

Density perturbations of quantum-mechanical origin and anisotropy of the microwave background

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If the large-angular-scale anisotropy in the cosmic microwave background radiation is caused by the long-wavelength cosmological perturbations of quantum mechanical origin, they are, most likely, gravitational waves, rather than density perturbations or rotational perturbations.

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I. INTRODUCTION

The ongoing and planned high-precision measurements of the anisotropy in the cosmic microwave background radiation (CMBR) [1] may have serious impact on our views about the very early Universe. The large-angular-scale anisotropy in the CMBR is most likely caused by cosmological perturbations with wavelengths of the order and longer than the present-day Hubble radius l_H . It is reasonable to expect that such longer-wavelength perturbations are "primordial" and have survived from the epochs when the Universe was much younger. The wavelengths of the perturbations have enormously grown up since the time of generation but other physical characteristics of the perturbations can still carry imprints of their origin. This cannot be said with the same degree of certainty about the relatively short-wavelength cosmological perturbations (unless they are gravitational waves) which could have been distorted and contaminated in the course of their life by many physical processes occurring in the Universe.

It is remarkable that the origin of all three possible types of cosmological perturbations, that is, the origin of density perturbations, rotational perturbations, and gravitational waves, may be of purely quantum-mechanical nature. Cosmological perturbations can be treated as excitations in a gravitational field. In the case of gravitational waves, they are just excitations in the gravitational field itself. In the case of density perturbations and rotational perturbations, they are excitations in a gravitational field which accompany excitations in matter. In the very distant past, the density and rotational perturbations were excitations in the primeval medium that was filling the Universe at that time. The quantum-mechanical generation mechanism of cosmological perturbations relies only upon the existence of their zero-point quantum fluctuations and the nonvanishing parametric coupling of the perturbations to the variable gravitational field of the homogeneous isotropic Universe. The strong variable gravitational field of the very early Universe played the role of the pump field. It supplied energy to the zero-point quantum fluctuations and amplified them. More precisely, the initial vac-

uum quantum state of each mode of the perturbations has been transformed, as a result of the quantum mechanical Schrödinger evolution, into a multiparticle state known as a squeezed vacuum quantum state. The generated perturbations have formed a collection of standing waves. The gravitational field of each of the three types of quantum mechanically generated perturbations can affect the propagating photons of the CMBR and produce anisotropy in the CMBR.

It was already emphasized [2] that there is a significant qualitative difference between gravitational waves on one side and density and rotational perturbations on the other side, with regard to the possibility of their quantum mechanical generation. Gravitational waves oscillate in the absence of external gravitational fields, and their appropriate parametric coupling to the pump field follows directly from the Einstein equations. The parametric excitation vanishes only if the cosmological scale factor obeys the equation $a''/a = 0$, that is, when there is no pump field at all, $a(\eta) = \text{const}$, or when the coupling a' is time independent. Up to this exception, one can say that the quantum mechanical generation of gravitational waves (relic gravitons) is unavoidable [3]. As for density and rotational perturbations, they are perturbations in matter being accompanied by perturbations of gravitational field. The ability to support oscillations of density and/or rotation and the form of their coupling to the pump field depend on a particular model of matter and its energy-momentum tensor. The very possibility of the quantum mechanical generation of these perturbations is model dependent. Recalling Einstein's definition of two pillars supporting general relativity, one can say that the quantum mechanically generated gravitational waves are associated with the pillar made of marble, while density and rotational perturbations are associated with the other one.

A particular sort of matter that has received much attention in the recent cosmological literature, especially in the literature on inflation [4], is one or another version of a scalar field. The scalar fields is a nice theoretical model that has been used in physics in many different studies. Whether the global scalar fields do really exist in nature and, if so, whether they couple to gravity in the way

we want, is presently unknown. However, we will follow the modern tradition in theoretical physics which states that everything that is not forbidden is allowed. One can at least guarantee that a sort of inflationary expansion is a typical feature (attracting separatrix) in the space of homogeneous isotropic solutions to the Einstein equations with certain scalar fields [5]. Scalar fields cannot support rotational perturbations but they can support density perturbations.

Specifically, we will study a scalar field $\varphi(t, x^1, x^2, x^3)$ with the energy-momentum tensor

$$T_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} + V(\varphi)\right], \quad (1)$$

where $V(\varphi)$ is an arbitrary scalar field potential and a comma denotes a partial derivative. This model of the primeval cosmological medium satisfies both conditions for the quantum mechanical generation of density perturbations be possible. First, the field can obviously support free oscillations in Minkowski space-time. Second, the explicit form of the energy-momentum tensor (1) reflects the appropriate (minimal, the same as for gravitational waves) coupling of the scalar field to gravity which was chosen by our will. So, on general grounds and by analogy with gravitational waves, one can expect that some amount of density perturbations might have been generated by strong variable gravitational field of the early Universe. The problem is to quantify this expectation and to derive the observational predictions, as reliable and detailed as possible, including the expected variations in the CMBR.

Scalar fields and scalar field perturbations is a very popular subject in the framework of inflationary cosmologies. So popular, that many believe that the inflationary type of expansion is conditioned by the existence of scalar fields and that the very possibility to generate perturbations quantum mechanically relies on the existence of the de Sitter event horizon. This is not so. Inflation, if understood as a statement about the behavior of the time-dependent cosmological scale factor, and not about creating and resolving the particle physics paradoxes, is a phenomenon more general than one particular realization of it with the help of a scalar field. (The attitude toward the archetype inflationary solution, exponential expansion, has changed over the years. Astronomers of the older generation were embarrassed with the de Sitter solution but tried to apply it for the explanation of the galaxies' redshifts and statistics of quasars in the most recent Universe. Cosmologists of our time take the exponential expansion as something almost proven but apply it to the very remote stages of evolution, somewhere near the Planck time.) And the quantum mechanical generation of perturbations is a phenomenon more general and universal than such concepts as global scalar fields, event horizons, and inflation. If it turns out that the inflationary hypothesis contradicts observations, the quantum mechanical generating mechanism will not die together with inflation. There is little doubt, for instance, that the search for relic gravitational waves will continue, with maybe larger emphasis on relatively short waves rather than on long waves [6]. And a test of the quantum mechanical origin of cosmological perturbations will be a

test of their origin, not a test of inflation specifically.

The generation of density perturbations in inflationary models governed by the scalar field (1) was a subject of discussion in many research and review articles, and books. If one consults the most recent literature, one can find that the current situation is often summarized in the following, or similar, words: "Exponential inflation predicts a scale-invariant, Gaussian spectrum of scalar fluctuations . . . , and a smaller amount of tensor fluctuations Other inflationary models, for instance power-law inflation . . . , predict spectra slightly tilted away from scale invariance." (See, for example, [7] and references therein.) The expected amplitudes of density perturbations are usually quoted in the following, or equivalent, form (see, for instance, [8] and references therein):

$$\left(\frac{\delta\rho}{\rho}\right)_{\lambda}^{\text{hor}} = \frac{m\kappa^2}{8\pi^{3/2}} \frac{H^2(\varphi)}{|H'(\varphi)|},$$

where the quantities on the right-hand side are supposed to be evaluated "when the scale λ crossed the Hubble radius during inflation." The denominator of this expression depends on the derivative of the Hubble parameter and goes to zero in the limit of exponential inflation. Apparently, this formula says that the predicted amplitudes of the scale-invariant spectrum are arbitrarily close to infinity, and the amplitudes of nearby spectra are "slightly tilted away" from infinity. According to this formula, the amplitudes of density perturbations are many orders of magnitude larger than the amplitudes of gravitational waves, if the expansion rate is sufficiently close to the exponential inflation. The belief that the amplitudes of density perturbations are larger, or much larger, than the amplitudes of gravitational waves is considered to be a strong prediction of inflationary models based on the scalar field (1). For instance, the author of Ref. [8] concludes "An observation violating this condition at any scale would immediately rule out the general class of models we are considering." The expected contribution of density perturbations and gravitational waves to the quadrupole anisotropy of the CMBR was also under study. The authors of Ref. [9] (see also [10] and references therein) say that "The ratio of gravitational wave (T) to energy-density perturbations (S) contributions to the CMB quadrupole anisotropy is predicted to be $T/S = 21(1 + \gamma)$," where γ , in that paper, is the parameter in the equation of state for matter governing the inflationary expansion, $p = \gamma\rho$. According to this formula, T/S vanishes in the limit of $\gamma = -1$, that is in the limit of strictly exponential (de Sitter) inflation. Apparently, this formula for T/S is based on the author's assumption that the effectiveness of generation of density perturbations is the higher the closer the expansion law to the exponential inflation, and goes to infinity in the limit of $\gamma = -1$. Apparently, this is why T/S goes to zero in this limit. The statements about density perturbations are sometimes characterized as such that have been "widely studied and there is broad agreement regarding both methods and results . . ." [11].

I suspect that this paper will not belong to that category of studies that enjoyed the "broad agreement"; my

conclusions are considerably different from what was described in the preceding paragraph. I will be arguing that there is no linear density perturbations at all at the purely exponential (de Sitter) inflationary stage for models governed by the scalar field (1). Density perturbations can only arise as a result of violation of the purely exponential expansion and transition to the radiation-dominated stage. Regardless of how close to zero was the derivative of the Hubble parameter "when the scale λ crossed the Hubble radius during inflation," the today's amplitudes of density perturbations are finite. The amplitudes of gravitational waves are typically a little larger than the amplitudes of density perturbations, at least in the long wavelength limit where spectra are smooth and have the power-law behavior. Correspondingly, the contribution of density perturbations to the quadrupole anisotropy is never much larger than the contribution of gravitational waves. In fact, it is somewhat smaller in the limit of long waves.

As for the statistical properties of cosmological perturbations and, hence, the statistical properties of the CMBR fluctuations caused by them, it was already emphasized [12] that they are determined by the statistics of quantum states being generated, namely, by the statistics of squeezed vacuum quantum states.

Since the conclusions of this paper are in disagreement with other publications, we will present detailed derivations (which could have been omitted otherwise) in order to make it possible for the interested reader to compare the present calculations with those of other authors. The structure of the paper is the following. In Sec. II we present the general equations for density perturbations. The equations are applicable for matter with the energy-momentum tensor of arbitrary form. They use only the defining property of density perturbations, namely, that the perturbations are based on the scalar functions of spatial coordinates. In Sec. III we apply these equations specifically to the initial stage (i stage) of cosmological evolution which is assumed to be governed by the scalar field with the energy-momentum tensor (1). No assumptions about a particular form of the scalar field potential $V(\varphi)$ or a particular (for example, inflationary) type of expansion are being made *a priori*. The time-dependent coefficients of the differential equations for the perturbations are expressed in terms of the scale factor (and its derivatives) only. This reflects the underlying interaction of the perturbations with the variable gravitational pump field. The determination of all unknown functions describing density perturbations is reduced to solving a single differential equation which is very similar to the equation for gravitational waves. The behavior of solutions during a more or less gradual transition from the i stage to the radiation-dominated stage (e stage) is studied in Sec. IV. In order to deal with simple exact solutions at both stages we will be interested in a sharp transition from the i stage to the e stage. In Sec. V we apply the perturbation equations to the perfect fluid matter with arbitrary velocity of sound. We present solutions to these equations at the e stage and the matter-dominated stage (m stage) in the form convenient for matching the solutions at all three stages

(i, e, m). In Sec. VI we join the solutions, find the coefficients which were undetermined so far, and express the solution at the m stage entirely in terms of the functions (and their first time derivatives) describing the perturbations at the time of joining the i and e stages. As a preparation for quantization of the perturbations, we briefly discuss the density and rotational perturbations of matter placed in the Minkowski space-time, that is neglecting gravitational fields (Sec. VII). The quantization of density perturbations is performed in Sec. VIII. This procedure essentially repeats the steps which have been previously done for gravitational waves and rotational perturbations [12,2]. The quantum mechanically generated perturbations are placed in squeezed vacuum quantum states. Classically, one can think of the perturbations as of a stochastic collection of standing waves. The justification and necessity of the so-called Sakharov oscillations in the spectra of density perturbations is discussed. In order to get the analytic results as detailed as possible, we specialize the scale factor of the i stage to the η -time power laws which include inflationary models. This allows us to derive concrete power-law spectra of density perturbations at the m stage. In Sec. IX we derive an exact formula for the angular correlation function of the CMBR temperature variations $\delta T/T$ caused by squeezed density perturbations. The multipole decomposition of the correlation function begins from the monopole term. The contributions to the monopole and dipole terms produced by individual waves with wavelengths exceeding l_H are strongly suppressed, which is in agreement with previous results [13]. Nevertheless, one should be aware that not only the entire quadrupole but also some little portions of the measured mean temperature of the CMBR and its dipole variation may be caused by perturbations of quantum mechanical origin. For one and the same cosmological model, the contribution of density perturbations to the quadrupole anisotropy of the CMBR is never much larger than the contribution of gravitational waves, at least for models considered here. (In Ref. [28] one can find more details about the origin of disagreements with the previously published papers.)

II. GENERAL EQUATIONS FOR DENSITY PERTURBATIONS

The unperturbed spatially flat FLRW (Friedmann-Lemaître-Robertson-Walker) cosmological models are described by the metric

$$ds^2 = -c^2 dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \\ = -a^2(\eta)[d\eta^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2]. \quad (2)$$

The scale factor $a(\eta)$ is governed by matter with the unperturbed values of energy density ϵ_0 , $T_0^0 = -\epsilon_0$, and pressure p_0 , $T_i^k = p_0 \delta_i^k$:

$$\frac{3}{a^2} \left(\frac{a'}{a} \right)^2 = \kappa \epsilon_0, \\ -\frac{1}{a^2} \left[2 \left(\frac{a'}{a} \right)' + \left(\frac{a'}{a} \right)^2 \right] = \kappa p_0, \quad (3)$$

where $\kappa = 8\pi G/c^4$ and a prime is $d/d\eta$, $d/d\eta = (a/c)d/dt$. The Hubble parameter is $H = \dot{a}/a = ca'/a^2$.

