might naively assume that the one-parameter freedom could be important there. If the attractor solution exists, then solutions for a wide range of initial conditions will converge, and subsequently all exit inflation in the same way. Hence, an expansion approximating one particular solution serves as an excellent approximation to them all, provided the initial condition for that solution is not pathological (which is prevented by the assumption that energies are less than the Planck energy).

We note that there is a formal problem in attempting to prove "no-hair" theorems; if inflation is to end, there is no formal asymptotic regime [14]. However, our requirement is just that enough inflation occurs to ensure that the range of values of  $\dot{\phi}$  needed to encompass the entire family of solutions is sufficiently small to validate the Eq. (8). In typical models so much inflation occurs that this situation is easily achieved.

#### A. Traditional Taylor-series approach

We start with a potential  $V(\phi)$  for which a solution is desired. Typically, we cannot solve exactly for  $H(\phi)$ ; instead, we aim to find an approximate solution, in terms of  $V(\phi)$  and a multivariate Taylor expansion in the PSR parameters, all of which can be computed analytically given an analytic  $V(\phi)$ . First recast Eq. (4) as

$$H^{2}(\phi) = \frac{8\pi}{3m_{\rm Pl}^{2}}V(\phi)\left(1 - \frac{1}{3}\epsilon_{H}(\phi)\right)^{-1}.$$
 (44)

Then seek an approximate solution for  $H^2(\phi)$  of the form

$$H^{2}(\phi) = \frac{8\pi}{3m_{\rm Pl}^{2}}V(\phi)(1 + a\epsilon_{V} + b\eta_{V} + c\epsilon_{V}^{2} + d\epsilon_{V}\eta_{V} + e\eta_{V}^{2} + f\xi_{V}^{2} + \cdots), \qquad (45)$$

where  $a, b, \ldots$  are constants to be determined. [In fact, Eq. (14) already guarantees that b = e = f = 0, by forcing every term in  $\epsilon_H$  to contain at least one power of  $\epsilon_V$ . Note also that, for reasons discussed in Sec. IV A,  $\xi_H^2$  is the lowest-order term involving  $\xi_H$  to appear in Eq. (45), and that no further PSR parameters appear at second order. Thus, we require an expansion of  $(1 - \epsilon_H/3)^{-1}$  in PSR parameters of ascending order. This is most readily achieved by assuming general expressions for the HSR parameters, in terms of PSR parameters, with a similar form to Eq. (45). Starting with general first-order forms, these may be substituted into Eq. (14), which is then solved for the unknown constants by comparing coefficients.<sup>2</sup> After first-order results have been obtained, this procedure may be repeated iteratively order by order. It is a straightforward (albeit tedious) matter to invert the expansion of  $(1 - \epsilon_H/3)$  using the binomial theorem, and hence obtain the corresponding series for  $H^2(\phi)$ .

We have done this to fourth order, although the results may be truncated at lower order to obtain more manageable expressions, as desired. We find

$$\epsilon_{H} = \epsilon_{V} - \frac{4}{3}\epsilon_{V}^{2} + \frac{2}{3}\epsilon_{V}\eta_{V} + \frac{32}{9}\epsilon_{V}^{3} + \frac{5}{9}\epsilon_{V}\eta_{V}^{2} - \frac{10}{3}\epsilon_{V}^{2}\eta_{V} + \frac{2}{9}\epsilon_{V}\xi_{V}^{2} - \frac{44}{3}\epsilon_{V}^{4} + \frac{530}{27}\epsilon_{V}^{3}\eta_{V} - \frac{62}{9}\epsilon_{V}^{2}\eta_{V}^{2} + \frac{14}{27}\epsilon_{V}\eta_{V}^{3} - \frac{16}{9}\epsilon_{V}^{2}\xi_{V}^{2} + \frac{2}{3}\epsilon_{V}\eta_{V}\xi_{V}^{2} + \frac{2}{27}\epsilon_{V}\sigma_{V}^{3} + O_{5}.$$
(46)

