# Nonlinear behaviors of cosmological inhomogeneities with a standard fluid and inflationary matter

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Nonlinear behaviors of cosmological inhomogeneities are studied in the anti-Newtonian or long-wavelength approximation. They consist of a standard fluid and inflationary matter. The latter is an inflationary fluid or a scalar field. After the initially dominant local anisotropy has decreased, they become quasi-isotropic. If the density of the standard fluid is dominant at this stage, the inhomogeneities grow with time, but, as the density of the inflationary matter increases, it is shown that the growth is gradually prevented and finally they decay. After the inflation they again grow due to the irregular spatial curvature. The generality of the solutions is also discussed.

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## I. INTRODUCTION

The evolution of inhomogeneous cosmological models has so far been studied using approximations appropriate to their various phases, such as the linearized theory of perturbations [1,2], the post-Newtonian theory [3,4], and the anti-Newtonian theory, also called the long-wavelength approximation or gradient expansion [5–8].

To derive the higher-order terms of the expansion in the latter approximation scheme is difficult in practice if the anisotropies are taken into account (see below for a precise definition). In the quasi-isotropic stage after the initial local anisotropies have decreased, however, the analysis is much simplified. In the case of dust, the solution can be obtained iteratively to arbitrary order [8] and was explicitly written down up to sixth order in the gradients (third iteration, the zeroth being the isotropic Friedmann solution) by Parry [9]. Comer *et al.* [10] worked out the second iteration in the case of a perfect fluid with an arbitrary adiabatic index or a scalar field. They showed that the obtained quasi-isotropic solutions can be regarded as attractors of the generic, anisotropic solutions which were derived by one of the present authors in [5] and [11], and that the decay or growth of the inhomogeneities depended on their equation of state, that is, whether or not the fluid or scalar field was inflationary. Finally the homogenization of local inhomogeneities has been discussed in [11] in connection with the problem of the behaviors of local inhomogeneities including black holes in de Sitter spacetime [12].

Now in more realistic situations there can be two matter components: a standard fluid and inflationary matter (an inflationary fluid or scalar field). Since it is not self-evident whether or not inhomogeneities can continue to grow in the inflationary stage, we shall study in this paper their behaviors in the long-wavelength iteration scheme and compare the results to the linear analysis and thus extend the "cosmic no hair" theorem.

## II. EXPONENTIAL INFLATION

The spacetime is described in the synchronous condition with the line element

$$ds^2 = -dt^2 + \gamma_{ij}dx^i dx^j, \tag{2.1}$$

where we use the units $c = 8\pi G = 1$, and the Einstein equations for two perfect fluids are expressed as

$$\tfrac{1}{2}\dot{\kappa}^i_i + \tfrac{1}{4}\kappa^j_i\kappa^i_j = -\sum_a\{\tfrac{1}{2}(\epsilon_a + 3p_a) + (\epsilon_a + p_a)[(u^0_{(a)})^2 - 1]\}, \tag{2.2}$$

$$\kappa^j_{j;i} - \kappa^j_{i;j} = 2\sum_a(\epsilon_a + p_a)u^0_{(a)}u_{(a)i}, \tag{2.3}$$

$$2P^j_i + \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}\kappa^j_i)^{\cdot} = \sum_a\{2(\epsilon_a + p_a)u_{(a)i}u^j_{(a)} + \delta^j_i(\epsilon_a - p_a)\}, \tag{2.4}$$

where $P^j_i$ are components of the Ricci tensor in the three-dimensional space with metric $dl^2 = \gamma_{ij}dx^i dx^j$, $\kappa^j_i$ is defined by $\gamma^{jl}\dot{\gamma}_{li}$, an overdot denotes the derivative with respect to $t$, $\gamma$ is the determinant $|\gamma_{ij}|$, and the four velocities $u^\mu_{(a)}(\mu = 0-3)$ satisfy $u_{(a)\mu}u^\mu_{(a)} =$

 