It is convenient to introduce two new functions of the scale factor:

$$\alpha(\eta) = \frac{a'}{a}, \quad \gamma(\eta) = 1 - \frac{\alpha'}{\alpha^2}. \quad (4)$$

In terms of t time the function γ is $\gamma(t) = -(\dot{H}/H^2)$. Because of Eqs. (3) one has

$$\kappa(\epsilon_0 + p_0) = \frac{2\alpha^2}{a^2} \gamma. \quad (5)$$

The function $\gamma(\eta)$ becomes a constant if $a(\eta)$ is governed by matter with the effective equation of state $p_0 = q\epsilon_0$, where $q = \text{const}$. The scale factor $a(\eta)$ takes on the η -time power-law behavior

$$a = l_0 |\eta|^{1+\beta} \quad (6)$$

(η must be negative for expanding models with $1+\beta < 0$) where l_0 and β are constants. The constant l_0 has the dimensionality of length. It follows from Eqs. (3) and (4) that $\gamma = (2+\beta)/(1+\beta)$, $q(\beta) = (1-\beta)/3(1+\beta)$, where the parameter β can vary in the interval $-\infty < \beta < \infty$. In particular, $\gamma = 2$ at the radiation-dominated stage, and $\gamma = 3/2$ at the matter-dominated stage. Note that $\gamma = 0$ in case of purely exponential (de Sitter) expansion for which $a(t) \sim e^{Ht}$, $H = \text{const}$, $a(\eta) = l_0 |\eta|^{-1}$, $\beta = -2$.

One can also derive from Eqs. (3) the relationship

$$\frac{p'_0}{\epsilon'_0} = -1 + \frac{2}{3} \gamma - \frac{\gamma'}{3\alpha\gamma} = -\frac{1}{3\alpha} (\ln a\alpha^2\gamma)'. \quad (7)$$

The ratio p'_0/ϵ'_0 becomes a constant for the scale factors (6), namely, $p'_0/\epsilon'_0 = q(\beta)$. In particular, p'_0/ϵ'_0 goes to -1 in the limit of $\beta = -2$.

The construction of density perturbations [14,15] (see also [16]) is based on the scalar functions $Q(x^1, x^2, x^3)$ satisfying the equation

$$Q_{,i}^i + n^2 Q = 0 \quad (8)$$

valid in three-space $dl^2 = dx^{1^2} + dx^{2^2} + dx^{3^2}$. For each wave vector \mathbf{n} , one can choose two linearly independent solutions to Eq. (8) in the form $e^{i\mathbf{n}\cdot\mathbf{x}}$ and $e^{-i\mathbf{n}\cdot\mathbf{x}}$. From a given scalar field Q one can construct a vector field $Q_{,i}$ and two tensor fields $\delta_{ik}Q$ and $Q_{,i,k} = -n_i n_k Q$. For each \mathbf{n} , the general perturbation of the energy-momentum tensor and the accompanying perturbation of the gravitational field can be written as a sum of products of time-dependent amplitudes and spatial functions introduced above.

Without restricting in any way the physical content of the problem, it is convenient to work in the class of synchronous coordinate systems. (At this point the reader may have to be ready to exhibit certain resistance to the pressure from the proponents of the "gauge-invariant" formalisms.) Using the η -time coordinate, one can write the general expression for the metric tensor including perturbations as

$$\begin{aligned} ds^2 &= -a^2 [d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j], \\ g_{00} &= -a^2, \quad g_{0i} = 0, \\ g_{ij} &= a^2 [(1 + hQ)\delta_{ij} + h_i n^{-2} Q_{,i,j}]. \end{aligned} \quad (9)$$

The function $h(\eta)$ represents the scalar (proportional to $\delta_{ij}Q$) perturbation of the gravitational field while the function $h_i(\eta)$ represents the longitudinal-longitudinal (proportional to $n_i n_j Q$) perturbation. The general expression for T_μ^ν including perturbations can be written as

$$\begin{aligned} T_0^0 &= -\epsilon_0 - \frac{1}{a^2} \epsilon_1 Q, \quad T_i^0 = \frac{1}{a^2} \xi' Q_{,i}, \quad T_0^i = -\frac{1}{a^2} \xi' Q^{,i}, \\ T_i^k &= p_0 \delta_i^k + \frac{1}{a^2} (p_1 + p_l) Q \delta_i^k + \frac{1}{a^2} n^{-2} p_l Q_{,i}^k. \end{aligned} \quad (10)$$

The form of Eqs. (9) and (10) is based solely on the definition of density perturbations and our choice of synchronous coordinate systems. In all other respects, the representation (9) and (10) is general. The particular notations for arbitrary functions describing the perturbations are chosen for later convenience.

The arbitrary functions $h(\eta)$, $h_i(\eta)$, $\epsilon_1(\eta)$, $p_1(\eta)$, $p_l(\eta)$, $\xi'(\eta)$ should satisfy altogether the perturbed Einstein equations

$$3\alpha h' + n^2 h - \alpha h_i' = \kappa \epsilon_1, \quad (11)$$

$$h' = \kappa \xi', \quad (12)$$

$$-h'' - 2\alpha h' = \kappa p_1, \quad (13)$$

$$\frac{1}{2}(h_i'' + 2\alpha h_i' - n^2 h) = \kappa p_l. \quad (14)$$

There are too many unknown functions to be found from Eqs. (11)–(14). This requires us to specify a model for matter and its energy-momentum tensor. We will consider three consecutive stages of expansion: i stage governed by the scalar field (1), and the subsequent e and m stages governed by perfect fluid with the energy-momentum tensor

$$T_\mu^\nu = (\epsilon + p)u_\mu u^\nu + p\delta_\mu^\nu. \quad (15)$$

The equation of state at the e and m stages is $p = \frac{1}{3}\epsilon$ and $p = 0$, respectively.

III. DENSITY PERTURBATIONS AT THE INITIAL STAGE OF EXPANSION GOVERNED BY A SCALAR FIELD

At the i stage, the evolution of the scale factor $a(\eta)$ is determined by the unperturbed homogeneous scalar field $\varphi = \varphi_0(\eta)$. The unperturbed values ϵ_0 , p_0 are given by Eq. (1):

$$\epsilon_0 = \frac{1}{2a^2} (\varphi'_0)^2 + V(\varphi), \quad (16)$$

$$p_0 = \frac{1}{2a^2} (\varphi'_0)^2 - V(\varphi). \quad (17)$$

By summing up Eqs. (16) and (17) and comparing the result with Eq. (5) one can derive the equation

$$\kappa(\varphi'_0)^2 = 2\alpha^2 \gamma. \quad (18)$$

It follows from this equation that $\gamma \geq 0$ for the scale factors governed by the scalar field (1). The de Sitter case corresponds to $\varphi'_0 = 0$, $\varphi_0 = \text{const}$ and $\epsilon_0 = -p_0 = V(\varphi_0) = \text{const}$.

If $\varphi'_0 \neq 0$, one can use the equation

$$\epsilon'_0 = -3\alpha(\epsilon_0 + p_0)$$

which is a consequence of Eq. (3), and obtain, with the help of Eqs. (16) and (17),

$$\varphi''_0 + 2\alpha\varphi'_0 + a^2V_{,\varphi} = 0, \quad (19)$$

where $V_{,\varphi} = dV(\varphi)/d\varphi$, the derivative is taken at $\varphi = \varphi_0$. The further useful relationships following from Eqs. (18) and (19) are

$$\frac{\varphi''_0}{\varphi'_0} = \frac{\alpha'}{\alpha} + \frac{1}{2} \frac{\gamma'}{\gamma}, \quad (20)$$

$$-\frac{a^2}{\varphi'_0} V_{,\varphi} = 2\alpha + \frac{\alpha'}{\alpha} + \frac{1}{2} \frac{\gamma'}{\gamma}. \quad (21)$$

The perturbations of the gravitational field are associated with the perturbations of the scalar field. We will write the perturbations of the scalar field as

$$\varphi = \varphi_0(\eta) + \varphi_1(\eta)Q. \quad (22)$$

Having at our disposition the energy-momentum tensor (1) and the definitions (22), (9), (10) we can directly calculate the functions ϵ_1 , ξ' , p_1 , p_l :

$$\epsilon_1 = \varphi'_0\varphi'_1 + a^2\varphi_1V_{,\varphi}, \quad (23)$$

$$\xi' = -\varphi'_0\varphi_1, \quad (24)$$

$$p_1 = \varphi'_0\varphi'_1 - a^2\varphi_1V_{,\varphi}, \quad (25)$$

$$p_l = 0. \quad (26)$$

We will now assume that $\varphi'_0 \neq 0$. The de Sitter case $\varphi'_0 = 0$ will be considered separately at the end of this section. It follows from Eq. (24) that $\varphi_1 = -\xi'/\varphi'_0$. Inserting this value of φ_1 into Eqs. (23) and (25) and using Eqs. (20) and (21), one can express ϵ_1 , p_1 , in terms of $h(\eta)$:

$$\kappa\epsilon_1 = -h'' + h' \left(2\alpha + 2\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} \right), \quad (27)$$

$$\kappa p_1 = -h'' - 2\alpha h'. \quad (28)$$

We should now return to the perturbed Einstein equations (11)–(14) making use of Eqs. (27) and (28). Equation (11) can be written as an expression for $h'_l(\eta)$ in terms of $h(\eta)$:

$$h'_l = \frac{1}{\alpha} \left[h'' + h' \left(\alpha - 2\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} \right) + n^2 h \right]. \quad (29)$$

Equation (12) expresses ξ' in terms of h' . Equation (13) is satisfied identically. Equation (14) reads as

$$h''_l + 2\alpha h'_l - n^2 h = 0. \quad (30)$$

Thus, if one knows the function $h(\eta)$, all other functions describing the density perturbations, namely, $h_l(\eta)$, $p_1(\eta)$, $\epsilon_1(\eta)$, and $\xi'(\eta)$, can be found with the help of Eqs. (29), (28), (27) [or, equivalently, (11)], and (12). To derive the equation for $h(\eta)$ we substitute Eq. (29) into Eq. (30) and obtain

$$h''' + h'' \left(3\alpha\gamma - \frac{\gamma'}{\gamma} \right) + h' \left[n^2 - 2\alpha' + 2\gamma\alpha^2 - \frac{\alpha'}{\alpha} \frac{\gamma'}{\gamma} - \left(\frac{\gamma'}{\gamma} \right)' \right] + n^2\alpha\gamma h = 0. \quad (31)$$

Equation (31) is a third-order differential equation. There should be no wonder (and no panic) on this occasion. One of the solutions to this equation we know in advance; this solution is

$$h = C \frac{\alpha}{a}, \quad (32)$$

where C is an arbitrary constant. We could have expected the existence of this solution, even before solving the equation for $h(\eta)$, because the perturbation of this form can be generated by a coordinate transformation which does not violate our choice of synchronous coordinate systems and, hence, does not destroy our initial form (9) of the perturbed metric. [One can easily check that the function (32) is indeed a solution to Eq. (31).] Concretely, one can perform a small coordinate transformation

$$\bar{\eta} = \eta - \frac{C}{2a}Q, \quad \bar{x}^i = x^i - \frac{C}{2}Q^i \int a^{-1}d\eta.$$

In terms of new coordinates $\bar{\eta}$, \bar{x}^i the transformed components (9) take on the form

$$\bar{g}_{00} = -a^2(\bar{\eta}), \quad \bar{g}_{0i} = 0, \\ \bar{g}_{ik} = a^2(\bar{\eta}) \left[(1 + \bar{h}Q)\delta_{ik} - \bar{h}_l n^{-2} Q_{,i,k} \right],$$

where

$$\bar{h} = h + C \frac{\alpha}{a}, \quad \bar{h}_l = h_l + Cn^2 \int a^{-1}d\eta. \quad (33)$$

The same transformation should be applied to the components of the energy-momentum tensor. Even if the original h , h_l are zero, the transformed \bar{h} , \bar{h}_l are not zero. After erasing the overbars in the transformed components of the metric, one returns to Eq. (9). The freedom of choosing different freely falling coordinate systems and corresponding spatial slices $\eta = \text{const}$ gets represented in the form of freedom to choose different solutions from the family of all solutions for the perturbations. All choices of C are equally well “physical.” The integral in Eq. (33) produces an additional integration constant

which reflects the possibility to make a purely spatial transformation and to shift h_l by a constant value, but we will not actually need this coordinate freedom. It follows from Eq. (33) that there are two functions (and many algebraic and differential combinations constructed from them) that do not contain C at all:

$$u = h' + \alpha\gamma h, \quad v = h'_l - \frac{1}{\alpha}n^2 h .$$

The solution (32) allows one to reduce the third-order differential equation (31) to the second-order differential equation. To reach this goal we use the function $u(\eta)$:

$$u = h' + \alpha\gamma h . \quad (34)$$

Obviously, this function vanishes on the solution (32). By substituting Eq. (34) into Eq. (31) we derive the equation for $u(\eta)$:

$$u'' + u' \left(2\alpha\gamma - \frac{\gamma'}{\gamma} \right) + u \left[n^2 - 2\alpha' - \alpha \frac{\gamma'}{\gamma} - \left(\frac{\gamma'}{\gamma} \right)' \right] = 0 . \quad (35)$$

Note that the coefficients of this differential equation depend exclusively on the scale factor and its derivatives. This fact is a manifestation of the underlying interaction of the perturbations with the cosmological pump field. No special assumptions about the shape of the potential $V(\varphi)$ or such things as “nonsimultaneous rolling the scalar field down the hill” have been made whatsoever.

