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Discrete self-similarity and critical point behavior in fluctuations about extremal black holes

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The issues of scaling symmetry and critical point behavior are studied for fluctuations about extremal charged black holes. We consider the scattering and capture of the spherically symmetric mode of a charged, massive test field on the background spacetime of a black hole with charge Q and mass M. The spacetime geometry near the horizon of a $|Q| = M$ black hole has a scaling symmetry, which is absent if $|Q| < M$, a scale being introduced by the surface gravity. We show that this symmetry leads to the existence of a self-similar solution for the charged field near the horizon, and further, that there is a one parameter family of discretely self-similar solutions. The scaling symmetry, or lack thereof, also shows up in correlation length scales, defined in terms of the rate at which the influence of an external source coupled to the field dies off. It is shown by constructing the Green's functions that an external source has a long range influence on the extremal background, compared to a correlation length scale which falls off exponentially fast in the $|Q| < M$ case. Finally it is shown that, in the limit of $\Delta \equiv (1 - Q^2/M^2)^{1/2} \to 0$ in the background spacetime, infinitesimal changes in the black hole area vary like $\Delta^{1/2}$.

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I. INTRODUCTION

Recently, numerical studies of gravitational collapse have shown scaling and critical-point-type behavior in the formation of zero mass, neutral black holes [1,2]. These studies showed several interesting properties; in the zero mass limit, the wave form of the collapsing wave always evolved to a particular form, which was discretely self-similar in appropriate time and space variables. The mass of the formed black hole and a nonanalytic dependence on a variety of parameters measuring the difference in the strength of the wave from some critical value, and the exponent was found to be universal.

Suppose that charged particles collapse to form a black hole. In this case, the black hole must have a mass which is greater than or equal to its charge. Would critical-point-type behavior be seen in fluctuations about the minimal area? Or more generally, would such behavior be seen in the interactions between charged wave packets and an already existing charged black hole, in the extremal limit?

There are two geometrical reasons why this might occur. First, the spacetime geometry near the horizon of a $Q = M$ black hole has an infinite throat (the metric approaches a Robinson-Bertotti metric). The throat has no scale, and the metric has a dilatation symmetry, which means that test fields on this background will have a scale invariance. By contrast, the geometry near the horizon of a nonextremal black hole has a scale set by the surface gravity κ. Second, dust with a mass density equal to its charge density can be placed in arbitrary configurations and will stay in equilibrium with other configurations of such dust, and with arbitrary distributions of charge equal to mass black holes. There is no particular size of charge equal to mass dust that is needed for a force balance. This is reminiscent of the picture of fluctuations on all length scales occurring at a critical point.

In this paper we will focus on the questions of scaling invariance and self-similar solutions, correlation length scales, and how fluctuations in the area of the black hole depend on $\Delta \equiv \sqrt{1 - Q^2/M^2}$, as $\Delta \to 0$. Of course one would like to have exact solutions describing wave packets of charged fields scattering off a charged black hole, analogous to the numerical work [1,2]. Here, in order to make some progress analytically, we will study a charged, massive, test field scattering off a fixed black hole background with charge Q and mass M. This is a consistent approach since (1) we are interested in the limit where the change in the area is infinitesimal, and (2) because there is already a black hole present to do perturbation theory around, unlike the neutral black hole case. We imagine an initial wave packet which heads towards the black hole, part of which is scattered and part of which is captured. One is interested in the form of the captured wave, in particular to see if it shows scaling behavior when the background spacetime approaches extremality. We will first show that near the horizon of an extremal black hole, the wave does have a scaling symmetry when the background spacetime is extremal. Among these solutions there is one which is self-similar, and has a translation invariance in logarithmic time and logarithmic radial coordinates. Further, we will show that near the horizon there exists a set of eigenfunctions of the wave equation which has a discrete self-similarity.

Second, addressing the issue of correlation lengths is a bit confusing—this is not (at least apparently) a sta-

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tistical system with degrees of freedom to average over. However, one can construct the Green's functions for the wave equations, and these tell what the response of the test field is to a source. The Green's functions show that on the extremal background the influence of a source is long range, whereas on a background with $Q < M$, it falls off exponentially fast. Third, the captured part of the wave adds mass and charge to the black hole. We will show that the resulting change in the area of the black hole goes like $\Delta^{1/2}$ in the limit where $\Delta \to 0$.