$-(u^0_{(a)})^2[1 - \Sigma_{i=1}^3 (u^i_{(a)})^2/(u^0_{(a)})^2] = -1$. Moreover, the total energy and pressure as the sums of the standard ones $(\epsilon_s, p_s)$ and the inflationary ones $(\epsilon_i, p_i)$ are given by $\epsilon$ and $p$:

$$\epsilon = \epsilon_s + \epsilon_i, \quad p = p_s + p_i. \tag{2.5}$$

Their equations of state are

$$p_s/\epsilon_s = w_s \ (= \Gamma_s - 1), \quad p_i/\epsilon_i = w_i \ (= \Gamma_i - 1), \tag{2.6}$$

where $w_s$ and $w_i$ or $\Gamma_s$ and $\Gamma_i$ are constants satisfying

$$1 > w_s \geq -1/3 \ (2 > \Gamma_s \geq 2/3)$$

and

$$-1/3 > w_i \geq -1 \ (2/3 > \Gamma_i \geq 0). \tag{2.7}$$

In this section we are confined to the case $w_i = -1$, in which $\epsilon_i = -p_i$ is a constant corresponding to the cosmological constant. Following the notation in the previous paper [11] we express them also as $\epsilon_i = \epsilon_\Lambda$.

When we neglect the spatial curvature $P_{ij}$ and the square of spatial velocities $(u^i_{(a)})^2$, we obtain, from Einstein equations (2.2) and (2.4),

$$\kappa_i^j = (\dot{X}/X)\delta_i^j + A_i^j X^{-3/2}, \tag{2.8}$$

$$\epsilon - p = \ddot{X}/X + \tfrac{1}{2}(\dot{X}/X)^2, \tag{2.9}$$

$$\epsilon + 3p = -3\ddot{X}/X + \tfrac{3}{2}(\dot{X}/X)^2 - 6\xi/X^3, \tag{2.10}$$

as was shown in the previous paper, where $X \equiv \gamma^{1/3}$, $\gamma = \det(\gamma_{ij})$, and $A_i^j$ is an arbitrary traceless function of only spatial variables with $\xi \equiv A_i^j A_j^i/12$. From Eqs. (2.9) and (2.10) we obtain

$$\ddot{Y} + \tfrac{9}{8}(1 - w_s{}^2)\xi Y^{(w_s-3)/(w_s+1)} - \tfrac{3}{4}(1 + w_s)^2 \epsilon_\Lambda Y = 0, \tag{2.11}$$

where $Y \equiv X^{3(1+w_s)/4} = \gamma^{(1+w_s)/4}$. The integration of Eq. (2.11) and the derivation of metric components, density and velocity components are shown in Ref. [11].

In the quasi-isotropic case $A_i^j = 0$, which we consider in the following, the solution $Y$ is expressed as

$$Y = X^{\frac{3}{4}(1+w_s)} = Y_0(\mathbf{x}) \ \sinh(Ht), \tag{2.12}$$

where $H \equiv (1 + w_s)(\frac{3}{4}\epsilon_\Lambda)^{1/2}$ and the time delay (being an integration function) is neglected. Then the energy density and velocity are

$$\epsilon_s = \epsilon_\Lambda / \sinh^2(Ht), \quad u_{(s)i} = 0, \tag{2.13}$$

and the metric components are

$$\gamma_{ij} = A_0 h_{ij}, \quad A_0{}^{\frac{3}{4}(1+w_s)} = \sinh(Ht). \tag{2.14}$$

When we treat higher-order solutions with respect to the spatial curvature, the above solution is regarded as the first-order solution ${}^{(1)}\gamma_{ij}$, ${}^{(1)}\epsilon_s$, and ${}^{(1)}u_{(s)i}$. Let us derive the next third-order solution in Comer *et al.*'s procedure [10]:

$${}^{(3)}\gamma_{ij} = A_0\{f_2(t)R_{ij} + \tfrac{1}{3}[g_2(t) - f_2(t)]Rh_{ij}\}, \tag{2.15}$$

where $R_{ij} = P_{ij}, R_i^j \equiv h^{ik}R_{kj}$, and $R = R_i^i$, and the independence of functions $R_{ij}$ and $Rh_{ij}$ is used for the derivation of separate equations dependent only on time. Thus the Einstein equations (2.2) and (2.4) for the metric $\gamma_{ij} = {}^{(1)}\gamma_{ij} + {}^{(3)}\gamma_{ij}$ lead to

$$\ddot{f}_2 + \frac{3}{2}\frac{\dot{A}_0}{A_0}\dot{f}_2 + 2A_0^{-1} = 0$$
$$\ddot{g}_2 + \tfrac{3}{2}(1 + w_s)\frac{\dot{A}_0}{A_0}\dot{g}_2 + \tfrac{1}{2}(1 + 3w_s)A_0^{-1} = 0. \tag{2.16}$$