Our next move is to transform Eq. (35) to the form similar to the equation for gravitational waves. This will allow us to use certain results derived previously for gravitational waves and rotational perturbations [17,2]. In order to get rid of u' we introduce the function $\mu(\eta)$ according to

$$u = \frac{\alpha\sqrt{\gamma}}{a} \mu . \quad (36)$$

It follows from Eq. (18) that the function $\gamma(\eta)$ is non-negative if the scale factor is governed by the scalar field (1) which we study here. However, Eq. (35) is formally applicable to negative γ as well. It may happen (as the author thinks) that Eq. (35) has a wider domain of validity and can be used, for other models of matter, with negative γ too. If this is the case, one is free to modify Eq. (36) by using $\sqrt{|\gamma|}$ instead of $\sqrt{\gamma}$. Anyway, with the help of Eq. (36) one derives the equation

$$\mu'' + \mu[n^2 - U(\eta)] = 0 ,$$

where

$$\begin{aligned} U(\eta) &= \alpha^2 + \alpha' + \alpha \frac{\gamma'}{\gamma} + \frac{1}{4} \left(\frac{\gamma'}{\gamma} \right)^2 + \frac{1}{2} \left(\frac{\gamma'}{\gamma} \right)' \\ &= U_0(\eta) + U_1(\eta) , \\ U_0(\eta) &= \alpha^2 + \alpha' = \frac{a''}{a} , \\ U_1(\eta) &= \frac{1}{\gamma^2} \left[\alpha\gamma\gamma' - \frac{1}{4}\gamma'^2 + \frac{1}{2}\gamma\gamma'' \right] . \end{aligned} \quad (37)$$

The effective potential $U(\eta)$ can also be written as $U(\eta) = (a\sqrt{\gamma})''/a\sqrt{\gamma}$ which reduces our basic equation to the form

$$\mu'' + \mu \left[n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] = 0 . \quad (38)$$

(The function $a\sqrt{\gamma}$ can be related with the function z discussed in [18], see also the early papers [27].)

Let us recall [3] that in the case of gravitational waves the potential $U(\eta)$ consists only of the $U_0(\eta)$ term, so that the basic equation is

$$\mu'' + \mu \left[n^2 - \frac{a''}{a} \right] = 0 . \quad (39)$$

The gravitational wave potential U_0 depends only on the first and second time derivatives of the logarithm of the scale factor: $(\ln a)'$, $(\ln a)''$. The potential $U(\eta)$ for density perturbations is more complicated and includes also $(\ln H)'$, $(\ln H)''$, and $(\ln H)'''$. We note, however, that the potentials are exactly the same, and, therefore, the basic equations and solutions for density perturbations and gravitational waves are exactly the same, if γ is constant, that is for the scale factors (6). For this class of pump fields, the general solution to Eq. (39) can be written in the form (for non-half-integer β)

$$\mu(\eta) = (n\eta)^{1/2} \left[A_1 J_{\beta+\frac{1}{2}}(n\eta) + A_2 J_{-(\beta+\frac{1}{2})}(n\eta) \right] . \quad (40)$$

Having two linearly independent solutions to Eq. (38) one can construct $h(\eta)$ and, hence, find the rest of functions describing density perturbations. It follows from Eqs. (34) and (36) that

$$h' = -\gamma\alpha h + \frac{\alpha\sqrt{\gamma}}{a} \mu \quad (41)$$

and

$$h(\eta) = \frac{\alpha}{a} \int_{\eta_0}^{\eta} \mu\sqrt{\gamma} d\eta + \frac{\alpha}{a} C_i , \quad (42)$$

where η_0 is some initial time where the initial conditions are to be imposed. The constant C entering Eq. (32) is denoted C_i at the i stage and will have the labels e and m at the e and m stages. All (complex) solutions to our perturbation problem for a given wave vector \mathbf{n} are completely determined by three arbitrary and independent (complex) constants. Two of them define a solution to Eq. (38) [these constants are A_1, A_2 when Eq. (40) is applicable]. The third constant, C_i , describes the remaining freedom in our choice of coordinates. This remaining freedom is not a misfortune of the theory. On the contrary, it will later allow us to join our coordinate system right to the comoving synchronous coordinate system at the m stage.

One can show by using Eqs. (24), (12), and (41) (and assuming $\varphi'_0 \neq 0$) that

$$\varphi_1(\eta) = \frac{1}{\sqrt{2\kappa}} \left[\frac{1}{a} \mu - \sqrt{\gamma} h \right] . \quad (43)$$

One can also find with the help of Eqs. (11), (13), and (41) that

$$\frac{p_1}{\epsilon_1} = \frac{\mu' + \mu \left(2\frac{\alpha'}{\alpha} + \frac{1}{2}\frac{\gamma'}{\gamma} \right) - a\sqrt{\gamma} \left(\alpha + 2\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} \right) h}{\mu' - \mu \left(4\alpha + \frac{1}{2}\frac{\gamma'}{\gamma} \right) + 3a\sqrt{\gamma} \alpha h}. \quad (44)$$

The quantity c_l , where $c_l^2/c^2 = p_1/\epsilon_1$, plays the role of the velocity of sound for the high-frequency scalar field oscillations (see also Sec. VII).

Similarly to what is true for gravitational waves, solutions to Eq. (38) are different for the high-frequency and low-frequency regimes. In the former case, $n^2 \gg |U(\eta)|$ and $\mu \sim e^{\pm i n \eta}$. In the latter case, $n^2 \ll |U(\eta)|$ and two independent solutions are $\mu_1 \sim a\sqrt{\gamma}$, $\mu_2 \sim a\sqrt{\gamma} \int d\eta / (a\sqrt{\gamma})^2$. The functions μ_1, μ_2 generalize the corresponding solutions for gravitational waves by replacing a with $a\sqrt{\gamma}$. Specifically for the scale factors (6), the solutions μ_1, μ_2 are $\mu_1 \sim \eta^{1+\beta}$ and $\mu_2 \sim \eta^{-\beta}$, in agreement with Eq. (40).

In the high-frequency regime, the term μ' dominates the other terms in Eq. (44). As one could expect, in this regime, the velocity of sound is equal to the velocity of light, $c_l^2 = c^2$. In the low-frequency regime, that is when a given mode enters the underbarrier region, the dominant solution is μ_1 (for a review, see [19]). Using this solution in Eq. (44) one can show that, in this regime, $p_1/\epsilon_1 \approx q(\beta)$, that is the “velocity of sound” is the same as the one defined by p'_0/ϵ'_0 . In particular, p_1/ϵ_1 goes to -1 for the low-frequency scalar field solutions at the de Sitter stage, $\beta = -2$.

Equations (31), (35), and (38) have been derived under the condition $\varphi'_0 \neq 0$. However, the final formula (42) gives the correct result $h(\eta) = C_i/l_0 = \text{const}$ in the de Sitter limit $\gamma = 0$, $\varphi'_0 = 0$. One can analyze this case separately, referring to the starting Eqs. (23)–(26). One can see that Eqs. (24) and (12) give $\xi' = 0$, $h = \text{const}$. Equations (13), (25), and (23) give $p_1 = 0$, $V_{,\varphi} = 0$, $\epsilon_1 = 0$. Finally, Eq. (11) requires $h'_i = -\eta n^2 h = -\eta n^2 C_i/l_0$. But this solution for h, h'_i is precisely solution (33) which can be eliminated by a coordinate transformation. In the de Sitter case governed by the scalar field (1) there is no density perturbations at all. Note that the function $\varphi_1(\eta)$, Eq. (22), remains arbitrary and the wavelengths of these fluctuations are growing in the course of expansion. If one wishes, one can attach to φ_1 such words as “inflation is pushing the waves beyond the de Sitter horizon.” Nevertheless, the result will be zero, as long as φ_1 is not accompanied by perturbations of the gravitational field. This is an instructive example in order to realize that to “stretch the waves outside the causal horizon” is not all we need for generation of density perturbations (likewise, it is not sufficient to simply stretch the electromagnetic waves “beyond the horizon” in order to generate photons).

IV. LATE TIME EVOLUTION OF THE PERTURBATIONS AT THE INITIAL STAGE

The main uncertainties about the evolution of the very early Universe refer to the times that preceded the epoch of the primordial nucleosynthesis. Whatever was the initial stage, it supposedly went over by that epoch into the radiation dominated stage governed by the scale factor $a(\eta) = l_0 a_e(\eta - \eta_e)$. The constants a_e, η_e are to be determined from the continuous joining of $a(\eta)$ and $a'(\eta)$ at the time $\eta = \eta_1$ of transition from the i stage to the e stage. If the i stage is described by the scale factors (6), one derives

$$a_e = -(1 + \beta)|\eta_1|^\beta, \quad \eta_e = \frac{\beta}{1 + \beta}\eta_1.$$

In further applications, we intend to use simple solutions (40), (42) and to make their appropriate joining with perturbations at the e stage. However, the function $\gamma(\eta)$, being equal to the constant $\gamma = (2 + \beta)/(1 + \beta)$ at the i stage, and to the constant $\gamma = 2$ at the e stage, experiences a finite jump at the transition point $\eta = \eta_1$. This presented no problem for gravitational waves, since $\gamma'(\eta)$ did not enter the gravitational wave potential $U_0(\eta)$. But this becomes important for density perturbations, since the $U_1(\eta)$ part of the potential acquires increasingly growing values at the end of the i stage for steeper and steeper transitions.

To deal with the problem, we introduce a parametrized set of smooth functions $\gamma(\eta)$ that approximate the step function in the limit of the parameter ϵ going to infinity:

$$\gamma(\eta) = \frac{4 + 3\beta}{2(1 + \beta)} + \frac{\beta}{2(1 + \beta)} \tanh[\epsilon(\eta - \eta_1)]. \quad (45)$$

For large negative values of η the function (45) goes to $(2 + \beta)/(1 + \beta)$, and for large positive values of η it goes to 2. We may surround the transition time $\eta = \eta_1$ by a thin “sandwich” with boundaries at $\eta_1 - \sigma$ and $\eta_1 + \sigma$. The asymptotic values of $\gamma(\eta)$ are already reached with arbitrary accuracy at the boundaries, if ϵ is sufficiently large, $\epsilon \gg 1/\sigma$.

The function (45) can be integrated, see Eq. (4), to produce the function $\alpha(\eta)$:

$$\frac{1}{\alpha(\eta)} = \frac{\eta}{1 + \beta} + \frac{\beta}{2(1 + \beta)} \left[\eta - \eta_1 + \frac{1}{\epsilon} \ln \frac{e^{\epsilon(\eta - \eta_1)} + e^{-\epsilon(\eta - \eta_1)}}{2} \right]. \quad (46)$$

Again, for sufficiently large ϵ , the function $\alpha(\eta)$ quickly approximates $(1 + \beta)/\eta$ to the left of the transition point, and $1/(\eta - \eta_e)$ to the right of the transition point. These are the values of $\alpha(\eta)$ that are appropriate for the i stage (6) and the e stage, respectively.

The divergent functions γ', γ'^2 , and γ'' participate in the potential $U_1(\eta)$ [Eq. (37)]. The function γ' grows as ϵ at the point $\eta = \eta_1$. The function γ'' is equal to zero at $\eta = \eta_1$, but it grows as ϵ^2 slightly to the left of this point, and it grows as $-\epsilon^2$ slightly to the right of this point. We

assume that the transition to the e stage has completed at $\eta = \eta_1 + \sigma$, and we let σ to go to zero. The function $U_1(\eta)$ compresses and stretches to the arbitrarily large positive and negative values when σ goes to zero and ϵ goes to infinity. Examining Eq. (38), one can expect that the value of $\mu'(\eta)$ at $\eta = \eta_1 + \sigma$ will be different from the value of $\mu'(\eta)$ at $\eta = \eta_1 - \sigma$. The integration of μ'' in the limits from $\eta_1 - \sigma$ to $\eta_1 + \sigma$ gives a jump in μ' which depends on the value of the integral from the divergent part of the potential $U_1(\eta)$. Fortunately, it is not $\mu'(\eta)$ itself, but a particular combination $(\sqrt{\gamma}/a)[\mu' - \mu(\alpha + \gamma'/2\gamma)]$ that we will need to know in our further calculations. This simplifies the analysis. Because of Eqs. (29) and (41) this combination is precisely the function $v(\eta)$ introduced in Sec. III.