The second-order truncation concurs with the result derived in [19]. Hence, we find the approximate solution

$$H^{2}(\phi) = \frac{8\pi}{3m_{\rm Pl}^{2}}V(\phi)\left[1 + \frac{1}{3}\epsilon_{V} - \frac{1}{3}\epsilon_{V}^{2} + \frac{2}{9}\epsilon_{V}\eta_{V} + \frac{25}{27}\epsilon_{V}^{3} + \frac{5}{27}\epsilon_{V}\eta_{V}^{2} - \frac{26}{27}\epsilon_{V}^{2}\eta_{V} + \frac{2}{27}\epsilon_{V}\xi_{V}^{2} - \frac{327}{81}\epsilon_{V}^{4} + \frac{460}{81}\epsilon_{V}^{3}\eta_{V} - \frac{172}{81}\epsilon_{V}^{2}\eta_{V}^{2} + \frac{14}{81}\epsilon_{V}\eta_{V}^{3} - \frac{44}{81}\epsilon_{V}^{2}\xi_{V}^{2} + \frac{2}{9}\epsilon_{V}\eta_{V}\xi_{V}^{2} + \frac{2}{81}\epsilon_{V}\sigma_{V}^{3} + O_{5}\right].$$
 (47)

<sup>&</sup>lt;sup>2</sup>Although the slow-roll functions are all interdependent, as all are based on derivatives of  $V(\phi)$ , their values at a given  $\phi$  are independent. Consequently, one should imagine that this procedure is being carried out separately at each  $\phi$ . The results are then used to construct a function of  $\phi$ .

0.2



0.1 0.05 () H 0.02 roli Dotted: 2nd onle cted al 0.01 ----d: [1/1] rational 10-3 <u>x</u>-2 -1.5 -1 -0.5 0 ¢/m<sub>Pl</sub>

FIG. 1. A comparison of different analytic approximations with the exact, numerically generated, solution  $H(\phi)$  for a potential  $V(\phi) \propto \phi^2$ , near the end of inflation. The normalization of H is arbitrary. The slow-roll approximation and its first and second-order corrected versions are shown, together with the [1/1] rational approximant introduced in Sec. V B, which is also a second-order correction. The rational approximant performs the best, as is more clearly seen in Fig. 2.

Thus, we have generated an analytic solution for inflation in the potential  $V(\phi)$  that is accurate up to fourth order in the slow-roll parameters, rather than the usual lowest order.

We illustrate this with the specific example of the potential  $V(\phi) = m^2 \phi^2/2$ . To second order in the HSR parameters, one finds

$$\epsilon_{H} = \frac{m_{\rm Pl}^2}{4\pi\phi^2} \left[ 1 - \frac{m_{\rm Pl}^2}{6\pi\phi^2} + O\left(\frac{m_{\rm Pl}^4}{\phi^4}\right) \right] \,, \tag{48}$$

and hence

$$H^{2}(\phi) = \frac{4\pi m^{2} \phi^{2}}{3m_{\rm Pl}^{2}} \left[ 1 + \frac{m_{\rm Pl}^{2}}{12\pi\phi^{2}} - \frac{m_{\rm Pl}^{4}}{144\pi^{2}\phi^{4}} + O\left(\frac{m_{\rm Pl}^{6}}{\phi^{6}}\right) \right] \,. \tag{49}$$

Results accurate to one order less, as given in Ref. [4], can be obtained by removing the last term from both expressions. The behavior of  $H(\phi)$  is indicated in Fig. 1, where for comparison, the exact numerical solution is also shown.

For this particular potential, it is necessary to include both first- and second-order corrections in order to maintain, as a sensible definition of the end of inflation, the condition  $\epsilon_H(\phi) = 1$  [here, we refer to the precise  $\epsilon_H(\phi)$ derived from the approximate solution  $H(\phi)$ , and not to  $\epsilon_H(\phi)$  truncating to the same order]. To lowest order, this is guaranteed because the potential has a minimum where  $V(\phi) = 0$ ; but, because corrections become large near the end of inflation, they may spoil this. In fact, for a  $\phi^2$  potential, if one includes only the first-order corrections, they conspire so that  $\epsilon_H(\phi)$  fails to reach unity for any  $\phi$ , despite the solutions being closer to the exact numerical solution than the lowest-order results for the bulk of the evolution. Including the second-order corrections removes this problem for the  $\phi^2$  potential. The exact behavior of  $\epsilon_H(\phi)$  is shown in Fig. 2 in each case.