To avoid repeated absolute value signs, we will take the charge of the black hole to be positive. We will use units with $G = 1$.

II. DESCRIPTION OF THE SYSTEM

Consider a charged scalar field on a Reissner-Nordström background spacetime, with the metric given by

$$ds^2 = \lambda^2(-dt^2 + dy^2) + R^2 d\Omega^2, \quad \lambda^2 = 1 - \frac{2M}{R} + \frac{Q^2}{R^2} . \tag{1}$$

Here y is the usual tortoise coordinate, $dy = dR/\lambda^2, R = R(y)$, and Q and M are the electric charge and mass of the spacetime. Assume that the charge Q is positive. Choose the gauge of the electrostatic potential such that it vanishes on the horizon, $R = R_H$:

$$A_t = \phi_0 - \frac{Q}{R}, \quad \phi_0 \equiv \frac{Q}{R_H} . \tag{2}$$

The quantity ϕ_0 is then the difference in the electrostatic potential between the horizon and infinity. In the context of the first law of black hole thermodynamics, ϕ_0 is conjugate to the electric charge Q. Note that, for $Q = M$,

$$A_t = \lambda = 1 - \frac{M}{R}, \quad \phi_0 = 1 . \tag{3}$$

The matter action, for a scalar field of mass m and charge e, is given by

$$S_m = \int \sqrt{-g}(D_a\Phi D^a\Phi^* + m^2\Phi\Phi^*) , \tag{4}$$

where the gauge-covariant derivative $D_a\Phi = (\partial_a - ieA_a)\Phi$. The equation of motion for the scalar field is

$$\nabla_a\nabla^a\Phi - 2ieA_a\nabla^a\Phi - (m^2 + e^2A_aA^a)\Phi = 0 . \tag{5}$$

We will consider spherically symmetric waves on the background (1). Let $\psi = R(y)\Phi$. Then the equation of motion for the field (5) becomes

$$\partial_y^2\psi - \partial_t^2\psi + 2ieA_t\partial_t\psi + (e^2A_t^2 - m^2\lambda^2 - V_{\rm grav})\psi = 0 , \tag{6}$$

with the potential $V_{\rm grav}$ given by

$$V_{\rm grav} = \frac{\partial_y^2 R}{R} = \frac{2M}{R^3}\lambda^2\left(1 - \frac{Q^2}{MR}\right) . \tag{7}$$

Last, we summarize the behavior of the system near the horizon. In the tortoise coordinates, the black hole horizon $R = R_H$ is at $y \to -\infty$. One finds near the horizon that, for $Q = M$,

$$R - M \to -\frac{M^2}{y} \text{ and } A_t = \lambda \to -\frac{M}{y} , \tag{8}$$

while, for $Q < M$,

$$R - R_H \to R_H e^{2\kappa y}, \quad \lambda^2 \to 2\kappa R_H e^{2\kappa y}, \quad A_t \to \phi_0 e^{2\kappa y} , \tag{9}$$

where $\kappa = \frac{1}{2}\partial_R\lambda^2|_{R_H}$ is the surface gravity at the horizon, which vanishes for the extremal black hole.

III. SCALING BEHAVIOR AND SELF-SIMILAR SOLUTIONS NEAR THE HORIZON OF EXTREMAL BLACK HOLES

From the asymptotic behaviors of the gauge potential and metric function, it follows that the wave equation (6) for the scalar field is invariant under the rescaling

$$y \to ay, \quad t \to at, \quad a = \text{const.} \tag{10}$$

This implies that there are solutions to (6) which are functions only of the ratio t/y. Equivalently, in terms of logarithmic coordinates $\bar{t} = \ln t$, $\bar{y} = \ln(-y)$, there are solutions of the from $\psi = F(\bar{t} - \bar{y})$. These solutions have the translation invariance, corresponding to self-similarity in the original t, y coordinates, $\psi(\bar{t} + D, \bar{y} + D) = \psi(\bar{t}, \bar{y})$ for any D. By contrast, the field equation on the nonextremal black hole spacetime does not have scaling invariance—the surface gravity κ introduces a scale.