In terms of the function $v(\eta)$, where

$$v = \frac{\sqrt{\gamma}}{a} \left[\mu' - \mu \left(\alpha + \frac{1}{2} \frac{\gamma'}{\gamma} \right) \right] = \gamma \left(\frac{\mu}{a\sqrt{\gamma}} \right)' , \quad (47)$$

the basic equation (38) takes on the form

$$(a^2 v)' = -n^2 a \sqrt{\gamma} \mu . \quad (48)$$

The integration of this equation over the thin "sandwich" shows that $v|_{\eta=\eta_1+0} = v|_{\eta=\eta_1-0}$. In other words, the function $\mu' - \mu(\alpha + \frac{1}{2} \frac{\gamma'}{\gamma})$ taken right at the beginning of the e stage is equal to the value of this function taken right at the end of the i stage (6) times the factor $\frac{1}{\sqrt{2}} \sqrt{\frac{2+\beta}{1+\beta}}$. In addition to the conditions: $\gamma|_{\eta=\eta_1-0} = \sqrt{\frac{2+\beta}{1+\beta}}$, $\gamma|_{\eta=\eta_1+0} = 2$, $\mu|_{\eta=\eta_1-0} = \mu|_{\eta=\eta_1+0}$, this establishes the rules for going through the "sandwich."

$$u'' + u' \left[3\alpha + 3\alpha \frac{c_i^2}{c^2} - \frac{(c_i^2)'}{c_i^2} \right] + u \left[n^2 \frac{c_i^2}{c^2} + \left(\alpha + \frac{\alpha'}{\alpha} \right) \left(3\alpha \frac{c_i^2}{c^2} - \frac{(c_i^2)'}{c_i^2} \right) + \frac{a''}{a} + 2 \left(\frac{\alpha'}{\alpha} \right)' \right]$$

$$+ h \left\{ 3\alpha^2 \gamma \frac{c_s^2}{c^2} \left[3\alpha \left(\frac{c_i^2}{c^2} - \frac{c_s^2}{c^2} \right) - \frac{(c_i^2)'}{c_i^2} + \frac{(c_s^2)'}{c_s^2} \right] \right\} = 0 . \quad (50)$$

The last term in this equation vanishes if

$$3\alpha \frac{c_i^2}{c^2} - \frac{(c_i^2)'}{c_i^2} = 3\alpha \frac{c_s^2}{c^2} - \frac{(c_s^2)'}{c_s^2} , \quad (51)$$

which integrates to

$$\frac{c_s^2}{c_i^2} = 1 - \frac{\text{const}}{a\alpha^2\gamma} .$$

An assumption which is usually made for perfect fluids is

$$c_i^2 = c_s^2 . \quad (52)$$

V. DENSITY PERTURBATIONS IN THE PERFECT FLUID MATTER

We will now consider Eqs. (11)–(14) at the perfect fluid stages governed by the energy-momentum tensor (15). Similarly to the scalar field case, the longitudinal-longitudinal part of stresses vanishes, $p_l = 0$. For the easier handling of arbitrary ϵ_1 , p_1 one can introduce the following notations: $p_1/\epsilon_1 = c_l^2/c^2$ and $p'_0/\epsilon'_0 = c_s^2/c^2$, see Eq. (7). These definitions are convenient but, generally speaking, they have only formal meaning, since both p_1/ϵ_1 and p'_0/ϵ'_0 can be negative. However, in certain regimes, the quantity c_l is a genuine longitudinal velocity of sound (see Sec. VII). Our first intention is to derive the equation for $h(\eta)$, analogous to Eq. (31) and valid for arbitrary nonzero p_1/ϵ_1 . Specific cases $c_l^2 = \frac{1}{3}c^2$ and $c_l^2 = 0$ will be considered separately.

In order to derive the equation for $h(\eta)$ one can essentially repeat the steps that have lead to Eq. (31). Find $h'_i(\eta)$ from Eq. (11) and plug it into Eq. (14). Use Eq. (13), the first derivative of this equation, and the definition of c_l^2/c^2 . As a result, one arrives at the equation

$$h''' + h'' \left[3\alpha + \alpha\gamma + 3\alpha \frac{c_l^2}{c^2} - \frac{(c_l^2)'}{c_l^2} \right] + h' \left[n^2 \frac{c_l^2}{c^2} + 4\alpha^2 + 6\alpha^2 \frac{c_l^2}{c^2} - 2\alpha \frac{(c_l^2)'}{c_l^2} \right] + hn^2 \alpha \gamma \frac{c_l^2}{c^2} = 0 . \quad (49)$$

Now, introduce the function $u(\eta)$ according to Eq. (34) and use it in Eq. (49). Equation (49) can be reduced to

Note that Eq. (52) is certainly true for matter with the equation of state $p = q\epsilon$, where q is a constant, but Eq. (52) is not true in general, and it is not true for the scalar field matter (1) [unless one considers the underbarrier region where Eq. (52) is true approximately, see Sec. III]. Because of Eq. (51) the last term in Eq. (50) cancels out. [If we have not assumed (51), the function $h(\eta) = C(\alpha/a)$ would not have been a solution to Eq. (49).]

We can now introduce the function $\nu(\eta)$ according to [compare with Eq. (36)]

$$u = \frac{\alpha\sqrt{\gamma}}{a} c_s \nu . \quad (53)$$

In terms of $\nu(\eta)$, Eq. (50) takes on the form

$$\nu'' + \nu \left[n^2 \frac{c_i^2}{c^2} - W(\eta) \right] = 0, \quad (54)$$

where

$$W(\eta) = \frac{a''}{a} - \frac{\alpha''}{\alpha} - \frac{(\alpha^2 \sqrt{\gamma} c_s)''}{\alpha^2 \sqrt{\gamma} c_s} + 2 \left[\frac{(\alpha^2 \sqrt{\gamma} c_s)'}{\alpha^2 \sqrt{\gamma} c_s} \right]^2.$$

The potential $W(\eta)$ depends exclusively on the scale factor and its derivatives. Having a solution $\nu(\eta)$ to this equation and using Eqs. (53) and (34), one can construct the function $h(\eta)$ and the rest of perturbations. Similarly to the scalar field case, all solutions for perturbations with a given \mathbf{n} are defined by three constants one of which describes the remaining coordinate freedom. These constants are expressible in terms of the constants given at the preceding i stage through the joining of the perturbations at the transition time $\eta = \eta_1$ from the i stage to the perfect fluid stage.

We will now consider Eq. (54) specifically at the radiation-dominated stage $p = \frac{1}{3}\epsilon$. We have $c_i^2 = c_s^2 = (1/3)c^2$, $\gamma = 2$, and the scale factor

$$a(\eta) = l_0 a_e (\eta - \eta_e), \quad (55)$$

where the constants a_e , η_e are to be determined from the continuous joining of $a(\eta)$ and $a'(\eta)$ at the transition time $\eta = \eta_1$. The potential $W(\eta)$ vanishes. Equation (54) simplifies to the familiar equation

$$\nu'' + \frac{1}{3} n^2 \nu = 0 \quad (56)$$

which describes the time-dependent part of sound wave oscillations in the radiation-dominated fluid. In what follows, we will be using the general solution to this equation written in the form

$$\nu = B_1 e^{-i \frac{n}{\sqrt{3}} (\eta - \eta_e)} + B_2 e^{i \frac{n}{\sqrt{3}} (\eta - \eta_e)}, \quad (57)$$

where B_1 , B_2 are arbitrary and independent (complex) numbers for each individual wave vector \mathbf{n} .

The function $h(\eta)$ is determined by a known solution for $\nu(\eta)$ and a coordinate solution with arbitrary constant C_e :

$$h(\eta) = \frac{\alpha}{a} \int_{\eta_1}^{\eta} \nu d\eta + \frac{\alpha}{a} C_e. \quad (58)$$

All other functions are expressible in terms of $h(\eta)$. In particular,

$$h'_i = \frac{1}{\alpha} [3h'' + 9\alpha h' + n^2 h]. \quad (59)$$

The general solution (57) is always oscillatory in η time. The perturbed energy density, pressure, and the associated gravitational field $h(\eta)$, $h_i(\eta)$ oscillate in space

and time as they should do for sound waves. However, if one considers these oscillations at intervals of time shorter than their period, they may appear as consisting of "growing" and "decaying" solutions. In particular, this happens if one considers relatively long waves, $\frac{n}{\sqrt{3}}(\eta_1 - \eta_e) \equiv y_1 \ll 1$, at their first oscillation since the beginning of the e stage, that is while the condition $\frac{n}{\sqrt{3}}(\eta - \eta_e) \ll 1$ is satisfied. Under this condition, the function $h(\eta)$, Eq. (58), can be approximated as

$$h(\eta) = \frac{\bar{C}_e}{\bar{\eta}^2} + \frac{\bar{B}_1}{\bar{\eta}} + \bar{B}_2 + \dots, \quad (60)$$

where

$$\bar{\eta} = \frac{1}{\sqrt{3}}(\eta - \eta_e), \quad \bar{B}_1 = \frac{B_1 + B_2}{\sqrt{3} l_0 a_e},$$

$$\bar{B}_2 = \frac{-in(B_1 - B_2)}{2\sqrt{3} l_0 a_e},$$

and

$$\bar{C}_e = \frac{1}{3l_0 a_e} \left\{ C_e + \frac{i\sqrt{3}}{n} \left[B_1 \left(1 - e^{-iy_1} \right) - B_2 \left(1 - e^{iy_1} \right) \right] \right\}.$$

The common practice [15] is to use the coordinate freedom for elimination of the "most divergent" term in Eq. (60), which is also the "most decaying" term, if one goes forward in time. This is achieved by such a choice of C_e that $\bar{C}_e = 0$, and the first term in Eq. (60) vanishes. Then, the energy density perturbation $\delta\epsilon/\epsilon_0$ [use the definition

$$\delta\epsilon/\epsilon_0 = \frac{\kappa\epsilon_1}{3\alpha^2} Q$$

and calculate $\kappa\epsilon_1$ according to Eq. (11)] can be approximated as

$$\frac{\delta\epsilon}{\epsilon_0} = \left(\frac{1}{9} n^2 \bar{B}_1 \bar{\eta} + \frac{1}{2} n^2 \bar{B}_2 \bar{\eta}^2 + \dots \right) Q. \quad (61)$$

The part of Eqs. (60) and (61) which depends on the coefficient \bar{B}_1 is usually called the "decaying" solution, while the part with the coefficient \bar{B}_2 is called the "growing" solution.

Despite the possibility of identifying (quite artificially) the "growing" and "decaying" solutions, density perturbations at the e -stage form a collection of traveling sound waves with arbitrary amplitudes and arbitrary phases, as long as constants B_1 , B_2 are arbitrary and independent. One should not think that simply because the sound waves have spent some time "beyond the horizon," $n\bar{\eta} \ll 1$, they would transform into standing waves at later times of their history after they came "inside the horizon," $n\bar{\eta} \gg 1$. To illustrate this point, let us take into account the spatial part of the perturbations and consider the contribution $h_{\mathbf{n}}(\eta, \mathbf{x})$ of a given mode \mathbf{n} to the total field $h(\eta, \mathbf{x}) = \sum_{\mathbf{n}} h_{\mathbf{n}}(\eta, \mathbf{x})$. This contribution can be written as

$$\begin{aligned}
h_{\mathbf{n}}(\eta, \mathbf{x}) &\sim h_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}} + h_{\mathbf{n}}^* e^{-i\mathbf{n}\cdot\mathbf{x}} \sim (B_{1\mathbf{n}} e^{-i\mathbf{n}\bar{\eta}} - B_{2\mathbf{n}} e^{i\mathbf{n}\bar{\eta}}) e^{i\mathbf{n}\cdot\mathbf{x}} + (B_{1\mathbf{n}}^* e^{i\mathbf{n}\bar{\eta}} - B_{2\mathbf{n}}^* e^{-i\mathbf{n}\bar{\eta}}) e^{-i\mathbf{n}\cdot\mathbf{x}} \\
&= 2|B_{1\mathbf{n}}| \cos(n\bar{\eta} - \mathbf{n} \cdot \mathbf{x} - \varphi_{1\mathbf{n}}) - 2|B_{2\mathbf{n}}| \cos(n\bar{\eta} + \mathbf{n} \cdot \mathbf{x} + \varphi_{2\mathbf{n}}), \quad (62)
\end{aligned}$$

where $B_{1\mathbf{n}} = |B_{1\mathbf{n}}|e^{i\varphi_{1\mathbf{n}}}$, $B_{2\mathbf{n}} = |B_{2\mathbf{n}}|e^{i\varphi_{2\mathbf{n}}}$. The last line in Eq. (62) shows explicitly that, in general, one is dealing with waves traveling in opposite directions with arbitrary amplitudes and arbitrary phases. A standing wave can only be “generated by hand,” by assuming that the constants $B_{1\mathbf{n}}$, $B_{2\mathbf{n}}$ are strictly related. This happens if one declares that he/she is only interested in the “growing” solution and puts $\bar{B}_1 = 0$. Then, the complex amplitudes $B_{1\mathbf{n}}$, $B_{2\mathbf{n}}$ become related: $|B_{2\mathbf{n}}| = (-1)^{k+1}|B_{1\mathbf{n}}|$, $\varphi_{2\mathbf{n}} = \varphi_{1\mathbf{n}} - k\pi$, ($k = 0, 1, 2, \dots$), and the last line in Eq. (62) can be transformed to

$$h_{\mathbf{n}}(\eta, \mathbf{x}) \sim 4|B_{1\mathbf{n}}| \cos n\bar{\eta} \cos(\mathbf{n} \cdot \mathbf{x} + \varphi_{1\mathbf{n}})$$

which is a standing wave indeed.

Standing sound waves at the e stage are responsible for the so-called Sakharov oscillations [20] in the power spectrum of density perturbations in the present Universe, at the m stage. As we have shown, standing sound waves cannot originate somehow automatically at the e stage, simply because of the transition from the “growing”/“decaying” regime to the oscillating regime (see also [21]). If one works with classical density perturbations at the e stage and makes no additional assumptions, one can say nothing about the necessity of standing waves, except of postulating this. The point of this discussion is that the quantum-mechanical generating mechanism, which we are considering in this paper, does really create standing waves. Standing waves arise for gravitational waves, rotational perturbations, and density perturbations. The physical reason for this is that the waves (particles) are generated in correlated pairs with equal and oppositely directed momenta (the two-mode squeezed vacuum quantum states). This is true for waves of any wavelength, as soon as conditions for their generation are satisfied. Technically, as we will see later, the second term in Eq. (60) taken at the beginning of the e stage turns out to be much smaller than the third term in Eq. (60), that is $B_1 + B_2 \approx 0$ for $y_1 \ll 1$.