FIG. 2. The same comparison as Fig. 1, but this time showing the exact  $\epsilon_H(\phi)$  corresponding to each of the solutions. Recall that the end of inflation is at  $\epsilon_H = 1$ . The pathological behavior of the first-order corrected solution, for which inflation never ends in this potential, is clear; all other solutions have a satisfactory end to inflation, with the rational approximant providing the best overall approximation to the exact solution. For potentials with  $\phi^{\alpha}$  behavior with  $\alpha \geq 4$ , the end of inflation is well defined even at first order.

#### **B.** Rational-approximant approach

In the chaotic inflationary example presented above, it was seen how, as the field rolls toward the minimum of the potential, the Taylor-series expansion becomes progressively less accurate due to the dependence of the slowroll parameters on inverse powers of  $\phi$ . It is not true, in general, that  $\epsilon_H(\phi)$  will be the first parameter to become large in this manner. Hence, it is not guaranteed that the approximation will describe the evolution through, or even up to, the end of inflation. At  $\phi = 0$ , where  $V(\phi) = 0$ , the Taylor expansion diverges, although inflation necessarily finishes before this.

In cases such as this, rational-approximant techniques [8,9] can be effective. For the single-variable case, instead of using a single Taylor polynomial, we approximate using the Padé approximant, a quotient of *two* polynomials, in the hope of achieving a better range and rate of convergence. A particular application of this technique to inflation was made in Ref. [20]. Unfortunately, Padé approximants can only be directly applied to single-variable expansions, and we require the extension of this theory to multiple-variable problems.

#### 1. Canterbury approximant

The Canterbury approximant [8] supplements a Padé quotient approximant in many variables with a minimal Taylor series, where minimal means containing as few terms as possible. So, for a function of r variables  $f(x_1, x_2, \ldots, x_r)$  expanded to *n*th order, we postulate an approximant of the form

$$[L/M]_{f} \equiv \frac{A(x_{1}, x_{2}, \dots, x_{r})}{1 + B(x_{1}, x_{2}, \dots, x_{r})} + \sum_{p_{1}=0}^{n} \sum_{p_{2}=0}^{n} \cdots \sum_{p_{r}=0}^{n} e_{p_{1}p_{2}\cdots p_{r}} \left[\prod_{i=1}^{r} x_{i}^{p_{i}}\right], \quad (50)$$

where  $A(x_1, x_2, ..., x_r)$  and  $B(x_1, x_2, ..., x_r)$  are multiple-variable polynomials of order L and M, respectively, the  $e_{p_1p_2...p_r}$  are constants, and  $p_i$  are the powers of  $x_i$ , ranging between 0 and n. The orders of the approximants, L and M, are constrained by the relation L + M = n, and the polynomial  $B(x_1, x_2, ..., x_r)$  contains no zeroth-order term. The undetermined constants in the approximant are fixed by expanding  $[L/M]_f$  to *n*th-order, and matching it with the *n*th-order expansion of f(x). Such an approximant often yields faster convergence because the quotient contains better estimates of the higher-order terms than a truncated Taylor series. Reference [8] provides a useful introduction to approximant techniques.

This approximant is not unique. Any attempt to neglect the Taylor terms in Eq. (50) and solve for the remaining constants fails in general, as unfortunately there will be an insufficient number of free constants to do this. The choice of which values we assign to the constants in the Padé term (and hence the form of the corrective Taylor series) is in fact completely arbitrary, demonstrating the nonuniqueness of Eq. (50). We find, however, that keeping the Taylor-correction terms purely *n*th-order simplifies the analysis somewhat, and also ensures that they are small in slow roll.

We are now in a position to recast the PSRA in the form of Eq. (50). In terms of PSR parameters, during slow-rolling inflation, an arbitrary function  $f(H(\phi))$  may be approximated by

$$[L/M]_{f} = \frac{A(\epsilon_{V}, \eta_{V}, \dots, {}^{L}\beta_{V})}{1 + B(\epsilon_{V}, \eta_{V}, \dots, {}^{M}\beta_{V})} + \sum_{p_{0}=0}^{n} \sum_{p_{1}=0}^{n} \cdots \sum_{p_{n}=0}^{n} e_{p_{0}p_{1}\cdots p_{n}} \left[ \prod_{i=0}^{n} ({}^{i}\beta_{V})^{p_{i}} \right], \quad (51)$$

where the  ${}^{n}\beta_{V}$  are the general PSR parameters defined in Eq. (36); recall, also, that we have  ${}^{0}\beta_{V} \equiv \epsilon_{V}$ .