The scaling invariance is a reflection of an additional dilatation symmetry of the metric near the horizon, $ds^2 \to \frac{M^2}{y^2}(-dt^2 + dy^2) + M^2 d\Omega^2$, with the dilatation Killing vector $\xi^a = t(\frac{\partial}{\partial t})^a + y(\frac{\partial}{\partial y})^a$. The symmetry is actually best seen in a slightly different set of coordinates, in which the metric function is used as a coordinate. Let $x = 1 - M/R$. Then the wave equation (5) becomes

$$-\partial_t^2\Phi + 2iex\partial_t\Phi + x^2\frac{(1-x)^4}{M^2}\partial_x(x^2\partial_x\Phi) + x^2(e^2 - m^2)\Phi = 0 . \tag{11}$$

As $x \to 0$ the horizon is approached and the differential equation becomes invariant under transformations of the form $x \to \frac{1}{a}x$, $t \to at$, with $a = \text{const}$. The utility of the coordinate x is that it is consistent to include the term $(e^2 - m^2)\Phi$. In the tortoise coordinates above it is not clear that one can legitimately retain this term, while ignoring higher-order terms in the inversion relation between y and R.

Now let $w = xt$ and look for solutions of the form

$$\Phi(x,t) = t^{i\nu} F_\nu(w) \ . \qquad (12)$$

In terms of the logarithmic time $\bar{t}$ introduced above, the prefactor is $e^{i\nu\bar{t}}$, and so the solution will have a discrete self-similarity $\bar{t} \to \bar{t} + \frac{2\pi}{\nu}$ and $\bar{x} = -\ln x \to \bar{x} + \frac{2\pi}{\nu}$. For the eigenvalue $\nu = 0$ this is the continuously self-similar solution discussed above, depending only on w.

The wave equation (11) becomes, for $x \ll 1$,

$$\left(1 - \frac{w^2}{M^2}\right) F_\nu''(w) + 2\left(-ie - \frac{w}{M^2} + \frac{i\nu}{w}\right) F_\nu'$$
$$+ \left(m^2 - e^2 + \frac{2e\nu}{w} - \frac{\nu^2 + i\nu}{w^2}\right) F_\nu = 0 \ . \qquad (13)$$

Analyzing the solutions to this equation is a topic for future study; however, we have shown that the wave equation for a massive charged field on the background spacetime of an extremal black hole has discretely self-similiar solutions, as the horizon is approached. It is worth noting that for $\nu = 0$ the differential equation becomes

$$\frac{d}{dw}\left[\left(1 - \frac{w^2}{M^2}\right) f'(w) - 2ief\right] - (e^2 - m^2) f = 0 \ . \qquad (14)$$

For $e = m$ it is simple to find the solution

$$f(w) = C_1 + C_2 \left(\frac{M+w}{M-w}\right)^{ieM} \ . \qquad (15)$$

In terms of the tortoise coordinate y this is

$$\Phi = C_1 + C_2 \left(\frac{1 - t/y}{1 + t/y}\right)^{ieM} \quad \text{as } y \to -\infty \ . \qquad (16)$$

A general solution to the wave equation near the horizon can be written as a sum of the eigenfunctions (12). However, an arbitrary sum will no longer be discretely self-similar. So, perhaps the most interesting question is the following: do generic wave packets starting in the flat region (or any packets for that matter) evolve into a packet which is a special sum of the modes (12), such that the sum is discretely (or continuously) self-similar? We do not yet know the answer to this, but one can imagine at least two ways in which this could happen. When the eigenvalue problem is solved with appropriate boundary conditions for a wave packet incoming from the flat region, it may be that the eigenvalues ν are actually quantized, say $\nu_n = n\frac{C}{M}$. Then the solution would have a discrete self-similarity with $D = \frac{2\pi M}{C}$. Alternatively, it could be that the imaginary part of the frequency ν is positive, so that the lowest frequency dominates at late times. Or, it may be that the evolution from wave packets "at infinity" is not self-similar. It will certainly be of interest to resolve this question.