We should now discuss a great difference between sound waves and gravitational waves with regard to their evolution in time. The velocity of sound waves at the e stage is only $\sqrt{3}$ times smaller than the velocity of gravitational waves, but their amplitudes behave drastically different. The amplitude of a gravitational wave decays as a^{-1} in course of time. The amplitude of a sound wave, as one can see from Eq. (58), decays as a^{-2} , since $\alpha \sim a^{-1}$. This leads to a difference in solutions even for relatively long waves which did not complete even one cycle of oscillations during the entire e stage from $\eta = \eta_1$ to $\eta = \eta_2$. As an illustration, let us consider sound waves which barely reached the oscillating regime by the end of the e stage. Their wave numbers satisfy the condition $\frac{n}{\sqrt{3}}(\eta_2 - \eta_e) \equiv \frac{n}{n_c} \equiv y_2 \approx 1$. These are the waves whose wavelength was of the order of the Hubble

radius at the time of transition from the e stage to the matter dominated m stage. Assuming that the present-day Hubble radius is $l_H \approx 6 \times 10^3$ Mpc and that the present-day scale factor $a(\eta_R)$ is $a(\eta_R) \approx 10^4 a(\eta_2)$, their wavelength today $\lambda_c = 2\pi a(\eta_R)/n_c$ is about 220 Mpc. The usual practice, in addition to eliminating \bar{C}_e , is to concentrate on the “growing” solution, that is to assume that at the beginning of the e stage the second term in Eq. (60) is smaller, or at least not larger, than the third term. Under these conditions, the final numerical value of the function $h(\eta)$ is of the same order of magnitude as the initial value, $h(\eta_2) \approx h(\eta_1)$, for the wavelengths of our interest. In other words, if the preceding i stage produced $h(\eta)$ with some initial numerical value $h(\eta_1)$, this number will effectively be transmitted to the beginning of the m stage.

However, in the very same coordinate system where the “most decaying” term in Eq. (60) was eliminated, the time derivative $h'(\eta)$ was left large. The final value of $h'(\eta)$ for the waves of our interest is $h'(\eta_2) \approx nh(\eta_2)$. According to Eq. (12), the function $h'(\eta)$ describes velocity of matter. This velocity will be inherited by matter at the matter-dominated stage. But this is not velocity of the fluid elements with respect to each other, this is not velocity describing deformations of the medium, and the functions $h'(\eta)$, $h'_l(\eta)$ are not the ones that we may use for our later calculation of the variations in the CMBR. The function $h'(\eta)$ describes velocity of matter with respect to the coordinate system that we have chosen for our convenience of eliminating the “most decaying” term. At the m stage, however, we are more interested in the comoving coordinate system. We are interested in the density contrasts and deformations of the fluid itself, we are interested in the components of the accompanying gravitational field that we may use for calculations of $\delta T/T$. We need a coordinate system which provides vanishing of $h'(\eta)$ by the end of the e stage. This requires a different choice of C_e . Under this new choice of C_e , the first term in Eq. (60) survives, and numerically the same, as in the previous example, initial value $h(\eta_1)$ transforms into a small number $h(\eta_2) \approx y_1^2 h(\eta_1)$ by the beginning of the m stage. This is what we need to keep in mind when we will compare the amplitudes of density perturbations and gravitational waves.

Finally, let us consider density perturbations at the m stage, $p = 0$. At this stage, one has $c_l^2 = c_s^2 = 0$, $\gamma = \frac{3}{2}$, and the scale factor

$$a(\eta) = l_0 a_m (\eta - \eta_m)^2 \quad (63)$$

The scale factor and its first time derivative are continuous at the time $\eta = \eta_2$ of transition from the e stage to the m stage. Therefore, $a_m = a_e/4(\eta_2 - \eta_e)$, $\eta_m = -\eta_2 + 2\eta_e$. It follows from Eqs. (13) and (14) that the general solution for $h(\eta)$, $h_l(\eta)$ has the form

$$h = C_1 + \frac{\alpha}{a} C_m, \quad h' = -\frac{3}{2} \frac{\alpha^2}{a} C_m,$$

$$h'_i = \frac{1}{5} C_1 n^2 (\eta - \eta_m) + \frac{1}{a} n^2 C_m + C_2 \frac{(\eta_2 - \eta_m)^3}{(\eta - \eta_m)^4}. \quad (64)$$

The energy-density and velocity perturbations can be found from Eqs. (11) and (12). Similarly to the preceding i and e stages, the perturbations are completely determined by three constants C_1, C_2, C_m one of which, C_m , reflects the remaining coordinate freedom.

The constant C_m is entirely responsible for a possible relative velocity of our fluid with respect to a chosen synchronous coordinate system, see Eqs. (12) and (10). A given coordinate system is not comoving, $T_0^i \neq 0$, as long as $C_m \neq 0, h' \neq 0$. But we know that for a dustlike fluid without rotation one can always introduce a coordinate system which is both synchronous and comoving. This is reflected in our ability to remove the C_m term from $h(\eta)$ and $h_i(\eta)$ by a coordinate transformation (33). Thus, the choice $C_m = 0$ in Eq. (64) is not a restriction of the physical content of the problem, it is an allowed choice of the coordinate system.

It is here that we will eventually restrict our coordinate freedom; we will put

$$C_m = 0. \quad (65)$$

The constants C_e and C_i will not be arbitrary any longer, they will be determined from the continuous joining of solutions. We need the comoving coordinate system for simple and appropriate formulation of the $\delta T/T$ problem. We are interested in the temperature of the CMBR and its anisotropy seen by a comoving observer, that is by an observer whose world line is one of the matter's world lines. One of these idealized comoving observers is an observer on Earth (up to accuracy of some nonzero peculiar velocity, local rotation, etc.). We are much less interested in the feelings of an observer who wanders in the Universe with an arbitrary time-dependent velocity. In the comoving coordinate system, the world line of a comoving observer is described by the simple equation $x^i = \text{const}$.

Upon the choice of $C_m = 0$, the perturbations reduce to

$$h(\eta) = C_1,$$

$$h_i(\eta) = \frac{1}{10} C_1 n^2 (\eta - \eta_m)^2 - \frac{1}{3} C_2 \frac{(\eta_2 - \eta_m)^3}{(\eta - \eta_m)^3}. \quad (66)$$

From Eqs. (66) and (11) one can derive the familiar expression [15]

$$\frac{\delta \epsilon}{\epsilon_0} = \left[\frac{1}{20} C_1 n^2 (\eta - \eta_m)^2 - \frac{1}{6} C_2 \frac{(\eta_2 - \eta_m)^3}{(\eta - \eta_m)^3} \right] Q. \quad (67)$$

(One may wish to correct a misprint in Eq. (115.21) of Ref. [15]: the decaying solution behaves as η^{-3} , not as the printed η^{-2} .) As long as the constants C_1, C_2 are arbitrary, the power spectrum of the density perturbations is arbitrary. In particular, there are no Sakharov oscilla-

tions, *a priori*, and they do not arise simply because of the transition from the e stage to the m stage. For instance, one can start from a perfectly smooth spectrum at $\eta = \eta_2$ and extrapolate these data back in time up to the beginning of the e stage.

For the further calculations of $\delta T/T$ we will need the first time derivatives of the gravitational field perturbations at the m stage in the comoving coordinates. Since for the scalar component of the perturbations one has $h' = 0$, it is only the longitudinal-longitudinal component h'_i that is effective.

VI. JOINING THE PERTURBATIONS AT THE THREE STAGES

We are now in the position to start our operation of joining the solutions at i, e , and m stages. We want to derive from the first principles the expected density perturbations at the m stage. Of course, the result will depend on the unknown behavior of $a(\eta)$ at the i stage. But this is precisely why we are doing this study: we try to learn something about the evolution of the very early Universe by deriving the expected variations in the CMBR and comparing them with the observations.

The general rule for joining solutions to Einstein's equations is to match from the both sides the intrinsic and extrinsic curvatures of the transition hypersurface [22]. For our solutions written in the class of synchronous coordinate systems, this translates into the continuity of the spatial metric and its first time derivative. Since we have already assumed that $a(\eta)$ and $a'(\eta)$ join continuously, it is the continuous joining of $h(\eta), h_i(\eta), h'(\eta)$, and $h'_i(\eta)$ that should be ensured. In fact, it is sufficient to follow $h(\eta), h'(\eta)$, and $h'_i(\eta)$ as $h_i(\eta)$ is derivable from $h'_i(\eta)$ at all three stages up to the integration constant which can be removed by the remaining integration constant in Eq. (33) anyway. It is convenient to write h'_i at the i and e stages, respectively, in the form [use Eqs. (29), (59) and (42), (58)]

$$h'_i = \frac{n^2}{\alpha} h + \frac{\sqrt{\gamma}}{a} \left[\mu' - \mu \left(\alpha + \frac{1}{2} \frac{\gamma'}{\gamma} \right) \right],$$

$$h'_i = \frac{n^2}{\alpha} h + \frac{3}{a} (\nu' - \alpha \nu). \quad (68)$$

We will denote $a(\eta), \alpha(\eta)$ at $\eta = \eta_1$ and $\eta = \eta_2$ by a_1, α_1 and a_2, α_2 , respectively. It is also convenient to introduce the parameter $y = \frac{n}{\sqrt{3}} (\eta - \eta_e)$ and its values y_1, y_2 at the transition points.

We will first make the joining of solutions in general form, without adopting any particular coordinate system, any particular behavior at the i stage, and any particular wavelength of the perturbations.

Let us start from the i - e transition. Whatever was the i stage and its late time behavior, it produced certain $\mu(\eta_1), \mu'(\eta_1)$ and ended at $\eta = \eta_1$ with some values of the scale factor and its derivatives. For generality, we do not assume, for the time being, that $\gamma(\eta_1)$ is exactly 2 and $\gamma'(\eta_1)$ is exactly zero.

From the continuous joining of $h(\eta), h'(\eta)$, and $h'_i(\eta)$

one derives

$$C_e = \int_{\eta_0}^{\eta_1} \mu \sqrt{\gamma} d\eta + C_i, \quad (69)$$

$$B_1 e^{-iy_1} + B_2 e^{iy_1} = \sqrt{\gamma} \mu + \alpha(2 - \gamma) \left[\int_{\eta_0}^{\eta_1} \mu \sqrt{\gamma} d\eta + C_i \right], \quad (70)$$

$$B_1 e^{-iy_1} (1 + iy_1) + B_2 e^{iy_1} (1 - iy_1) = -\frac{1}{3} a \sqrt{\gamma} \left[\mu' - \mu \left(\alpha + \frac{1}{2} \frac{\gamma'}{\gamma} \right) \right]. \quad (71)$$

All functions in these equations are taken at $\eta = \eta_1$ so that, for instance, γ means $\gamma(\eta_1)$, μ' means $\mu'(\eta_1)$, γ' means $\gamma'(\eta_1)$, etc. For a more compact record we will also use the notations

$$u_1 = \frac{\alpha_1}{a_1} \sqrt{\gamma(\eta_1)} \mu(\eta_1),$$

$$v_1 = \frac{\sqrt{\gamma(\eta_1)}}{a_1} \left[\mu'(\eta_1) - \mu(\eta_1) \left(\alpha_1 + \frac{1}{2} \frac{\gamma'}{\gamma}(\eta_1) \right) \right],$$

and

$$\bar{C}_i = C_i + \int_{\eta_0}^{\eta_1} \mu \sqrt{\gamma} d\eta.$$

Equations (69)–(71) allow us to express the constants B_1 , B_2 , and C_e describing the perturbations at the e stage entirely in terms of the output values of the functions defined at the i stage.

Let us now turn to the e - m transition. Again, from the joining of $h(\eta)$, $h'(\eta)$, and $h'_i(\eta)$ one derives

$$C_m = \frac{4}{3} C_e - \frac{2}{\sqrt{3}n} \left(B_1 e^{-iy_1} s + B_2 e^{iy_1} s^* \right), \quad (72)$$

$$C_1 = -\frac{\alpha_2}{3a_2} C_e + \frac{1}{6a_2 y_2} \left[B_1 e^{-iy_1} (s + 3y_2 e^{-id}) + B_2 e^{iy_1} (s^* + 3y_2 e^{id}) \right], \quad (73)$$

$$C_2 = -\frac{2\sqrt{3}ny_2}{5a_2} C_e + \frac{6}{5a_2} \left\{ B_1 e^{-iy_1} [e^{-id}(-5 + 2y_2^2 - 6iy_2) + iy_2] + B_2 e^{iy_1} [e^{id}(-5 + 2y_2^2 + 6iy_2) - iy_2] \right\}, \quad (74)$$

where $d = y_2 - y_1 = \frac{n}{\sqrt{3}}(\eta_2 - \eta_1)$, $s = 2i + (y_2 - 2i)e^{-id}$. Everything at the m stage is known as soon as the coefficients C_1 , C_2 , and C_m are known. They are expressed in terms of the coefficients B_1 , B_2 , and C_e attributed to the e stage. Since these numbers, in turn, are known implicitly in terms of the coefficients attributed to the i stage, we have linked the very beginning with the very end.