This complicated formalism is clarified by showing it at work to a given order. We present a [2/2] Canterbury approximant to  $(1 - \epsilon_H/3)$ , and indicate how this may be used to construct the approximant to  $H^2(\phi)$ . For  $(1 - \epsilon_H/3)$ , we obtain

$$[2/2]_{(1-\epsilon_H/3)} = \frac{1 + \frac{21}{4}\epsilon_V - \frac{7}{3}\eta_V - \frac{53}{18}\epsilon_V\eta_V + \frac{89}{36}\epsilon_V^2 + \eta_V^2 - \frac{2}{9}\xi_V^2}{1 + \frac{67}{12}\epsilon_V - \frac{7}{3}\eta_V - \frac{7}{2}\epsilon_V\eta_V + \frac{35}{36}\epsilon_V^2 + \eta_V^2 - \frac{2}{9}\xi_V^2} - \frac{2}{81}\epsilon_V\sigma_V^3 + \frac{1}{162}\epsilon_V^3\eta_V - \frac{35}{324}\epsilon_V^2\eta_V^2 + \frac{13}{162}\epsilon_V^2\xi_V^2 + \frac{1}{27}\epsilon_V\eta_V^3.$$
(52)

Diagonal Canterbury approximants (i.e., the [L/L] cases) share many of the useful properties of standard Padé approximants (duality, homographic invariance, unitarity, etc.), and substantially simplify the application of the Canterbury technique [8]. Use of diagonal approximants, where possible, is thus recommended. In particular, the duality property may be exploited to save considerable effort when calculating the corresponding expression for  $H^2(\phi)$  from Eq. (52). The duality prop-

erty is as follows: If  $f(x) = [g(x)]^{-1}$  and  $g(0) \neq 0$ , then

$$[L/L]_{f(x)} = \left\{ [L/L]_{g(x)} \right\}^{-1} .$$
(53)

We may thus obtain a  $[2/2]_{H^2(\phi)}$  from Eq. (52), via

$$[2/2]_{H^2(\phi)} = \frac{8\pi}{3m_{\rm Pl}^2} V(\phi) \left\{ [2/2]_{(1-\epsilon_H/3)} \right\}^{-1} .$$
 (54)

#### 2. Simplified Canterbury approximants

The Canterbury approximant provides a powerful technique for calculations of inflationary dynamics. However, in its full higher-order glory, it can be quite cumbersome and unwieldy. It is therefore useful to find circumstances in which the corrective Taylor series is not required. The simplest way to bring this about is to take the [0/n] approximant which never needs correcting. However, it may also be true that at low orders there is sufficient freedom in the diagonal approximants, due to the vanishing of some of the terms in the original Taylor series.

In fact, the [1/1] approximant has this property. Assuming a general form for the  $[1/1]_{1-\epsilon_H/3}$  approximant, and matching to the Taylor expansion for  $(1-\epsilon_H/3)$  from Eq. (46) to second order, yields

$$[1/1]_{(1-\epsilon_{H}/3)} \equiv \frac{a+b\epsilon_{V}+c\eta_{V}}{1+d\epsilon_{V}+e\eta_{V}} = 1 - \frac{\epsilon_{V}}{3} \left\{ 1 - \frac{4}{3}\epsilon_{V} + \frac{2}{3}\eta_{V} + O_{2} \right\}.$$
(55)

Comparing coefficients, we arrive at the result

$$[1/1]_{(1-\epsilon_H/3)} = \frac{1+\epsilon_V - \frac{2}{3}\eta_V}{1+\frac{4}{3}\epsilon_V - \frac{2}{3}\eta_V},$$
(56)

which, by construction, agrees with the Taylor series to second order. The corresponding  $H(\phi)$  is easily obtained. For the  $\phi^2$  potential, examined earlier, this does indeed improve on the second-order Taylor series given in Eq. (49), as shown in Figs. 1 and 2.

As a final observation, note that if the slow-roll parameters all have the same functional form, permitting us to write

$${}^{n}\beta_{V} = \sum_{i} C_{i}f^{i}(\phi) , \qquad (57)$$

where  $C_i$  are constants and  $f(\phi)$  is an arbitrary function of  $\phi$  (which is small in the PSRA), then we can always circumvent the need for a corrective Taylor series by expanding in powers of  $f(\phi)$ , thus reducing the problem to the unmodified Padé case.

