Microcanonical fermionic average method in the Schwinger model: A realistic computation of the chiral condensate

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The microcanonical fermionic average method has been used so far in the context of lattice models with phase transitions at finite coupling. To test its applicability to asymptotically free theories, we have implemented it in two-dimensional QED, i.e., the Schwinger model. We exploit the possibility, intrinsic to this method, of studying the whole β, m plane without extra computer cost, to follow constant physics trajectories and measure the $m \to 0$ limit of the chiral condensate. We recover the continuum result within three decimal places. Moreover, the possibility, intrinsic to the method, of performing simulations directly in the chiral limit allows us to compute the average plaquette energy at $m = 0$, the result being in perfect agreement with the expected value.

The microcanonical fermionic average (MFA) method for performing lattice simulations with dynamical fermions [1] is ideally suited for discussing the phase structure of theories with phase transitions at finite couplings, and it has been applied so far in this context [2, 3].

The conventional wisdom, however, requires that physically interesting theories are asymptotically free like QCD. It is then interesting to test the applicability of the MFA method to a theory without phase transitions at finite coupling [4]. In this paper we present an analysis of the Schwinger model on the lattice. Strictly speaking, the Schwinger model in the continuum is not asymptotically free, since it is super-renormalizable and the Callan-Symanzik β function vanishes. However, in the lattice version, since the continuum coupling is dimensionful, the continuum theory is reached at infinite lattice coupling, much in the same way as four-dimensional asymptotically free theories such as QCD.

The continuum model is confining; it is exactly solvable at zero fermionic mass, so that we can compare the results of our simulations with exact ones. This check has become standard for any proposal for simulating dynamical fermions in lattice gauge theories.

We have simulated the (unquenched) model in lattices ranging from 16^2 to 150^2; we present here results for the average plaquette and for the chiral condensate, in the nonsymmetric ($\theta = 0$) vacuum of the model.

The evaluation of the chiral condensate has been made easier by the fact that, in the MFA approach, the main computer cost resides in the evaluation of an effective fermionic action at fixed pure gauge energy by evaluating *all* the eigenvalues of the fermionic matrix at $m = 0$. It is then essentially possible, at no extra cost, to move in the plane β, m to follow constant physics trajectories in approaching the correct continuum limit. This is easier in this model since here the renormalization group amounts to simple dimensional analysis, and constant physics lines are exactly known. This approach differs from the way the chiral condensate is generally computed, which requires arbitrary extrapolations in the lattice mass; in this model a direct comparison of the two methods is possible.

The MFA method is fully described in [1]. Starting from the partition function in terms of the total action $S = S_F + S_G$, the sum of the fermionic and pure gauge contributions, we define the density of states at a fixed pure gauge action (i.e., Euclidean energy)

$$N(E) = \int DU\delta(S_G(U) - VE) \tag{1}$$

and an effective fermionic action through

$$\begin{aligned} e^{-S^F_{\rm eff}(m,n_f,E)} &= \langle \det \Delta^{\frac{n_f}{2}} \rangle_E \\ &= \frac{\int DU \det \Delta^{\frac{n_f}{2}} \delta(S_G(U) - VE)}{N(E)}, \end{aligned} \tag{2}$$

which is the microcanonical average of the fermionic determinant.

In terms of the effective action the partition function can thus be rewritten as

$$\mathcal{Z} = \int dEN(E) e^{-\beta VE - S^F_{\rm eff}(m,n_f,E)}. \tag{3}$$

Massless electrodynamics in $1 + 1$ dimensions is confining, super-renormalizable and exactly solvable.

Its partition function is

$$\mathcal{Z} = \int DA_\mu D\bar{\psi} D\psi e^{\int d^2x[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}\not{D}\psi]} \tag{4}$$

with the usual definitions of $F_{\mu\nu}$ and $\not{D}$. The electric charge is dimensionful in this model.

The partition function (in the bosonic sector) can be rewritten as [5]

$$\mathcal{Z} = \int DA_\mu e^{\int d^2x[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{e^2}{2\pi}A_\mu A_\mu]}, \tag{5}$$

i.e., as that of a theory of free massive vector bosons of mass $M = \frac{e}{\sqrt{\pi}}$. In particular the Green's functions of purely bosonic operators are the same in both theories. This fact can be exploited for obtaining the average plaquette in the lattice (see later).

As for the chiral properties of the model, the chiral current is anomalous. If the chiral limit is obtained from $m \neq 0$, then the $\theta = 0$ vacuum is selected. In this vacuum the chiral condensate is (with one flavor)

$$\frac{1}{e}\langle\bar{\psi}\psi\rangle_c = \frac{e^{\gamma_e}}{2\pi\sqrt{\pi}} = 0.159\,95, \tag{6}$$

while it diverges at zero flavor (i.e., the quenched limit) and is zero with two flavors. This is the value of the chiral condensate to be compared with the results of lattice simulations, where its chiral limit is obtained from $m \neq 0$.

In the present simulation the pure gauge part is described in terms of noncompact fields, while for the fermionic gauge term we use the standard staggered formulation with n_f species.