Our next step is to impose the requirement (65) and to choose the comoving coordinate system at the m stage. From Eqs. (72) and (69) one can find

$$\bar{C}_i D = \frac{i}{2n^2} y_1^2 a_1 \left\{ s [3u_1(1 - iy_1) + v_1] - s^* [3u_1(1 + iy_1) + v_1] \right\}, \quad (75)$$

where

$$D = 2y_1^2 + i \frac{2 - \gamma}{2} [s^*(1 + iy_1) - s(1 - iy_1)] = 2y_1^2 + (2 - \gamma) [2 - (2 + y_1 y_2) \cos d - (y_2 - 2y_1) \sin d]. \quad (76)$$

Equation (75) says how to choose C_i at the i stage in order to match right to the comoving coordinate system at the m stage. We can now put Eq. (75) into Eq. (70) and solve Eqs. (70) and (71):

$$B_1 e^{-iy_1} = \frac{i}{6D} \frac{a_1}{\alpha_1} \{ 6y_1(1 - iy_1)u_1 - [(2 - \gamma)s^* - 2y_1]v_1 \}, \quad (77)$$

$$B_2 e^{iy_1} = -\frac{i}{6D} \frac{a_1}{\alpha_1} \{ 6y_1(1 + iy_1)u_1 - [(2 - \gamma)s - 2y_1]v_1 \}. \quad (78)$$

Substituting these formulas and Eq. (75) into Eqs. (73) and (74), we reach our goal — the finding of C_1 , and C_2 in the comoving coordinates:

$$C_1 = \frac{a_1}{3a_2 \alpha_1 D} \{ 3u_1 y_1 (\sin d + y_1 \cos d) - v_1 [(2 - \gamma)(\cos d - 1) - y_1 \sin d] \}, \quad (79)$$

$$C_2 = -\frac{2a_1}{5a_2 \alpha_1 D} \{ 3u_1 y_1 [(10 - 3y_2^2 - 10y_1 y_2) \sin d + (-10y_2 + 10y_1 - 3y_1 y_2^2) \cos d] - v_1 \{ -2(2 - \gamma)(5 + y_2^2) - [y_1(10 - 3y_2^2) - 10y_2(2 - \gamma)] \sin d + [(2 - \gamma)(10 - 3y_2^2) + 10y_1 y_2] \cos d \} \}. \quad (80)$$

So far, no approximations have been made. We will start making them now.

The numerical value of the denominator D , and hence the absolute values of C_1 and C_2 , depend critically on how close the exiting value $\gamma(\eta_1) \equiv \gamma_1$ is to 2. We are interested in wavelengths that were longer than the Hubble

radius at $\eta = \eta_1$. Their wave numbers satisfy the requirement $y_1 \ll 1$. On the other hand, $y_2/y_1 = a_2/a_1 \gg 1$, and $d = y_2 - y_1 \approx y_2$. The wavelengths longer than the present-day $\lambda_c = 2\pi a(\eta_R)/n_c \approx 220$ Mpc correspond to small y_2 , and $n < n_c$. These are the wavelengths of the major interest for the discussion of the large-angular-

scale anisotropy in CMBR. In the approximation of small y_2 , two leading terms in D are

$$D \approx \gamma_1 y_1^2 + \frac{2 - \gamma_1}{12} y_2^4 .$$

The second term is much larger than the first one (and, hence, the expected C_1 , and C_2 are hopelessly small) for all

$$n_c \frac{a_1}{a_2} \sqrt{12\gamma_1/(2 - \gamma_1)} < n < n_c ,$$

unless the exiting value γ_1 is so close to 2 that the second term can be neglected. In order to deal with the most favorable situation and not proliferate complications, we will assume that $\gamma_1 = 2$ and $(\gamma'/\gamma)(\eta_1) = 0$. We know, see Sec. IV, that the transition from the very end of the i stage (after a thin "sandwich" interval) to the very beginning of the e stage can be made arbitrarily smooth (at least, in theory). So, we will be using $D \approx 2y_1^2 \ll 1$.

We should now take into account the fact that $v_1 \ll u_1$ for all wavelengths of our interest. Indeed, we are interested in modes that have interacted with the potential barrier in Eq. (38) and have been amplified at the i stage. For the scale factors (6), their wave numbers satisfy the requirement $(n\eta_1)^2 \ll \beta(\beta + 1)$ which translates into the condition $y_1^2 \ll \beta/3(\beta + 1)$, or simply $y_1^2 \ll 1$. We will derive the approximate formulas valid in the leading order by the parameter y_1 .

As we know, there are two independent solutions in the underbarrier region. Which of them dominates is the matter of choice of the initial conditions at $\eta = \eta_0$ [choice of constants A_1 , and A_2 in Eq. (40)]. For classical solutions, one can choose the initial data in such a way that there will be no amplification at all, or there will be even attenuation instead of amplification. However, a "typical" choice of initial data at $\eta = \eta_0$, which amounts to the averaging over the initial phase (or a rigorous quantum-mechanical treatment), always leads to the dominant solution $\mu \sim a\sqrt{\gamma}$, and to amplification [19]. This is the choice that we imply here and will justify later, Eq. (102) in Sec. VIII. Since $v \sim (\mu/a\sqrt{\gamma})'$ and $\mu \sim a\sqrt{\gamma}$, the quantity v_1 is relatively small. Concretely, for solutions (40), one has

$$\mu(\eta_1) \approx \frac{A_1}{2^{\beta+\frac{1}{2}} \Gamma(\beta + \frac{3}{2})} (n\eta_1)^{\beta+1} \quad (81)$$

and

$$(\mu' - \mu\alpha)(\eta_1) \approx -\frac{nA_1}{2^{\beta+\frac{1}{2}} \Gamma(\beta + \frac{3}{2})} (2\beta + 3)^{-1} (n\eta_1)^{\beta+2}$$

so that

$$\frac{v_1}{u_1} \approx -\frac{3(1 + \beta)}{2\beta + 3} y_1^2 \ll 1 .$$

Thus, we can neglect all the terms containing v_1 in Eqs. (77)–(80).

In the leading order, one has

$$B_1 \approx -B_2 \approx \frac{i}{2y_1} \sqrt{2} \mu(\eta_1) \equiv B ,$$

$$B_1 + B_2 \approx B y_1^3 .$$

These formulas ensure the standing wave pattern for all wavelengths at the e stage and the power spectrum modulation at the m stage (compare with Sec. V). The leading order expressions for C_1 , C_2 are

$$C_1 \approx \frac{1}{2a_2} \frac{1}{y_1} \sqrt{2} \mu(\eta_1) \sin d , \quad (82)$$

$$C_2 \approx -\frac{3}{5a_2} \frac{1}{y_1} \sqrt{2} \mu(\eta_1) \left[(10 - 3y_2^2) \sin d - 10y_2 \cos d \right] . \quad (83)$$

We will give a brief analysis of Eqs. (82) and (83). The growing and decaying components of $h_i(\eta)$ and $\delta\epsilon/\epsilon_0$, see Eqs. (66) and (67), are of the same order of magnitude at $\eta = \eta_2$ for all wavelengths. The coefficients C_1 , C_2 are smooth for long waves, $d \approx y_2 \ll 1$, $n \ll n_c$:

$$C_1 \approx \frac{1}{2a_1} \sqrt{2} \mu(\eta_1) , \quad C_2 \approx -\frac{6}{5a_1} \left(\frac{n}{n_c} \right)^2 \sqrt{2} \mu(\eta_1) , \quad (84)$$

and are oscillating for shorter waves, $n > n_c$ (Sakharov oscillations). At a series of frequencies, the factor $\sin d$ is zero, and the growing component totally vanishes: no gravitational field perturbations, no time derivatives of the perturbations, no energy density perturbations. These modes were highly excited, as others, at the end of the i stage, but were stripped off of their energy by the very late times of their evolution. In terms of quantum mechanics, one can say that these modes have been desqueezed, sent back to the vacuum state [23]. The position of zeros is determined by $\frac{n}{\sqrt{3}}(\eta_2 - \eta_1) = \pi k$, $k = 1, 2, 3 \dots$ or, approximately, by $\frac{n}{n_c} = \pi k$. If one defines the distance x traveled by sound waves between the barriers at $\eta = \eta_1$ and $\eta = \eta_2$ by $x = a\frac{1}{\sqrt{3}}(\eta_2 - \eta_1)$, the zeros arise when x is covered by an integer number of half-waves, $x = \frac{\lambda}{2} k$. The first zero in the spectrum of the growing component arises at $k = 1$ which corresponds to the present-day scale of the order of 70 Mpc.

The numerical values of C_1 and C_2 as functions of n are controlled by the n -dependent function $\mu(\eta_1)$. For simple scale factors (6) and solutions (40), $\mu(\eta_1)$ is given by Eq. (81) where the value of A_1 is determined by quantum mechanics, as will be discussed in Sec. VIII. The function $\mu(\eta_1)$ is exactly the same as the one used for gravitational wave calculations [17]. This allows us to make certain comparisons of density perturbations with gravitational waves. For instance, the growing component of gravitational waves taken at the beginning of the m stage, $h_{\text{gw}}(\eta_2)$, has the following amplitude in the low frequency limit, $n \ll n_c$:

$$h_{\text{gw}}(\eta_2) \approx \frac{3\sqrt{3\pi}}{\sqrt{2}} \frac{1}{a_1} \mu(\eta_1) .$$

This number is $3\sqrt{3\pi}$ times larger than C_1 , Eq. (84),

which gives the amplitude h for density perturbations in the same limit. We can also compare the growing component of h_l with the growing component of gravitational waves. Let us take $\eta = \eta_R$ and $n \ll n_H$ where $n_H \equiv 4\pi/(\eta_R - \eta_m)$ corresponds to the wavelength equal to the Hubble radius at $\eta = \eta_R$. One can derive

$$h_l(\eta_R) \approx \frac{8\pi\sqrt{\pi}}{15} \left(\frac{n}{n_H} \right)^2 h_{\text{gw}}(\eta_R)$$

As we see, gravitational waves and density perturbations "enter" the (time-dependent) Hubble radius with approximately equal amplitudes, regardless of the numerical values of parameters l_0 , β describing the i stage.

According to our definitions, see Sec. VIII, the Fourier component of the quantized field includes the factor $l_{P1}/\sqrt{2n}$ in addition to $h(\eta)$ or $h_l(\eta)$. We can give an estimate for the "characteristic" amplitude $h(n) \sim nl_{P1}C_1$ of the h field, which is a substitute for a more rigorously defined expectation value of the dispersion (square root of the variance) of the field per logarithmic frequency interval. Combining Eqs. (84) and (81) we can find in the low frequency limit $n \ll n_c$: $h(n) \sim (l_{P1}/l_0)n^{\beta+2}$, i.e., exactly the same behavior as for gravitational waves. The growing components of $h_l(\eta)$ and $\delta\epsilon/\epsilon_0(\eta)$ contain the additional factor $n^2(\eta - \eta_m)^2$ which gives two extra powers of n in their spectra. There is nothing spectacular about the de Sitter case $\beta = -2$. The derivative of the Hubble parameter can be arbitrarily close to zero at the time when the wavelength of our interest leaves the Hubble radius at the i stage. The perturbation will have a finite, not infinite, amplitude today.

VII. DENSITY AND ROTATIONAL PERTURBATIONS IN THE HIGH-FREQUENCY LIMIT

The normalization of the perturbations is determined by quantum mechanics. We intend to amplify the zero-point quantum fluctuations of the primeval matter which is, in our case, the scalar field (1). Before the amplification, the frequencies of the fluctuations were much higher than the frequency of the gravitational pump field. To the modes of our interest the surrounding space-time seemed at the beginning almost flat. As a preparation for quantization, we will first consider density and rotational perturbations in matter placed in the Minkowski space-time, that is when gravity is totally neglected [$a(\eta) = 1$ in Eq. (2)].

The deformation of an elastic medium is usually described [24] with the help of a displacement three-vector $u_i(t, \mathbf{x})$ which can be written as

$$u_i = \xi(t)Q_{,i} + \theta(t)Q_i \quad (85)$$

The scalar function Q is defined by Eq. (8), the vector function Q_i is defined by the equations

$$Q_{i,k}{}^{,k} + n^2 Q_i = 0, \quad Q^i{}_{,i} = 0 \quad (86)$$

The deformation tensor is

$$\begin{aligned} u_{ik} &= \frac{1}{2}(u_{i,k} + u_{k,i}) \\ &= \frac{1}{2}\xi(Q_{,i,k} + Q_{,k,i}) + \frac{1}{2}\theta(Q_{i,k} + Q_{k,i}) \end{aligned}$$

and its trace is $u = u^i{}_{,i} = -\xi n^2 Q$. The stress tensor can be written in the general form

$$\begin{aligned} \sigma_{ik} &= s(Q_{,i,k} + Q_{,k,i}) + \rho c_l^2 \theta(Q_{i,k} + Q_{k,i}) \\ &\quad + n^2(2s - \rho c_l^2 \xi)Q\delta_{ik}, \end{aligned} \quad (87)$$

where ρ is density, c_l is the longitudinal velocity of sound, c_t is the transverse (torsional) velocity of sound, and s is arbitrary function of time. The equations of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_i^k}{\partial x^k}$$

reduce to the oscillatory equations for elastic waves. In terms of η time, they can be written as

$$\xi'' + n^2 \frac{c_l^2}{c^2} \xi = 0, \quad \theta'' + n^2 \frac{c_t^2}{c^2} \theta = 0 \quad (88)$$

We shall now relate the theory of elasticity with the theory of cosmological perturbations. We shall consider the high-frequency limit of the perturbations, that is when the scale factor $a(\eta)$ is almost constant and its variability can be neglected in comparison with frequencies of the waves. As we know, the perturbed components of the energy-momentum tensor for density and rotational perturbations have the general form (see Eq. (10) and Ref. [2])

$$\begin{aligned} T_0^0 &= -\frac{1}{a^2} \epsilon_1 Q, \quad T_i^0 = -T_0^i = \frac{1}{a^2} \xi' Q_{,i} + \frac{1}{a^2} \theta' Q_i, \\ T_i^k &= \frac{1}{a^2} \frac{p_l}{2n^2} \left(Q_{,i}{}^{,k} + Q_{,k}{}^{,i} \right) - \frac{1}{a^2} \frac{\chi}{n^2} \left(Q_{i}{}^{,k} + Q^k{}_{,i} \right) \\ &\quad + \frac{1}{a^2} (p_1 + p_l) \delta_{ik} Q. \end{aligned} \quad (89)$$

The differential conservation laws $T_{\alpha,\beta}^\beta = 0$ can be reduced in the high-frequency limit to the following equations: the $\alpha = 0$ component gives the equation $-\epsilon_1' + \xi' n^2 = 0$, which integrates to

$$n^2 \xi = \epsilon_1 \quad (90)$$

the $\alpha = i$ components give

$$\xi'' + p_1 = 0, \quad \theta'' + \chi = 0 \quad (91)$$

Equations (90) and (91) can, of course, be obtained from the perturbed Einstein equations as well.