Their solutions are

$$f_2 = -2\int dt A_0^{-3/2}\int dt' A_0^{1/2}, \tag{2.17}$$

$$g_2 = -\tfrac{1}{2}(1 + 3w_s)\int dt A_0^{-3(1+w_s)/2}\int dt' A_0^{(1+3w_s)/2}, \tag{2.18}$$

which are of the same form as in Comer *et al.* [12], though our $A_0$ is different from theirs. The energy density derived from Eqs. (2.9) and (2.10) is

$${}^{(3)}\epsilon_s = \delta\epsilon_s = \frac{1}{2}\left(\dot{g}_2\frac{\dot{A}_0}{A_0} + A_0{}^{-1}\right)R. \tag{2.19}$$

By using Eq. (2.18), we can rewrite as

$${}^{(3)}\epsilon_s = \tfrac{1}{2}A_0{}^{-3(1+w_s)/2}\frac{\dot{A}_0}{A_0}R \times \int dt(1 - A_0\ddot{A}_0/\dot{A}_0^2)A_0{}^{(1+3w_s)/2}. \tag{2.20}$$

The general behaviors of $f_2$ and $g_2$ are found by solving numerically Eq. (2.16) in various initial conditions. Homogeneous part of solutions $f_2$ and $g_2$ show the components being constant or decaying with time always, and the constant terms can be included in $h_{ij}$ with the factors $R_{ij}$ and $Rh_{ij}$. This means that the changes in $h_{ij}$ and $R_{ij}$ are brought by the dynamical evolution of fluids. Next we pay attention to the special solutions which may bring growing components. They are shown in the following two extreme cases $Ht \ll 1$ and $Ht \gg 1$.

(a) $Ht \ll 1$. As $A_0 \propto t^{4/[3(1+w_s)]}$, we have

$$f_2 \propto g_2 \propto t^{2(1+3w_s)/[3(1+w_s)]}. \tag{2.21}$$

The energy density is

$${}^{(3)}\epsilon_s = \delta\epsilon_s \propto t^{-4/[3(1+w_s)]}, \tag{2.22}$$

so that the contrast is

$$\delta\epsilon_s/\epsilon \propto t^{2(1+3w_s)/[3(1+w_s)]}. \quad (2.23)$$

Therefore the inhomogeneities grow with time.

(b) $Ht \gg 1$. As $A_0 \simeq (\frac{1}{2}e^{Ht})^{4/[3(1+w_s)]}[1 - \frac{4}{3(1+w_s)}e^{-2Ht} + O(e^{-4Ht})]$, we have

$$f_2 \propto g_2 \propto e^{-4Ht/[3(1+w_s)]} \quad (2.24)$$

and

$$^{(3)}\epsilon_s = \delta\epsilon_s = e^{-\{2+4/[3(1+w_s)]\}Ht}, \quad (2.25)$$

where it is noted that the density in the special solution decays more rapidly than that ($\propto e^{-2Ht}$) in the homogeneous solution, because of the factor $(1 - A_0\ddot{A}_0/\dot{A}_0^2)$ in Eq. (2.20). Since $\epsilon$ is constant at this stage, the density contrast is proportional to $^{(3)}\epsilon_s$, and so the inhomogeneities including a standard fluid decay with time. The case when $A_0$ is exactly exponential, i.e., $A_0 = (\frac{1}{2}e^{Ht})^{4/[3(1+w_s)]}$, was treated by Starobinskii [13], in which the above special solution vanishes and we have only the homogeneous solution.

The third-order velocity $^{(3)}u_i$ is derived using Eq. (2.3), where the third-order components of $\kappa_i^j$ consist of terms $R_i^j$ and $Rh_i^j$. Because of the identity $R^j_{i;j} = R_{,j}$, the left-hand side of Eq. (2.3) vanishes, so that we obtain $^{(3)}u_{(s)_i} = 0$. It should be noted that $^{(2n+1)}u_{(s)}$ do not vanish for $n > 1$ in general, if $w_s \neq 0$. A numerical example of time evolution of $^{(3)}\epsilon_s/\epsilon$ is shown in Fig. 1.