IV. SCATTERING AND ASYMPTOTIC SOLUTIONS

The scaling symmetry can be displayed in terms of the Green's function for the scalar field equation. The Green's function describes how the field propagates in response to an external source. We will show that the Green's function for the $Q = M$ case has long range $1/y$ correlations, compared to the Green's function for the $Q < M$ case, in which correlations die off exponentially with the scale κ^{-1}. To this end, we will find eigenmodes of the wave equation in the asymptotic regimes, and use the solutions to construct the Green's function. Further, the scattering behavior of the eigenmodes will be analyzed to determine what scatters and what is captured. From this, the change in the horizon area δA will be found, when a small amount of mass and charge is captured. δA has nonanalytic behavior as extremality is approached, whereas, away from $Q = M$, δA is linear in the added mass and charge.

To put the wave equation in the form of a scattering problem, first, Fourier transform in time; let

$$\partial_t \psi = -i(\omega - e\phi_0)\psi \ . \qquad (17)$$

Then (6) becomes

$$\partial_y^2 \psi + [(\omega - e\phi_0)^2 + 2e(\omega - e\phi_0)A_t$$
$$+ e^2 A_t^2 - m^2\lambda^2 - V_{\rm grav}]\psi = 0 \ , \qquad (18)$$

or,

$$\partial_y^2 \psi + [k^2 - V - V_{\rm grav}]\psi = 0, \quad k^2 + m^2 \equiv \omega^2 \ , \qquad (19)$$

where

$$V \to \frac{2}{R}(e\omega Q - m^2 M) - (e^2 - m^2)\frac{Q^2}{M^2}, \quad R \to \infty \ . \qquad (20)$$

At large distances $R \to y$ and the form of the potential V is the same in all cases. $V_{\rm grav}$ falls off like R^{-3}. However, near the horizon there is a qualitative difference for the extremal and nonextremal cases.

For $Q < M$,

$$V \to 2\omega e\phi_0 - e^2\phi_0^2 - m^2 - [2e\phi_0(\omega - e\phi_0) - m^2 2\kappa R_H]e^{2\kappa y} + O(e^{4\kappa y}), \quad y \to -\infty \ . \qquad (21)$$

For $Q = M$,

$$V \to -m^2 - e^2 + 2e\omega + 2e(\omega - e)\frac{M}{y} - (e^2 - m^2)\frac{M^2}{y^2}, \quad y \to -\infty \ . \qquad (22)$$

An analysis of the potentials shows the following qualitative features of the scattering problem: At large y, the potential falls off quite slowly (like y^{-1}). A WKB approximation shows that the transmission is exponentially suppressed if the incident wave is under the barrier. For the purposes of the following discussion then, it will be sufficient to approximate the capture cross section as a step—if the wave is over the barrier, it is captured, and if it is under the barrier, the wave is scattered. Essentially, we are working in a geometrics optics approximation. (However, we know that the approximation is a good one here, because the scattering problem is similar to that in [3,4], in which the scattering is worked out in analytic and numerical detail.)

So what we need is the criterion for an eigenmode to be over the barrier. Now, this is not quite a standard scattering equation, because the height of the potential depends on the incident wave frequency ω. But studying the potential, one finds that to be over the barrier, a wave with frequency ω must satisfy

$$(\omega - e\phi_0) > \frac{m^2}{e}\frac{\Delta + \epsilon}{\sqrt{1-\Delta^2}}, \quad \epsilon > 0 , \tag{23}$$

where

$$\Delta \equiv \sqrt{1 - Q^2/M^2} . \tag{24}$$

Δ is a parameter which measures how close the spacetime is to extremal. ϵ is any number greater than zero, and merely ensures that a wave packet centered on the frequency ω reaches the horizon in finite time. This condition is the same as one finds from analyzing paths of charged particles, which is a much simpler way to see the results.

The scaling symmetry discussed above shows up in the form of the solutions near the horizon, and also in correlation lengths. Of course, this is not a statistical or quantum mechanical system (though the fact that Hawking radiation makes it seem like one is intriguing) so to find correlation lengths we cannot take an ensemble average. However, we can compute the Green's function for the wave equation, which tells how the classical field evolves in response to a general source. Quantum mechanically, the two point correlation function is the Green's function. Next we show via the Green's function that the influence of a source dies off exponentially for $Q < M$, whereas there is a long range tail for $Q = M$.