## VI. CONCLUSIONS

By defining a suitable hierarchy of parameters, we have extended the slow-roll approximation to a slow-roll expansion, allowing progressively more accurate *analytic* approximations to be constructed via an order-by-order decomposition in terms of slow-roll parameters. The use of rational approximants pushes the range of validity of the slow-roll expansion up to, and in many instances beyond, the end of inflation. With the accurate observational information becoming available, this allows an assessment of the accuracy of calculations within the slow-roll approximation, and is especially important with the present considerable emphasis focused on inflationary models which make predictions far from the standard (zeroth-order) case.

We have used these parameters to define an improved measure of the amount of inflation. However, present uncertainties regarding the physics of reheating make it useful only in rather extreme circumstances such as a temporary suspension of inflation, during which the Universe remains scalar field dominated, as in the hybrid inflation model of Ref. [17].

Let us caution the reader regarding the necessity of the attractor condition for the slow-roll expansion to make sense. By incorporating order-by-order corrections, we can only generate one solution  $H(\phi)$  out of the oneparameter family of actual solutions allowed by the freedom of H, or equivalently  $\phi$ , permitted by the initial conditions. If the attractor hypothesis is not satisfied, then the solution generated, while conceivably an accurate particular solution of the equations of motion, need have no relation to the actual dynamical solutions which might be attained. A case in point is the exact "intermediate" inflation solution [10, 13, 21]. For small  $\phi$ , this solution corresponds to the rather unnatural (and noninflationary) behavior of the field moving up the potential and over a maximum, beyond which inflation starts. If one attempts to use our procedure to describe this entrance to inflation, the solutions generated bear no particular resemblance to the exact solution until well into the inflationary regime.<sup>3</sup> This serves as a cautionary note, that known exact solutions are typically only late-time attractors, and unless a significant period of inflation occurs before the time of interest, so that the attractor solution is reached, they are of little relevance.

Importantly, with regard to the exit from inflation, we are on much safer ground. It is assumed that enough time has passed for the attractor to be reached, and hence all solutions exit from inflation in the same way. Therefore, when our expansion procedure supplies a particular solution, it provides an excellent description of the way in which the entire one-parameter family of initial conditions will exit inflation. Without this vital point, the generation of solutions via the slow-roll expansion would be fruitless.

We have concentrated on the dynamics of inflation, rather than on the perturbation spectra produced from them. However, the slow-roll expansion can also be brought into play there; as an example, we quote the results for the spectral indices n for the density perturbations and  $n_T$  for the gravitational waves (see [2] for precise definitions). These have long been taken as approximately 1 and 0, respectively; results to first order were given by Liddle and Lyth [5] and to second order by Stewart and Lyth [6]. With our definitions, these read, in the HSRA and PSRA, respectively

<sup>&</sup>lt;sup>3</sup>By contrast, the "intermediate solution" can also be employed as the slow-roll solution in the simple potential  $V \propto \phi^{-\beta}$  (with  $\beta$  and  $\phi$  both positive), where the attractor hypothesis can be applied, though the solution to which the expansion process tends would have to be found numerically again.

## FORMALIZING THE SLOW-ROLL APPROXIMATION IN INFLATION

$$1 - n = 4\epsilon_H - 2\eta_H + 2(1+c)\epsilon_H^2 + \frac{1}{2}(3-5c)\epsilon_H\eta_H - \frac{1}{2}(3-c)\xi_H^2 + \cdots$$
(58)

$$= 6\epsilon_V - 2\eta_V - \frac{1}{3}(44 - 18c)\epsilon_V^2 - (4c - 14)\epsilon_V\eta_V - \frac{2}{3}\eta_V^2 - \frac{1}{6}(13 - 3c)\xi_V^2 + \cdots,$$
(59)

$$n_T = -2\epsilon_H - (3+c)\epsilon_H^2 + (1+c)\epsilon_H\eta_H + \cdots$$
(60)

$$= -2\epsilon_V - \frac{1}{3}(8+6c)\epsilon_V^2 + \frac{1}{3}(1+3c)\epsilon_V\eta_V + \cdots,$$
(61)