The stress tensor σ_i^k is connected with the perturbed components T_i^k by $\sigma_i^k = -\rho c^2 T_i^k$ ($a(\eta) = 1$). From the comparison of Eqs. (87) and (89) one finds

$$p_1 = n^2 \frac{c_l^2}{c^2} \xi, \quad \chi = n^2 \frac{c_t^2}{c^2} \theta, \quad s = -\rho c^2 \frac{p_l}{2n^2}. \quad (92)$$

With these expressions for p_1 , χ , Eqs. (91) coincide with the wave equations (88). In cosmology and theory of elasticity, we are dealing essentially with the same physics.

The quantization of density and rotational perturba-

tions should be based on Eqs. (88). Rotational perturbations have been considered elsewhere [2]. The quantum-mechanically generated rotational perturbations can contribute to the CMBR anisotropy and may be important for the smaller scale astrophysics. (One should be aware, though, that there exists also an alternative view on the subject according to which rotational perturbations are “irrelevant for cosmology” [25].) We will concentrate here on density perturbations. For simple models of matter, such as perfect fluids and scalar fields, the functions χ and p_l vanish. One is left with the single variable ξ and isotropic stresses. The quantization of these oscillations in an elastic material placed in the Minkowski space-time would lead to the notion of phonons.

There is no wonder that in the case of scalar field matter the role of ξ is played by φ_1 . If one writes $\xi(\eta) = \xi_0 e^{-in\eta}$ and $\varphi_1(\eta) = \varphi_{10} e^{-in\eta}$, Eq. (24) gives

$$in\xi_0 = \varphi_0' \varphi_{10} \quad (93)$$

In the high-frequency limit (large n), the first term in Eqs. (23) and (25) dominates. With the help of Eq. (93) one derives $\epsilon_1 = p_1 = n^2 \xi$. In other words, for the high-frequency scalar field perturbations, the velocity of sound is almost equal to the velocity of light. The perturbations behave as massless scalar particles.

It is the scalar field oscillations that should be normalized by ascribing a “half of the quantum” to each mode. Because of the Einstein equations the scalar field perturbations are accompanied by the gravitational field perturbations. In the high-frequency limit, the second term in Eq. (43) can be neglected, the normalization of φ_1 transfers to the gravitational field variable μ and ultimately to h and h_l . This is how we will know the initial amplitude for density perturbations.

$$\delta\varphi(\eta, \mathbf{x}) = \frac{1}{a(\eta)} \sqrt{c\hbar} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{n} \frac{1}{\sqrt{2n}} [c_{\mathbf{n}} e^{-in\eta} e^{i\mathbf{n}\cdot\mathbf{x}} + c_{\mathbf{n}}^\dagger e^{in\eta} e^{-i\mathbf{n}\cdot\mathbf{x}}]. \quad (95)$$

This expression is only valid in the high-frequency limit, when the field can be regarded as free and its nonadiabatic interaction with gravity can be neglected. When the interaction becomes important, the time dependence of the field ceases to be so simple. The operator $c_{\mathbf{n}} e^{-in\eta}$, should be replaced by $c_{\mathbf{n}}(\eta)$, and the evolution of $c_{\mathbf{n}}(\eta)$, $c_{\mathbf{n}}^\dagger(\eta)$, should be found from the Heisenberg equations of motion.

The matter perturbations are accompanied by the gravitational field perturbations. Because of the Einstein equations they are linked together and form, in a sense, a united entity. As we have seen in Sec. III, the entire dynamical problem at the i stage reduces to a single wave equation (38) for a single variable $\mu(\eta)$. All perturbations can be found from a given solution to this equation. It follows from Eq. (43) that $\mu(\eta) \approx \sqrt{2\kappa a} \varphi_1(\eta)$ in the high-frequency limit. Since the positive frequency scalar

VIII. QUANTIZATION OF DENSITY PERTURBATIONS

In the limit of a free massless scalar field placed in the Minkowski space-time, we have, for each mode,

$$\delta\varphi_{\mathbf{k}} = \varphi_1 Q + \varphi_1^* Q^* = \varphi_1(t) e^{i\mathbf{k}\cdot\mathbf{y}} + \varphi_1^*(t) e^{-i\mathbf{k}\cdot\mathbf{y}}$$

The total field can be written as

$$\delta\varphi(t, \mathbf{y}) = C \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{k} \frac{1}{\sqrt{2w_{\mathbf{k}}}} \left[c_{\mathbf{k}} e^{-iw_{\mathbf{k}}t} e^{i\mathbf{k}\cdot\mathbf{y}} + c_{\mathbf{k}}^\dagger e^{iw_{\mathbf{k}}t} e^{-i\mathbf{k}\cdot\mathbf{y}} \right]. \quad (94)$$

The normalization constant C is to be found from the requirement

$$\langle 0 | \int_{-\infty}^{\infty} \epsilon d^3\mathbf{y} | 0 \rangle = \frac{1}{2} \hbar \int_{-\infty}^{\infty} d^3\mathbf{k} w_{\mathbf{k}} \langle 0 | c_{\mathbf{k}} c_{\mathbf{k}}^\dagger + c_{\mathbf{k}}^\dagger c_{\mathbf{k}} | 0 \rangle,$$

where ϵ is the energy density of the field. Since in our case,

$$\epsilon = \frac{1}{2} \left[(\delta\varphi_{,0})^2 + (\delta\varphi_{,1})^2 + (\delta\varphi_{,2})^2 + (\delta\varphi_{,3})^2 \right],$$

we derive $C = c\sqrt{\hbar}$. Obviously, the normalization coefficient C includes the Planck constant \hbar but does not include the gravitational constant.

To write the field operator (94) in the curved space-time (2) one should make the replacements

$$\mathbf{y} = a(\eta)\mathbf{x}, \quad \mathbf{k} = \frac{1}{a(\eta)}\mathbf{n}, \quad w_{\mathbf{k}} = \frac{cn}{a(\eta)},$$

$$c_{\mathbf{k}} = [a(\eta)]^{3/2} c_{\mathbf{n}}.$$

The field operator takes on the form

field \mathbf{n} -mode solution is $\varphi_1(\eta) \sim [1/a(\eta)] \sqrt{c\hbar} c_{\mathbf{n}} e^{-in\eta}$ this leads to $\mu(\eta) \sim 4\sqrt{\pi} l_{\text{Pl}} c_{\mathbf{n}} e^{-in\eta}$. Note that the normalization coefficient includes the gravitational constant and is proportional to the Planck length $l_{\text{Pl}} = (G\hbar/c^3)^{1/2}$.

Having in mind the basic wave equation (38), we can now introduce the “fundamental” scalar field $\Phi(\eta, \mathbf{x})$ which describes the whole quantum system interacting with the gravitational pump field:

$$\Phi(\eta, \mathbf{x}) = 4\sqrt{\pi} l_{\text{Pl}} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{n} \frac{1}{\sqrt{2n}} \left[c_{\mathbf{n}}(\eta) e^{i\mathbf{n}\cdot\mathbf{x}} + c_{\mathbf{n}}^\dagger(\eta) e^{-i\mathbf{n}\cdot\mathbf{x}} \right]. \quad (96)$$

The annihilation and creation operators $c_{\mathbf{n}}(\eta)$, $c_{\mathbf{n}}^\dagger(\eta)$ are governed by the Heisenberg equations of motion

$$\frac{dc_{\mathbf{n}}(\eta)}{d\eta} = -i[c_{\mathbf{n}}(\eta), H], \quad \frac{dc_{\mathbf{n}}^\dagger(\eta)}{d\eta} = -i[c_{\mathbf{n}}^\dagger(\eta), H] \quad (97)$$

The interaction Hamiltonian H is given by

$$H = nc_{\mathbf{n}}^\dagger c_{\mathbf{n}} + nc_{-\mathbf{n}}^\dagger c_{-\mathbf{n}} + 2\sigma(\eta)c_{\mathbf{n}}^\dagger c_{-\mathbf{n}}^\dagger + 2\sigma^*(\eta)c_{\mathbf{n}}c_{-\mathbf{n}}, \quad (98)$$

where the coupling function $\sigma(\eta)$ is $\sigma(\eta) = \frac{i}{2} \frac{(a\sqrt{\gamma})'}{a\sqrt{\gamma}}$. Equation (98) demonstrates explicitly the underlying parametric interaction of the field oscillators with the pump field. This is simply a generalization of a theory previously developed for gravitational waves (for a review, see Ref. [19]).

The common way of solving Eqs. (97) and (98) is to write the operators in the form

$$\begin{aligned} c_{\mathbf{n}}(\eta) &= u_{\mathbf{n}}(\eta)c_{\mathbf{n}}(0) + v_{\mathbf{n}}(\eta)c_{-\mathbf{n}}^\dagger(0), \\ c_{\mathbf{n}}^\dagger(\eta) &= u_{\mathbf{n}}^*(\eta)c_{\mathbf{n}}^\dagger(0) + v_{\mathbf{n}}^*(\eta)c_{-\mathbf{n}}(0), \end{aligned} \quad (99)$$

where $c_{\mathbf{n}}(0)$, $c_{\mathbf{n}}^\dagger(0)$ are the initial values of the operators taken at some $\eta = \eta_0$ long before the interaction became effective, and $[c_{\mathbf{n}}(0), c_{\mathbf{m}}^\dagger(0)] = \delta^3(\mathbf{n} - \mathbf{m})$. The classical complex functions $u_{\mathbf{n}}(\eta)$, $v_{\mathbf{n}}(\eta)$ [do not mix up with the functions $u(\eta)$, $v(\eta)$ introduced in Sec. III] obey the condition $|u_{\mathbf{n}}|^2 - |v_{\mathbf{n}}|^2 = 1$ and satisfy the equations

$$iu_{\mathbf{n}}' = nu_{\mathbf{n}} + 2\sigma v_{\mathbf{n}}^*, \quad iv_{\mathbf{n}}' = nv_{\mathbf{n}} + 2\sigma u_{\mathbf{n}}^*, \quad (100)$$

with the initial data $u_{\mathbf{n}}(0) = 1$, $v_{\mathbf{n}}(0) = 0$. If one introduces $\mu_{\mathbf{n}}(\eta) \equiv u_{\mathbf{n}}(\eta) + v_{\mathbf{n}}^*(\eta)$, it follows from Eqs. (100) that the function $\mu_{\mathbf{n}}(\eta)$ should satisfy precisely Eq. (38). The initial conditions for $\mu_{\mathbf{n}}(\eta)$ in the high-frequency limit $|n\eta| \rightarrow \infty$ are $\mu_{\mathbf{n}}(\eta) \rightarrow e^{-in(\eta-\eta_0)}$, $\mu_{\mathbf{n}}'(\eta) \rightarrow -in e^{-in(\eta-\eta_0)}$.

For each mode \mathbf{n} there exists the vacuum state $|0_{\mathbf{n}}\rangle$ defined by the condition $c_{\mathbf{n}}(0)|0_{\mathbf{n}}\rangle = 0$. As a result of the Schrödinger evolution with the Hamiltonian (98), the initial vacuum state $|0_{\mathbf{n},-\mathbf{n}}\rangle \equiv |0_{\mathbf{n}}\rangle|0_{-\mathbf{n}}\rangle$ transforms into a multiparticle two-mode squeezed vacuum state (see Ref. [19] and references cited therein). In other words, the perturbations (waves) are generated in correlated pairs. The statistical properties of the field are determined by $c_{\mathbf{n}}(\eta)$, $c_{\mathbf{n}}^\dagger(\eta)$. By using Eq. (99) one can rewrite Eq. (96) in the form

$$\Phi(\eta, \mathbf{x}) = 4\sqrt{\pi}l_{\text{Pl}} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{n} \frac{1}{\sqrt{2n}} \left[c_{\mathbf{n}}(0)\mu_{\mathbf{n}}(\eta)e^{i\mathbf{n}\cdot\mathbf{x}} + c_{\mathbf{n}}^\dagger(0)\mu_{\mathbf{n}}^*(\eta)e^{-i\mathbf{n}\cdot\mathbf{x}} \right], \quad (101)$$

where the functions $\mu_{\mathbf{n}}(\eta)$, $\mu_{\mathbf{n}}^*(\eta)$ should be taken with the appropriate initial conditions discussed above. For simple solutions (40), the initial conditions translate into the requirements

$$A_1 = -\frac{i}{\cos \beta\pi} \sqrt{\frac{\pi}{2}} e^{i(n\eta_0 + \frac{\pi\beta}{2})}, \quad A_2 = iA_1 e^{-i\pi\beta} \quad (102)$$

The quantized gravitational field perturbations are expressible entirely in terms of the $\Phi(\eta, \mathbf{x})$ field (96). There are many components of h_{ij} but there is only one sort of creation and annihilation operators. Let us introduce new notations $\overset{1}{h}(\eta) = h(\eta)$, $\overset{2}{h}(\eta) = h_l(\eta)$ and the polarization tensors $\overset{1}{P}_{ij} = \delta_{ij}$, $\overset{2}{P}_{ij} = -n_i n_j / n^2$, and $\overset{1}{P}_{ij} \overset{2}{P}^{ij} = -1$. The field operator $h_{ij}(\eta, \mathbf{x})$ can be written as

$$h_{ij} = 4\sqrt{\pi}l_{\text{Pl}} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{n} \frac{1}{\sqrt{2n}} \sum_{s=1}^2 \overset{s}{P}_{ij}(\mathbf{n}) \left[c_{\mathbf{n}}(0)\overset{s}{h}_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}} + c_{\mathbf{n}}^\dagger(0)\overset{s}{h}_{\mathbf{n}}^* e^{-i\mathbf{n}\cdot\mathbf{x}} \right], \quad (103)$$

where the classical complex functions $\overset{s}{h}_{\mathbf{n}}(\eta)$ should be derived from $\mu_{\mathbf{n}}(\eta)$ through the equations and initial conditions already discussed. A similar expression can be written for the operator of the energy density perturbation $\delta\epsilon/\epsilon_0$. Equation (103) is the starting point for the calculation of the expected angular anisotropy in the CMBR.