Let a wave be incident on the black hole, with $\psi \sim \frac{1}{\sqrt{2\omega}} e^{-i(\omega - e\phi_0)t - iky}$ as $y \to \infty$, with $(\omega - e\phi_0)$ satisfying (23). Then as $y \to -\infty$, $\psi \to \sqrt{[k/(\omega - e\phi_0)]} e^{-i(\omega-e\phi_0)(t+y)}$. The normalization follows because the Wronskian of (18) is constant, and using the fact that the amplitude of the captured wave is much greater than the amplitude of the scattered part. This is only the leading term in the asymptotic solution for ψ. More information will be needed, so next we find the leading nontrivial behavior of the wave near the horizon.

For $Q = M$ and for $y \ll -M$, the wave equation becomes

$$\partial_y^2 \psi + \left[(\omega - e)^2 - 2e(\omega - e)\frac{M}{y} \right] \psi = 0 . \tag{25}$$

The inward propagating solution correct through order $\frac{M}{y}$ is

$$\psi_1 = e^{-iS} e^{\frac{eM}{2(\omega - e)y}} , \tag{26}$$

where

$$S = (\omega - e)y + eM \ln(-y) + \frac{e^2 M^2}{2(\omega - e)} \frac{1}{y} . \tag{27}$$

The outward propagating mode is given by

$$\psi_2 = e^{iS} e^{\frac{-eM}{2(\omega - e)y}} . \tag{28}$$

For $Q < M$ and $\kappa y \ll -1$, the wave equation becomes

$$\partial_y^2 \psi + [(\omega - e\phi_0)^2 + P_\omega e^{2\kappa y}]\psi = 0 , \tag{29}$$

where $P_\omega = 2e\phi_0(\omega - e\phi_0) - 2\kappa m^2 R_H$. P_ω must be positive for the wave to actually get over the barrier, by (23). There are two (potentially) small parameters here, κ which goes to zero in the extremal limit, and $(\omega - e\phi_0)$ which we want small to be adding a small amount of mass to the black hole. The regime of interest will be $\kappa \ll (\omega - e\phi_0)$, to approach the extremal black hole, and this is included in the $P_\omega > 0$ range.

The inward propagating solution, correct through terms of order $e^{2\kappa y}$ is

$$\psi_3 = e^{-i(\omega - e\phi_0)y} e^{-Ae^{2\kappa y}} , \tag{30}$$

where

$$A = \frac{P}{4(\omega - e\phi_0)^2 + 2\kappa^2} \left(\frac{2(\omega - e\phi_0)}{2\kappa} + i \right) .$$

For $\kappa \ll (\omega - e\phi_0)$,

$$A \to \frac{e\phi_0}{2\kappa} - \frac{m^2 R_H}{2(\omega - e\phi_0)} + i \left(\frac{e\phi_0}{2(\omega - e\phi_0)} - \frac{2\kappa m^2 R_H}{4(\omega - e\phi_0)^2} \right) .$$

The outward propagating mode is the complex conjugate:

$$\psi_4 = \psi_3^* . \tag{31}$$

As a consistency check, the Wronskian of each of the above solutions $\psi_i(y)$ is constant, as it should be.

The (advanced) Green's function can now be constructed. For $Q = M$ this is

$$G(t,y;t',y') = -\tfrac{1}{2}\Theta(\Delta t)\Theta(-\Delta y)\Theta(\Delta t+\Delta y)e^{-ieM\ln(y/y')}\left\{-\frac{1}{2}(1-ieM)\left(\frac{y'}{y}-1\right)+\exp\left[-i\frac{eM}{y'}(\Delta t+\Delta y)\right]\right.$$
$$\left.\times\exp\left[\tfrac{1}{2}(1-ieM)\left(\frac{y'}{y}-1\right)\right]\right\} - \tfrac{1}{2}\Theta(\Delta t)\Theta(\Delta y)\Theta(\Delta t-\Delta y)e^{ieM\ln(y/y')}\left\{\frac{1}{2}(1-ieM)\left(\frac{y'}{y}-1\right)\right.$$
$$\left.+\exp\left(-i\frac{eM}{y'}(\Delta t-\Delta y)\right)\times\exp\left[-\frac{1}{2}(1-ieM)\left(\frac{y'}{y}-1\right)\right]\right\} . \quad (32)$$