where  $c = 4(\ln 2 + \gamma)$  with  $\gamma$  being Euler's constant. Notice the factors in 1-n change even at first order, due to the different definitions of  $\eta$  which have been used. Similarly, we reproduce the second-order result for the ratio R of tensor and scalar amplitudes [6],

$$R = \frac{25}{2} \epsilon_H \left[ 1 + 2c \left( \epsilon_H - \eta_H \right) + \cdots \right]$$
(62)

$$=\frac{25}{2}\epsilon_{V}\left[1+2\left(c-\frac{1}{3}\right)\left(2\epsilon_{V}-\eta_{V}\right)+\cdots\right]$$
(63)

though it should be noted that this is not a direct observable [20]. Unlike the relatively simple dynamics which we have emphasised in this paper, no way of extending these expressions analytically to arbitrary order is known.

Note added. As we were completing this paper we received a paper by Lidsey and Waga [22] which also discusses the slow-roll approximation, although with a considerably different emphasis.

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#### APPENDIX

We provide here a list of expressions, deemed too cumbersome and obtrusive to be imposed upon the main body of text, but which could prove useful in certain applications. The first four are exact extensions of Eq. (14) to higher-order parameters:

$$\eta_V = \left[3 - \epsilon_H\right]^{-1} \left(3\epsilon_H + 3\eta_H - \eta_H^2 - \xi_H^2\right),\tag{A1}$$

$$\xi_{V}^{2} = [3 - \epsilon_{H}]^{-2} \left( 27\epsilon_{H}\eta_{H} + 9\xi_{H}^{2} - 9\epsilon_{H}\eta_{H}^{2} - 12\eta_{H}\xi_{H}^{2} - 3\sigma_{H}^{3} + 3\eta_{H}^{2}\xi_{H}^{2} + \eta_{H}\sigma_{H}^{3} \right),$$
(A2)

$$\begin{aligned} \sigma_{V}^{V} &= [3 - \epsilon_{H}]^{-4} \left(81\epsilon_{H}\eta_{H}^{2} + 108\epsilon_{H}\xi_{H}^{2} + 27\sigma_{H}^{3} - 54\epsilon_{H}\eta_{H}^{2} - 72\epsilon_{H}\eta_{H}\xi_{H}^{2} - 54\eta_{H}\sigma_{H}^{3} \\ &- 27\xi_{H}^{4} - 9\tau_{H}^{4} + 9\epsilon_{H}\eta_{H}^{4} + 12\epsilon_{H}\eta_{H}^{2}\xi_{H}^{2} + 18\eta_{H}\xi_{H}^{4} + 27\eta_{H}^{2}\sigma_{H}^{3} + 6\eta_{H}\tau_{H}^{4} - 3\eta_{H}^{2}\xi_{H}^{4} - 4\eta_{H}^{3}\sigma_{H}^{3} - \eta_{H}^{2}\tau_{H}^{4}\right), \end{aligned}$$
(A3)  
$$\tau_{V}^{4} &= [3 - \epsilon_{H}]^{-4} \left(810\epsilon_{H}\eta_{H}\xi_{H}^{2} + 405\epsilon_{H}\sigma_{H}^{3} + 81\tau_{H}^{4} - 810\epsilon_{H}\eta_{H}^{2}\xi_{H}^{2} - 405\epsilon_{H}\eta_{H}\sigma_{H}^{3} \\ &- 270\xi_{H}^{2}\sigma_{H}^{3} - 216\eta_{H}\tau_{H}^{4} - 27\zeta_{H}^{5} + 270\epsilon_{H}\eta_{H}^{3}\xi_{H}^{2} + 135\epsilon_{H}\eta_{H}^{2}\sigma_{H}^{3} + 270\eta_{H}\xi_{H}^{2}\sigma_{H}^{3} \\ &+ 162\eta_{H}^{2}\tau_{H}^{4} + 27\eta_{H}\zeta_{H}^{5} - 30\epsilon_{H}\eta_{H}^{4}\xi_{H}^{2} - 15\epsilon_{H}\eta_{H}^{3}\sigma_{H}^{3} - 90\eta_{H}^{2}\xi_{H}^{2}\sigma_{H}^{3} - 48\eta_{H}^{3}\tau_{H}^{4} \\ &- 9\eta_{H}^{2}\zeta_{H}^{5} + 10\eta_{H}^{3}\xi_{H}^{2}\sigma_{H}^{3} + 5\eta_{H}^{4}\tau_{H}^{4} + \eta_{H}^{3}\zeta_{H}^{5}\right). \end{aligned}$$