IX. VARIATIONS OF THE CMBR TEMPERATURE CAUSED BY DENSITY PERTURBATIONS OF QUANTUM-MECHANICAL ORIGIN

The photons of the CMBR are emitted at $\eta = \eta_E$ and are received by us at $\eta = \eta_R$. A particular direc-

tion of observations is characterized by the unit vector $e^k = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. In the absence of perturbations, the temperature of the CMBR seen in all directions would be the same, T . Gravitational field h_{ij} associated with the density perturbations at the m stage causes a variation of the temperature with respect to the unperturbed value T [26]:

$$\frac{\delta T}{T}(e^k) = \frac{1}{2} \int_0^{w_1} \left(\frac{\partial h_{ij}}{\partial \eta} e^i e^j \right) dw, \quad (104)$$

where $w_1 = \eta_R - \eta_E$, and $\partial h_{ij}/\partial \eta$ is taken in the comoving synchronous coordinate system along the path

$x^k = e^k w$, $\eta = \eta_R - w$. In case of small perturbations, which we are actually dealing with, the emission time η_E can be regarded as being one and the same for all directions. Since the scale factor satisfies the approximate relationship $a(\eta_E)/a(\eta_R) \approx 10^{-3}$, and the η time can be chosen in such a way that $\eta_R - \eta_m = 1$, the quantity w_1 is close to 1. We will use $w_1 = 1$. The wavelength

equal to the present-day Hubble radius l_H corresponds to $n_H = 4\pi$.

For the quantized h_{ij} perturbations, the temperature variation $\delta T/T$, Eq. (104), becomes a quantum-mechanical operator. Since it is only the longitudinal-longitudinal part of h_{ij} that participates in producing $\delta T/T$, we can write

$$\frac{\delta T}{T}(e^k) = -2\sqrt{\pi} l_{P1} \frac{1}{(2\pi)^{3/2}} \int_0^1 dw \int_{-\infty}^{\infty} d^3 \mathbf{n} \frac{(n_i e^i)^2}{n^2} \left[c_{\mathbf{n}}(0) f_n(\eta_R - w) e^{in_{\mathbf{k}} e^k w} + c_{\mathbf{n}}^\dagger(0) f_n^*(\eta_R - w) e^{-in_{\mathbf{k}} e^k w} \right]. \quad (105)$$

where

$$f_n(\eta_R - w) = \frac{1}{\sqrt{2n}} \left. \frac{dh_l(\eta)}{d\eta} \right|_{\eta=\eta_R-w}$$

The individual observed distributions of the CMBR temperature over the sky should be compared with theoretical predictions based on the quantum-mechanical expectation values. The mean value of $\delta T/T(e^k)$ is obviously zero, $\langle 0 | \delta T/T(e^k) | 0 \rangle = 0$. The variance $\langle 0 | \delta T/T(e^k) \delta T/T(e^k) | 0 \rangle$ is not zero but does not depend on the point and direction of observations. To study the angular distribution of the temperature variations one should construct the angular correlation function K for two different directions e_1^k and e_2^k , $e_1^k e_2^i \delta_{ki} = \cos \delta$:

$$K = \left\langle 0 \left| \frac{\delta T}{T}(e_1^k) \frac{\delta T}{T}(e_2^k) \right| 0 \right\rangle.$$

By manipulating with the product of two expressions (105), one can derive

$$K = 4\pi l_{P1}^2 \frac{1}{(2\pi)^3} \int_0^1 dw \int_0^1 d\bar{w} \int_0^\infty n^2 |f_n|^2 dn \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \frac{(n_i e_1^i)^2 (n_i e_2^i)^2}{n^2 n^2} \cos(n_{\mathbf{k}} \zeta^k) d\phi, \quad (106)$$

where $\zeta^k = e_1^k w - e_2^k \bar{w}$. A lengthy calculation of the integrals over the angular variables ϕ , θ gives

$$\begin{aligned} & \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \frac{(n_i e_1^i)^2 (n_i e_2^i)^2}{n^2 n^2} \cos(n_{\mathbf{k}} \zeta^k) d\phi \\ &= 4\pi \sqrt{\frac{\pi}{2}} \left\{ \cos^2 \delta (n\zeta)^{-1/2} J_{1/2}(n\zeta) + (1-5 \cos^2 \delta) (n\zeta)^{-3/2} J_{3/2}(n\zeta) \right. \\ & \quad + 2 \cos \delta (1-\cos^2 \delta) (nw)(n\bar{w})(n\zeta)^{-5/2} J_{5/2}(n\zeta) + 4(3 \cos^2 \delta - 1) (n\zeta)^{-5/2} J_{5/2}(n\zeta) \\ & \quad \left. + 8 \cos \delta (\cos^2 \delta - 1) (nw)(n\bar{w})(n\zeta)^{-7/2} J_{7/2}(n\zeta) + (\cos^2 \delta - 1)^2 (nw)^2 (n\bar{w})^2 (n\zeta)^{-9/2} J_{9/2}(n\zeta) \right\}, \quad (107) \end{aligned}$$

where $\zeta = (w^2 - 2w\bar{w} \cos \delta + \bar{w}^2)^{1/2}$. For further calculations one may use the formula (valid for half-integer ν)

$$(n\zeta)^{-\nu} J_\nu(n\zeta) = \sqrt{2\pi} \sum_{k=0}^{\infty} (\nu+k) \frac{J_{\nu+k}(nw)}{(nw)^\nu} \frac{J_{\nu+k}(n\bar{w})}{(n\bar{w})^\nu} \frac{d^{\nu-1/2}}{dz^{\nu-1/2}} P_{k+\nu-1/2}(z), \quad (108)$$

where $z = \cos \delta$ and $P_l(z)$ are the Legendre polynomials. (I derived and used this formula in the course of studying the gravitational wave [17] and rotational [2] perturbations, but I believe that this formula may exist somewhere in the previously published literature.) With the help of Eq. (108) one can rearrange the correlation

function K to the final expression

$$K = l_{P1}^2 \sum_{l=0}^{\infty} K_l P_l(\cos \delta), \quad (109)$$

where

$$K_l = (2l+1) \int_0^\infty |I_{ln}|^2 dn, \\ I_{ln} = \int_0^n \frac{f_n(\eta_R - x/n)}{x^{1/2}} \left\{ \left[\frac{l(l-1)}{x^2} - 1 \right] J_{\frac{1}{2}+l}(x) + \frac{2}{x} J_{\frac{3}{2}+l}(x) \right\} dx, \quad (110)$$

and $x = nw$. The derived formula for K is general and can be used with arbitrary function f_n . In practice, the limits of integration over n are determined by the frequency interval within which the perturbations were really generated.

As we see, the decomposition of K consists of all multipoles including the monopole, $l = 0$, and dipole, $l = 1$, terms. To carry out the calculations up to a concrete number, we will consider scale factors (6) and solutions (40).

As we know, the growing and decaying components of $h_l(\eta)$ are of the same order of magnitude at $\eta = \eta_2$. However, the decaying component is decreasing since then and can be neglected in the calculation of $\delta T/T$. For the coefficient C_1 responsible for the growing solution, see Eq. (82), we have

$$|C_1|^2 \approx \frac{1}{2l_0^2} |\psi(\beta)|^2 n^{2\beta} n_c^2 \sin^2 \frac{n}{n_c}, \quad (111)$$

where

$$|\psi(\beta)|^2 = \frac{\pi}{2} \left[2^{\beta+\frac{1}{2}} \cos \beta \pi \Gamma \left(\beta + \frac{3}{2} \right) \right]^{-2}, \\ |\psi(\beta)|^2 = 1 \quad \text{for } \beta = -2.$$

The expression for K_l takes on the form

$$K_l = \frac{1}{100l_0^2} |\psi(\beta)|^2 n_c^2 (2l+1) \int_0^\infty n^{2\beta+1} \sin^2 \frac{n}{n_c} (i_{ln})^2 dn, \quad (112)$$

where

$$i_{ln} = \int_0^n \frac{n-x}{x^{1/2}} \left\{ \left[\frac{l(l-1)}{x^2} - 1 \right] J_{\frac{1}{2}+l}(x) + \frac{2}{x} J_{\frac{3}{2}+l}(x) \right\} dx. \quad (113)$$

We will now estimate the contribution of long waves, $n < 1$, to the lower order multipoles K_l . The integrals i_{ln} should be calculated separately for $l = 0$, $l = 1$, and $l \geq 2$, because of the factor $l(l-1)$ in Eq. (113). For $l \geq 2$, the term with this factor dominates. The approximate expression for $(i_{ln})^2$, $l \geq 2$, takes on the form

$$(i_{ln})^2 \approx A_l^2 n^{2l}, \quad (114)$$

where

$$A_l^2 = \left[2^{l+\frac{1}{2}} \Gamma \left(l + \frac{3}{2} \right) \right]^{-2}.$$

For $l = 0$ and $l = 1$, the term with the factor $l(l-1)$ does not contribute to $(i_{ln})^2$. The results still have the form of Eq. (114) but n^{2l} should be replaced by $\frac{1}{100} n^6$ for $l = 1$, and by $\frac{1}{36} n^4$ for $l = 0$. These results are in full agreement with Ref. [13]: in the limit of long waves, the monopole and dipole contributions of an individual wave are suppressed; the monopole component is of the same order of magnitude as the quadrupole component, while the dipole component is further suppressed by an extra power of n .

We should now use $(i_{ln})^2$ for the calculation of K_l . This is where the spectrum of the perturbations comes into play. We can write, for $l \geq 2$,

$$K_l = \frac{1}{100l_0^2} |\psi(\beta)|^2 (2l+1) A_l^2 \int_0^1 n^{2\beta+3+2l} dn, \quad (115)$$

and the appropriate replacements discussed above should be made for $l = 1$, $l = 0$. Since we are working in the limit of small n , the integration over n cannot be extended to the values $n > 1$. However, typically, short waves contribute little to the lower index multipoles. The values of n up to $n \approx 2l$ are more important, but for the purposes of simple evaluation we restrict the integration by $n = 1$.

The suppression of the monopole contributions of individual waves saves us from big trouble. If Eq. (115) were true for $l = 0$, the monopole term K_0 would be power-law divergent in the limit of $n \rightarrow 0$ for all $\beta < -2$. In order not to be in conflict with the finite observed 2.7 K temperature, we would need to resort to the fine-tuned minimally sufficient duration of the i stage, in which case the long waves with this spectrum are simply not being generated. From the correct Eq. (115) follows that the danger of divergence for K_0 and K_2 arises only in models with $\beta < -4$. (The quadrupole anisotropy produced by gravitational waves does also diverge in these models [17].) Thus, the interval $-2 \geq \beta > -4$ is potentially allowed.

We will now introduce the notations $l_{P1}\sqrt{K_0} = M$, $l_{P1}\sqrt{K_1} = D$, $l_{P1}\sqrt{K_2} = Q$, and will compare M , D , and Q . For the quadrupole Q , one can find, from Eq. (115),

$$Q \approx \frac{1}{30\sqrt{5}\pi} \frac{l_{P1}}{l_0} |\psi(\beta)| \frac{1}{\sqrt{\beta+4}}. \quad (116)$$

The monopole M and dipole D are related with Q by

$$\frac{M}{Q} = \frac{\sqrt{5}}{2}, \quad \frac{D}{Q} = \sqrt{\frac{3}{20}} \sqrt{\frac{\beta+4}{\beta+5}}.$$

We do not have an observational access to the unperturbed temperature T , but whatever is the measured Q , we can expect that a correction of about the same magnitude as Q is included in the measured T . The same is true for the dipole component D (there is little doubt, however, that the overwhelming part of the measured dipole anisotropy is accounted for by our peculiar motion).

We will now compare the contributions of density perturbations and gravitational waves to the components K_l of the correlation function K in the long wave limit, $n < 1$. We should compare Eq. (115) with the analogous expression for gravitational waves [17]. The ratio of the

gravity wave contribution $\frac{g}{d}K_l$ to the density contribution K_l has the form

$$\frac{\frac{g}{d}K_l}{K_l} \approx \frac{(l+1)(l+2)(2l+1)^2}{2l(l-1)}.$$

We see that the ratio is independent of the parameters l_0, β describing the i stage. For the quadrupoles $\frac{g}{d}Q$ and Q , we have, in the long wave limit,

$$\frac{\frac{g}{d}Q}{Q} \approx \sqrt{75},$$

that is a somewhat larger contribution of gravitational waves. It is necessary to take into account also the shorter waves in order to get a more accurate estimate.

In conclusion, there are no dimensionless ratios that could be adjusted in such a way that the contribution

of density perturbations to the quadrupole anisotropy would be much larger than the contribution of gravitational waves. These contributions are of the same order of magnitude while numerical coefficients are somewhat in favor of gravitational waves. At the same time, the very generation of density perturbations (and rotational perturbations) is more problematic than the generation of gravitational waves. On these grounds one can conclude that if the observed large-angular-scale anisotropy of the CMBR is caused by cosmological perturbations of quantum-mechanical origin (what else?), they are, most likely, gravitational waves.

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