For example, the field response to a Δ function source at y_0, t_0 is

$$\psi(y,t) \approx e^{-ieM\ln(y/y_0)}\left[\frac{1}{4}(1-ieM)\left(\frac{y_0}{y}-1\right)-\frac{1}{2}\exp[-\tfrac{1}{2}(1-ieM)]\exp\left(-i\frac{eM}{y_0}(\Delta t+\Delta y)\right)\left(1+\frac{1}{2}\frac{y_0}{y}\right)\right] . \quad (33)$$

The solution shows the scaling symmetry, and long-range correlations; i.e., the effect of the source falls off like y^{-1}. G (or ψ) has a piece which is independent of time, and a piece which goes to free oscillations at a frequency eM/y_0, which depends on the location of the source point.

By contrast, for $Q < M$ there is no scaling symmetry, and the influence of the source falls off exponentially fast, like $e^{2\kappa y}$. The Green's function has pieces which are oscillations at two frequencies: $\mu \equiv e\phi_0 e^{4\kappa y'}$ and $\kappa\gamma$, where $\gamma \equiv \frac{m^2 R_H}{e\phi_0}$. The first depends on the location of the source point, and the second goes to zero in the extremal limit. Let $h(\Delta y) = e^{2\kappa\Delta y} - 1$. Then the Green's function for $Q < M$ is

$$G(y,t;y',t') = \Theta(\Delta t)\Theta(-\Delta y)\Theta(\Delta t+\Delta y)\tfrac{1}{2}\exp\left(-i\frac{\mu}{2\kappa}h(\Delta y)\right)\left[\mu e^{i\mu(\Delta t+\Delta y)}\exp\left[\tfrac{1}{2}(1-i\gamma)h(\Delta y)\right]-\frac{\kappa\gamma}{\mu^2}e^{-i\kappa\gamma(\Delta t+\Delta y)}\right.$$
$$\left.\times\exp^{\left(-\frac{\mu(1-i\gamma)}{2\kappa\gamma}h(\Delta y)\right)}\right] + \Theta(\Delta t)\Theta(\Delta y)\Theta(\Delta t-\Delta y)\tfrac{1}{2}\exp\left(i\frac{\mu}{2\kappa}h(\Delta y)\right)$$
$$\times\left\{\mu e^{i\mu(\Delta t-\Delta y)}\exp\left[\tfrac{1}{2}(1+i\gamma)h(\Delta y)\right]-\frac{\kappa\gamma}{\mu^2}e^{-i\kappa\gamma(\Delta t+\Delta y)}\exp\left(-\frac{\mu(1+i\gamma)}{2\kappa\gamma}h(\Delta y)\right)\right\} . \quad (34)$$

For example, the field configuration at large negative values of y, due to a point source, can be read off from the first two lines, showing that ψ approaches free oscillations exponentially fast.

V. CRITICAL POINT BEHAVIOR?

For the formation of a mass, neutral black hole, the numerical studies [1,2] looked at how the mass of the new black hole varied with the parameters of the incident wave. It was found that this behavior was nonanalytic (and universal). When forming a charged black hole, the relevant quantity may be fluctuations about the minimal area. In the present context, let us look at infinitesimal changes in the area of an already existing black hole, in the limit where the black hole approaches extremal. (This is the analogue of the approach to the putative critical point.) For a black hole with general charge and mass, the horizon radius is $R_H = M(1+\Delta)$ and the area is $4\pi R_H^2$. If small amount of mass δM and charge δQ are added, the change in the horizon radius is

$$\delta R_H = \delta M + M\left(\Delta^2 + 2\frac{\delta M}{M} - 2\frac{Q}{M}\frac{\delta Q}{M}\right)^{1/2} - M\Delta , \quad (35)$$

where terms of order $\delta M^2, \delta Q^2$ have been neglected. For a wave carrying $\delta Q = e$ to be captured, as discussed in (23), it must have

$$\delta M \equiv \omega = e\phi_0 + \frac{m^2}{e}(\Delta+\epsilon) , \quad (36)$$

where $\epsilon \to 0$ to add the minimal possible mass. Now there are two cases. If one fixes Δ and then considers $\delta M, \delta Q \ll \Delta$, then as expected, one finds a formula for the change in radius which is linear in the perturbations to the mass and charge:

$$\delta R_H \approx e\left(\phi_0 - \frac{Q}{M\Delta}\right) + m\frac{m}{e}(\Delta+\epsilon)\left(1+\frac{1}{\Delta}\right) . \quad (37)$$