It is also possible to express any parameter we choose as a first-order differential relation in terms of lower-order parameters. This was done in Eqs. (16) and (17) for  $\eta$ ; we do this here for  $\xi$  and  $\sigma$ :

$$\xi_H^2 = \epsilon_H \eta_H - \sqrt{\frac{m_{\rm Pl}^2}{4\pi}} \sqrt{\epsilon_H} \eta'_H , \qquad (A5)$$

$$\xi_V^2 = 2\epsilon_V \eta_V - \sqrt{\frac{m_{\rm Pl}^2}{4\pi}} \sqrt{\epsilon_V} \eta_V' , \qquad (A6)$$

$$\sigma_H^3 = \xi_H^2 \left( 2\epsilon_H - \eta_H \right) - \sqrt{\frac{m_{\rm Pl}^2}{\pi}} \sqrt{\epsilon_H} \xi_H \xi'_H , \qquad (A7)$$

$$\sigma_V^3 = \xi_V^2 \left( 4\epsilon_V - \eta_V \right) - \sqrt{\frac{m_{\rm Pl}^2}{\pi}} \sqrt{\epsilon_V} \xi_V \xi_V' \,. \tag{A8}$$

These compressions allow us to express any PSR parameter of order n as a first-order differential relation involving HSR parameters of order not exceeding n, as was done in Eq. (15) for  $\eta_V$ . We present the case for  $\xi_V$ , although such a result may be derived for any of the higher-order parameters,

$$\xi_{V}^{2} = [3 - \epsilon_{H}]^{-2} \left\{ 27\epsilon_{H}\eta_{H} + 9\xi_{H}^{2} - 12\eta_{H}\xi_{H}^{2} - 9\epsilon_{H}\eta_{H}^{2} + 3\eta_{H}^{2}\xi_{H}^{2} + (\eta_{H} - 3)\xi_{H} \left[ \xi_{H} \left( 2\epsilon_{H} - \eta_{H} \right) - \sqrt{\frac{m_{\rm Pl}^{2}}{\pi}}\sqrt{\epsilon_{H}}\xi_{H}^{\prime} \right] \right\}.$$
(A9)

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Equations (A1)-(A9) are all exact. We now give some approximate formulas, inverting some of the above relations to yield expressions for HSR parameters in terms of PSR parameters. If necessary, these can be fitted to Padé or Canterbury approximants, using the methods outlined in Sec. V B. We have already stated the result for  $\epsilon_H$  [Eq. (46)]; here we give the higher-order parameters,

$$\eta_H = \eta_V - \epsilon_V + \frac{8}{3}\epsilon_V^2 + \frac{1}{3}\eta_V^2 - \frac{8}{3}\epsilon_V\eta_V + \frac{1}{3}\xi_V^2 - 12\epsilon_V^3 + \frac{2}{9}\eta_V^3 + 16\epsilon_V^2\eta_V$$

$$-\frac{46}{9}\epsilon_V\eta_V^2 - \frac{17}{9}\epsilon_V\xi_V^2 + \frac{2}{3}\eta_V\xi_V^2 + \frac{1}{9}\sigma_V^3 + \mathcal{O}_4, \qquad (A10)$$

$$\xi_{H}^{2} = \xi_{V}^{2} - 3\epsilon_{V}\eta_{V} + 3\epsilon_{V}^{2} - 20\epsilon_{V}^{3} + 26\epsilon_{V}^{2}\eta_{V} - 7\epsilon_{V}\eta_{V}^{2} - \frac{13}{3}\epsilon_{V}\xi_{V}^{2} + \frac{4}{3}\eta_{V}\xi_{V}^{2} + \frac{1}{3}\sigma_{V}^{3} + \mathcal{O}_{4},$$
(A11)

$$\sigma_H^3 = \sigma_V^3 - 3\epsilon_V \eta_V^2 + 18\epsilon_V^2 \eta_V - 15\epsilon_V^3 - 4\epsilon_V \xi_V^2 + \mathcal{O}_4 \,. \tag{A12}$$

Note that these inversions are only valid when the attractor condition, Eq. (8), holds. The second-order truncation of Eq. (A10) is compatible with the result presented in [19].

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