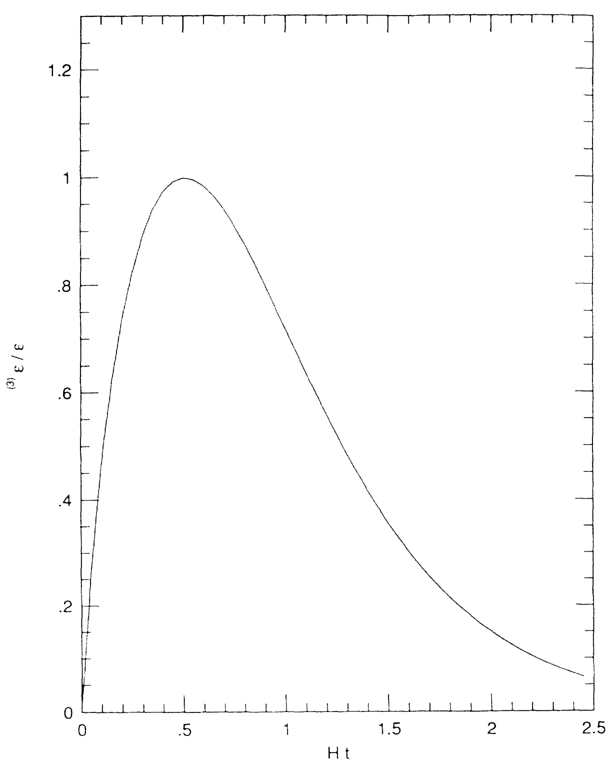


FIG. 1. Behavior of $^{(3)}\epsilon/\epsilon$ in the case $w_s = 0$ and $^{(3)}\epsilon/\epsilon \propto t^{2/3}$ for $Ht \ll 1$. The ordinate scale is arbitrary.

For higher-order terms, additional separate equations, dependent only on time, are derived similarly. We refer the reader to the Appendix in Comer *et al.* [10] for the details of the calculation. It is found that the $(2n+1)$th metric components are in proportion to $t^{2n(1+3w_s)/[3(1+w_s)]}$ or $e^{-4nHt/[3(1+w_s)]}$ for $Ht \ll$ or $Ht \gg 1$, respectively. In the intermediate case $Ht \sim 1$, the analysis is rather complicated.

From the above two cases, we can conclude that the inhomogeneities can grow at an early stage such as $Ht < 1$, the growth is prevented gradually at $Ht \sim 1$, and decay finally, when the exponential inflation becomes dominant. Accordingly gravitational collapse can occur only at the stage $Ht < 1$. The black holes which appeared at the stage $Ht < 1$ can no longer collide due to gravitation at the stage $Ht > 1$, consistent with the analysis of Nakao *et al.* [12].

## III. POWER-LAW INFLATION

Next we consider the case when the inflationary fluid has the equation of state $-1 < w_i < -1/3$ (or $0 < \Gamma_i < 2/3$), and assume that both fluids are adiabatic, because we know no realistic interactions between them. Then the energy densities are

$$\epsilon_s = \epsilon_{s0}(\mathbf{x})\ X^{-3(1+w_s)/2}, \quad \epsilon_i = \epsilon_{i0}(\mathbf{x})\ X^{-3(1+w_i)/2}, \quad (3.1)$$

which are derived from the relation of conservation, and from Eqs. (2.9) and (2.10) we obtain an equation for $X$:

$$(\dot{X}/X)^2 = \tfrac{4}{3}(\epsilon_s + \epsilon_i). \quad (3.2)$$

The solution is expressed in the two limits as

$$X \simeq [(\tfrac{3}{4}\epsilon_{s0})^{1/2}(1+w_s)t]^{4/[3(1+w_s)]} \quad \text{for } X \ll X_c, \quad (3.3)$$

$$X \simeq [(\tfrac{3}{4}\epsilon_{i0})^{1/2}(1+w_i)t]^{4/[3(1+w_i)]} \quad \text{for } X \gg X_c, \quad (3.4)$$

where $X_c \equiv (\epsilon_{i0}/\epsilon_{s0})^{2/[3(w_i-w_s)]}$ and we neglect the time delay being an integration function. The Universe has the power-law inflation for large $X$. The first-order metric is

$$^{(1)}\gamma_{ij} = A_0 h_{ij}, \quad (3.5)$$

where $A_0(t) = X(\mathbf{x}=\mathbf{0})$ and $A_{0c} = X_c(\mathbf{x}=\mathbf{0})$ for a representative point $\mathbf{x} = \mathbf{0}$.