On the other hand, for the case of interest here, m and e are still small compared to M and Q, but in addition,

$\Delta \to 0$. Precisely, for $\Delta \ll m^2/(Me)$,

$$\delta R_H \approx e + \sqrt{\frac{2Mm^2}{e}} \left[\left(1 - \frac{e^2}{m^2}\right) \Delta + \epsilon \right]^{1/2} . \quad (38)$$

Therefore, in the limit where the change in the area is as small as possible ($\epsilon \to 0$), the variation in the horizon area has nonanalytic behavior, as the extremal background is approached. Of course, (38) could be written as linear in δ, where $\delta = (1 - \frac{Q^2}{M^2})^{1/4}$. Here Δ appeared to be natural choice because the horizon area is polynomial in Δ. The suggestion in (38) is that tuning through the background spacetimes as $\Delta \to 0$ is like tuning the magnetic field to a critical value.

The first law states that $\delta M = \frac{\kappa}{8\pi}\delta A + \phi_0 \delta Q$. κ and ϕ_0 play the roles of the temperature and an electric (or chemical) potential, so derivatives of δR_H with respect to κ, ϕ_0 are also of interest. Instead of using M and Δ as the two independent variables to describe the state of the system, we switch to the variables $\kappa = \frac{2\Delta}{M(1+\Delta)^2}$ and $\phi_0 = \frac{\sqrt{1+\Delta^2}}{1+\Delta}$. Then, for example, the analogue of the specific heat is

$$\kappa \left(\frac{\partial \delta A}{\partial \kappa}\right)_{\phi_0} = 4\pi\kappa R_H \left(\frac{\partial \delta R_H}{\partial \kappa}\right)_{\phi_0} = -2\pi R_H \sqrt{\frac{2Mm^2}{e}} \left[\left(1 - \frac{e^2}{m^2}\right) \Delta + \epsilon \right]^{1/2} . \quad (39)$$

VI. DISCUSSION

Consider the class of spacetimes consisting of a charged black hole, parametrized by M and Δ, interacting with charged matter. We have been looking at various properties of this system, that suggests that the point $\Delta = 0$ is like a critical point. To examine this, we move away from this point (look at $Q < M$ spacetimes) and probe the system with charged test fields, to see the behavior as $\Delta \to 0$. It was seen that fluctuations in the area of the black hole have nonanalytic behavior in this limit. We showed that correlation lengths, defined in terms of the classical Green's function, are long range on the $\Delta = 0$ background, and decay exponentially for spacetimes with $\Delta > 0$. Further, we showed that spherically symmetric packets of the test field evolve to configurations which have a scaling symmetry near the horizon, if and only if the background has $\Delta = 0$. Near the horizon, the eigenmodes can be chosen such that each mode has a discrete self-similarity. This is interesting, because the numerical studies of formation of neutral black holes showed that the collapsing field was discretely self-similar, near the critical point. In the present study we do not know if the same is true; does an incident wave packet evolve into a special sum of the eigenmodes, such that the sum has a discrete self-similarity. This is an interesting open question.

Fluctuations about zero mass is a limiting case of fluctuations about the minimal area (irreducible mass), when the black hole is neutral. This would suggest that a key feature to criticality is extremality. However, there is another possibility which is interesting to think about. A black hole with $Q = M$ in a spacetime with positive cosmological constant is not extremal. However, it does have the property that the surface gravity of the black hole is equal in magnitude to the surface gravity of the de Sitter Cauchy horizon, i.e., the two temperatures are the same. Geometrically, the spacetime geometry has an infinite throat near the horizon of the black hole, similiar to the geometry of the throat discussed here. Therefore, one might expect that the behavior of a charged test field would be the same as in the present case. If this is true, then the key ingredient would be equality of the Hawking temperature with the background.

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