For the third-order metric (2.15), the Einstein equations lead to

$$(1+3w_s)\delta\epsilon_s + (1+3w_i)\delta\epsilon_i = R\left(\ddot{g}_2 + \frac{\dot{A}_0}{A_0}\dot{g}_2\right), \quad (3.6)$$

$$(1-w_s)\delta\epsilon_s + (1-w_i)\delta\epsilon_i = R\left[\left(\dot{g}_2 - \frac{1}{2}\dot{f}_2\right)\frac{\dot{A}_0}{A_0} + \frac{1}{3}(\ddot{g}_2 - \ddot{f}_2)\right], \quad (3.7)$$

$$\ddot{f}_2 + \frac{3}{2}\frac{\dot{A}_0}{A_0}\dot{f}_2 + 2A_0^{-1} = 0. \quad (3.8)$$

The last equation is the same as the first one in Eq. (2.16).

Here we assume for simplicity that the perturbations of the inflationary fluid are negligibly small and tend to decay, because of their decaying property which was shown by Comer *et al.* [12] in the single fluid case. Then we can neglect $|\delta\epsilon_i|$ and $|\delta u_{(i)_i}|$, comparing with $|\delta\epsilon_s|$ and $|\delta u_{(s)_i}|$. From two equations (3.6) and (3.7) we obtain

$$\ddot{g}_2 + \tfrac{3}{2}(1+w_s)\frac{\dot{A}_0}{A_0}\dot{g}_2 + \tfrac{1}{2}(1+3w_s)A_0^{-1} = 0, \quad (3.9)$$

which is equal to the second one in Eq. (2.16). Their solutions are therefore expressed as in Eqs. (2.17) and (2.18). When we put $H \equiv (1+w_s)[\frac{3}{4}\epsilon_s(0)]^{1/2}[\epsilon_s(0)/\epsilon_i(0)]^{(1+w_s)/[2(w_i-w_s)]}$, their behaviors of main special components are as follows.

(a) For $A_0 \ll A_{0c}$ or $Ht \ll 1$,

$$f_2 \propto g_2 \propto \delta\epsilon_s/\epsilon \propto t^{2(1+3w_s)/[3(1+w_s)]}. \quad (3.10)$$

All of them ($f_2, g_2$ and $\delta\epsilon_s/\epsilon$) grow with time.

(b) For $A_0 \gg A_{0c}$ or $Ht \gg 1$ ,

$$f_2 \propto g_2 \propto \delta\epsilon_s/\epsilon \propto t^{2(1+3w_i)/[3(1+w_i)]}. \quad (3.11)$$

All of them ($f_2, g_2$ and $\delta\epsilon_s/\epsilon_i$) decay with time. Though $\delta\epsilon_s/\epsilon_s(\propto t^{2(1+3w_s)/[3(1+w_i)]})$ increases with time, this does not mean any physical growth. The third-order velocities ${}^{(3)}u_i$ vanish in the same way as in the preceding section, and the behavior of ${}^{(3)}\epsilon_s/\epsilon = \delta\epsilon/\epsilon$ is also similar to that in Fig. 1.

The $(2n+1)$th higher-order terms are found to be in proportion to $t^{2n(1+3w_s)/[3(1+w_s)]}$ or $t^{2n(1+3w_i)/[3(1+w_i)]}$ for $Ht \ll 1$ or $\gg 1$, respectively, due to the iterative derivation of higher-order terms. This means that the behaviors of third-order terms in the expansion are common with the full nonlinear behaviors, as long as the expansion analysis is valid. As for gravitational collapse and the collision of black holes the situation is similar to the case $w_i = -1$.

## IV. STANDARD FLUID PLUS A SCALAR FIELD

So far we have considered the case when inflationary matter is a fluid. Next let us treat a case when it is a scalar field $\phi$. Then the Einstein equations in the synchronous gauge (2.1)–(2.4) are replaced by

$$\tfrac{1}{2}\dot{\kappa}^i_i + \tfrac{1}{4}\kappa^j_i\kappa^i_j = \rho(1 - 3\Gamma/2 - \Gamma u_k u^k) + (-\dot{\phi}^2 + V), \quad (4.1)$$

$$(1/2)(\kappa^j_{j;i} - \kappa^j_{i;j}) = \rho\Gamma u_i(1+u_k u^k)^{1/2} - \dot{\phi}\partial_i\phi, \quad (4.2)$$

$$(1/2)\left[2P^j_i + \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}\kappa^j_i)^{\cdot}\right] = \rho[\Gamma u^i u_j + \delta^i_j(1-\Gamma/2)] + (\partial_i\phi\partial^i\phi + V\delta^j_i), \quad (4.3)$$

where $V(\phi)$ is the potential for $\phi$.

In the first-order of the long-wavelength iteration scheme, one ignores $P^i_j, u^i u_j$ and $\partial_i\phi\partial^i\phi$ in Eqs. (4.1)–(4.3). The quasi-isotropic solution of the traceless part of Eq. (4.3) then is $\gamma_{ij} = A(t)h_{ij}$, where $A = \gamma^{1/3}$ and where the integration "constants" $h_{ij}(x)$ can be seen as a "seed" metric. Equation (4.1) and the trace of Eq. (4.3) then yield

$$2\dot{H} + \dot{\phi}^2 = -\Gamma\rho, \quad (4.4)$$

$$3H^2 - (\tfrac{1}{2}\dot{\phi}^2 + V) = \rho, \quad (4.5)$$

where $H \equiv (1/2)\dot{A}/A$. Assuming that the scalar field and the fluid do not interact, that is, that their stress-energy tensors are separately conserved, we obtain the additional equation

$$\dot{\rho} + \tfrac{3}{2}\Gamma\rho\frac{\dot{A}}{A} = 0 \quad \text{or} \quad \rho = \bar{\rho}A^{3\Gamma/2}, \quad (4.6)$$

where $\bar{\rho}(x)$ is an integration "constant." The conservation of $T_{\mu\nu}(\phi)$ yields the Klein-Gordon equation which is redundant, being a consequence of (4.4)–(4.6) through the Bianchi identities.

A detailed analysis of the evolution of $A(t), \phi(t)$ and $\rho(t)$ requires a numerical integration of Eqs. (4.4)–(4.6). The qualitative behavior of the solution however is easily obtained. Indeed the scalar field [for, say, $V \propto \phi^n$] behaves first like a stiff fluid ($\Gamma^{\rm eff} = 2$) near the big bang, then like a cosmological constant ($\Gamma^{\rm eff} = 0$) during the inflationary period, and then finally like dust ($\Gamma^{\rm eff} = 1$) at the end of inflation (see, e.g., Refs. [14,15]). Its energy density therefore evolves in the form (4.6) with an index $\Gamma^{\rm eff} = 2, 0$ or 1, depending on the period considered. The energy of the standard fluid, on the other hand, is given by Eq. (4.6) with a fixed index $2/3 < \Gamma < 2$. Therefore we can have the following scenario for the evolution of the scale factor: first the scalar field dominates ($\Gamma^{\rm eff} = 2$) and $A(t) \propto t^{2/3}$; then the standard fluid ($2/3 < \Gamma < 2$) dominates, so that $A(t) \propto t^{4/3\Gamma}$; then comes the inflationary stage ($\Gamma^{\rm eff} = 0$) and $A(t) \propto e^{Ht}$; finally, at the end of inflation, either the scalar field dominates [$A(t) \propto t^{4/3}$] or the perfect fluid dominates [$A(t) \propto t^{4/3\Gamma}$], depending on whether or not $\Gamma < 1$.

Let us now look at the third-order solution of Einstein's equations (4.1)–(4.3) (first iteration). The metric, the fluid density, its three-velocity, and the scalar field are of the form $\gamma_{ij} = A(t)[h_{ij} + a_2(t)Rh_{ij} + b_2(t)R_{ij}]$, $\rho = \rho_0(t) + \rho_2(t)R$, $u_i = u_3(t)\nabla_i R$, and $\phi = \phi_0(t) + \phi_2(t)R$, where $A(t), \rho_0(t), \phi_0(t)$ are the solutions of equations (4.4)–(4.6) previously described, where $R_{ij}$ is the Ricci tensor of the seed metric $h_{ij}$, $\nabla$ the corresponding covariant derivative, and where $a_2, b_2, u_3$, and $\phi_2$ satisfy the differential equations (see Ref. [10] for details):

$$\ddot{b}_2 + 3H\dot{b}_2 + 2/A = 0, \tag{4.7}$$

$$\ddot{a}_2 + H(6\dot{a}_2 + \dot{b}_2) = \rho_2(2-\Gamma) + 2\phi_2 V_0', \tag{4.8}$$

$$3\ddot{a}_2 + \ddot{b}_2 + 2H(3\dot{a}_2 + \dot{b}_2) = \rho_2(2-3\Gamma) + 2(-2\dot{\phi}_0\dot{\phi}_2 + \phi_2 V_0'), \tag{4.9}$$

$$\tfrac{1}{4}\dot{b}_2 + \dot{a}_2 = \rho_0\Gamma u_3 - \dot{\phi}_0\phi_2, \tag{4.10}$$

$$\ddot{\phi}_2 + 3H\dot{\phi}_2 + \tfrac{1}{2}(3\dot{a}_2 + \dot{b}_2)\dot{\phi}_0 + \phi_2 V_0'' = 0, \tag{4.11}$$

where $V' = dV/d\phi, V'' = d^2V/d\phi^2$. Equations (4.7) and (4.8) come from the traceless and trace parts of Eq. (4.3), Eqs. (4.9) and (4.10) come from Eqs. (4.1) and (4.2), respectively, and Eq. (4.11) is the third-order part of the Klein-Gordon equation for $\phi$.

A detailed analysis of the evolution of this inhomogeneous approximate solution of Einstein's equations requires a numerical integration of Eqs. (4.7)–(4.11). However, since in the first-order the scalar field behaves like a perfect fluid, we can extend the results of the preceding section to infer, qualitatively, what happens. The effective adiabatic index will evolve from 2 (stiff fluid), to $\Gamma_s$ (standard fluid), then to $\Gamma_i$ (inflationary matter), and finally to 1 or $\Gamma_s$ (standard fluid again). The inhomogeneities will therefore grow according to Eq. (3.10) before the inflationary stage, then will be wiped out during inflation following Eq. (3.11), and finally will grow again according to Eq. (3.10) after inflation. The inclusion of higher-order terms will not change the conclusion, which thus extends the cosmic no hair theorem to the nonlinear regime.

## V. COMPARISON WITH GAUGE-INVARIANT LINEARIZED PERTURBATIONS

If the inhomogeneities are regarded as small perturbations to the Friedmann universe, they can be analyzed as the linear perturbations. According to the gauge-invariant linearized perturbation theory [2], the density contrast for the total density, $\Delta(=\Delta_s + \Delta_i)$ is described by the second-order differential equation

$$\ddot{\Delta} + 2(\dot{a}/a)\left\{1 + \tfrac{3}{2}\left(\epsilon\frac{dq}{d\epsilon} - q\right)\right\}\dot{\Delta} + \left[\left(\frac{n}{a}\right)^2\frac{\delta p}{\delta\epsilon} - \tfrac{3}{2}(\dot{a}/a)^2\left(1 + 2q - 3q^2 - 6\epsilon\frac{dq}{d\epsilon}\right)\right]\Delta = 0, \tag{5.1}$$

where $a$ is the scale factor,

$$q \equiv p/\epsilon = \frac{w_s\epsilon_s + w_i\epsilon_i}{\epsilon_s + \epsilon_i} = \frac{w_s + w_i x}{1+x}, \tag{5.2}$$

$x \equiv \epsilon_i/\epsilon_s$, and $n$ is the wave number.

From the definition we have

$$\epsilon\frac{dq}{d\epsilon} = (w_s - w_i)^2 x/\{[1 + w_s + (1+w_i)x](1+x)\}. \tag{5.3}$$

For $x \ll 1$, Eq. (5.1) for $\Delta(\simeq \Delta_s)$ reduces to the equation for the density contrast $\Delta_s$ in a standard fluid and gives a growing solution as one of two independent solutions, when we exclude the term with $(n/a)^2$, which can be neglected in our superhorizon situation. The density contrast in this growing solution is equal to $\delta\epsilon_s/\epsilon$ [Eqs. (2.23) and (3.10)] in case (a) in Secs. II and III.

For $x \gg 1$, we obtain, in the case $w_i = -1$,

$$\ddot{\Delta} + (8 + 3w_s)\tilde{H}[1 + O(1/x)]\dot{\Delta} + 3(5 + 3w_s)\tilde{H}^2[1 + O(1/x)]\Delta = 0 \tag{5.4}$$

with $a \propto e^{\tilde{H}t}$ and $\tilde{H} \equiv 2H/[3(1+w_s)]$, and the solutions are

$$\Delta = e^{-\{2+4/[3(1+w_s)]\}Ht}, \quad e^{-2Ht/(1+w_s)}. \tag{5.5}$$

The first solution of these two is equal to a special solution [Eq. (2.25)] in case (b), Sec. II. In the same way, for $x \gg 1$ and $-1 < w_i < -1/3$, we obtain

$$\ddot{\Delta} + \frac{2(2-3w_i)}{3(1+w_i)t}[1 + O(1/x)]\dot{\Delta} - \frac{2(1 + 2w_i - 3w_i^2)}{3(1+w_i)^2 t^2}[1 + O(1/x)]\Delta = 0, \tag{5.6}$$

and their solutions are

$$\Delta = t^{2(1+3w_i)/[3(1+w_i)]}, \quad t^{-(1-w_i)/(1+w_i)}. \tag{5.7}$$

The first solution is consistent with a special solution [Eq. (3.11)] in case (b), Sec. III.

Thus there is a consistency between special solutions for the third-order inhomogeneities and those for the gauge-invariant density perturbations. This can be explained as coming from the consistency of their definitions in the sense that the comoving condition is commonly used in both of them (cf. ${}^{(3)}u_i = 0$ described earlier). Of course all their solutions are not equivalent because of the use of the synchronous coordinate condition in the former treatment.

## VI. CONCLUSION AND DISCUSSIONS

After the Universe starts with general anisotropic and inhomogeneous states, it gradually approaches the quasi-isotropic state through an inflationary stage and on the one hand the local inhomogeneities evolve further dynamically further, depending on the equations of state in constituent matter. In this paper we studied the behaviors of inhomogeneities consisting of a standard fluid and inflationary matter in the anti-Newtonian or long-wavelength approximation, and showed that the local dominance of the standard fluid causes their growth, but finally they always decay during the stage when the inflationary matter is dominant. However, the spatial curvature $P$ associated with the metric $h_{ij}$ remains spatially irregular. After the inflation and reheating, the inhomogeneities are caused, and they grow again due to local spatial curvature.

In Sec. III we derive the velocity components by use of Eq. (2.3), assuming that the perturbations in the inflationary fluid are negligibly small compared with those in the standard fluid. In general, however, we must use not only Eq. (2.3) but also the energy-momentum conservation law $T^{\mu\nu}_{(a);\nu} = V^{\mu}_{(a)}$ to determine each velocity component, where $V^{\mu}_{(a)}$ is the term due to the energy-momentum transfer satisfying $V^{\mu}_{(s)} + V^{\mu}_{(i)} = 0$ and we have $V^{\mu}_{(s)} = V^{\mu}_{(i)} = 0$ for the adiabatic state.

As long as the inhomogeneities satisfy our present approximation, this conclusion is general and has the attractor property, because of the generality of the initial condition. The number of arbitrary functions of spatial variables was examined by Comer *et al.* [10].

Finally, we emphasize superior points of the theories in the anti-Newtonian or long-wavelength approximation, by noting that they can treat the nonlinear perturbations to arbitrary orders, in contrast with the linear perturbation theory, and that their applications to various physical problems have been exploited also by employing the Hamilton-Jacobi method beginning with an action principle [8].

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