Since the continuum theory is equivalent to a theory of a free, massive vector boson, the average plaquette of the Schwinger model can be compared with that of the vector boson, which can be exactly computed on a finite lattice:

$$\langle E\rangle_L = \frac{1}{2V}\sum_{p_1 p_4}\frac{2 - \cos p_1 - \cos p_4}{2\beta\sum_\gamma(1-\cos p_\gamma) + M^2} \tag{7}$$

($p_\mu = \frac{2\pi}{Na}k_\mu$) and, for $V \to \infty$,

$$E = \frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\frac{2(1-\cos p_1) + 2(1-\cos p_4)}{M^2 + 2\beta\sum_\gamma(1-\cos p_\gamma)}. \tag{8}$$

The value $M = \frac{1}{\sqrt{\pi}}$ corresponds to the continuum Schwinger model, while the quenched value is

$$E(M=0) = \frac{1}{2\beta}. \tag{9}$$

Since e_c is dimensionful, β explicitly contains the lattice spacing, $\beta = \frac{1}{a^2 e_c^2}$, so that the continuum limit of the theory is approached at $\beta \to \infty$. The limit must be reached keeping fixed the dimensionless ratio $\frac{m_c}{e_c} = \sqrt{\beta}m$. This ratio defines constant physics trajectories.

We have performed simulations in lattices up to 150^2. We present here the results for the 64^2 lattice, where we have the best statistics (for a total of 70 Cray-equivalent hours). We will mainly discuss the one-flavor case.

As stated before, we compute all the eigenvalues of the fermionic matrix. This allows us to compute the effective action for all values of the mass, including $m = 0$. We have done so for 20 values of the energy, from 0.08 to 1.3.

One advantage of the MFA method is that the phase structure of the theory can be inspected directly from the fermionic effective action, whose derivatives must be discontinuous in order to generate a phase transition, at least for small n_f [2], if the underlying pure gauge theory has no transition. In the case of two-dimensional QED (QED_2) the continuum theory is obtained as $\beta \to \infty$ and one does not expect finite β transitions. The effective fermionic action numerically evaluated for the model does not show any sign of nonanalyticity and hence of phase transition.

The average plaquette is obtained as

$$\langle E\rangle_L = \frac{\int dEN(E)Ee^{-\beta VE}e^{-S^F_{\text{eff}}(m,n_f,E)}}{\mathcal{Z}} \tag{10}$$

and can be directly computed at $m = 0$. Since the underlying pure gauge theory is quadratic, the density of states is known analytically,

$$N(E) = C_G E^{\frac{1}{2}V-\frac{3}{2}}, \tag{11}$$

so that the integrals in (10) are simple one-dimensional integrals.

In Fig. 1 we report the value of the average plaquette energy (diamonds), multiplied by 2β to improve the visibility, compared with the exact result for a massive vector model on the lattice. It is important to notice that the Schwinger model is equivalent to a vector model in the continuum. On the lattice, there is no guarantee that the two models are related. From Fig. 1 one can see that at small β, where presumably we are far from the continuum, there is disagreement between the numerical results and the analytical ones. However, already at $\beta \sim 1$ the agreement becomes excellent, showing that, at least for this operator, the continuum physics is reached quickly. Notice also that these results do not rely on any kind of mass extrapolations.

The straight line in Fig. 1 is the quenched value $2\beta\langle E\rangle = 1$, and one can see that, as m increases, the asymptotic value of $\langle E\rangle$ moves towards it.

The chiral condensate

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{n_f V}\frac{\int dEe^{-S_{\text{eff}}}\frac{\partial}{\partial m}S^F_{\text{eff}}}{\int dEe^{-S_{\text{eff}}}} \tag{12}$$

at $m = 0$ vanishes on a finite lattice, so it must be obtained as the limit $m \to 0$. To reach the correct continuum value, this limit has to be taken simultaneously with the $\beta \to \infty$ one, keeping the product $\sqrt{\beta}m$ fixed. This can be easily done with this method, which does not re-

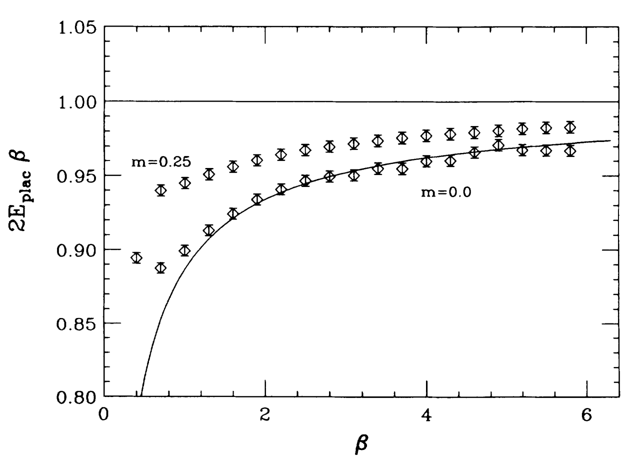

FIG. 1. Average plaquette $64^2, n_f = 1, m = 0$, and $m = 0.25$.

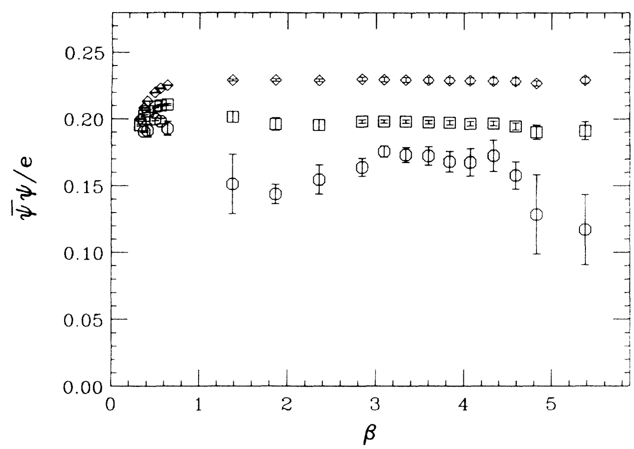

FIG. 2. Chiral condensate, 64^2, $\frac{m_c}{e_c} = 0.01$ (circles), 0.04 (squares), 0.08 (diamonds).

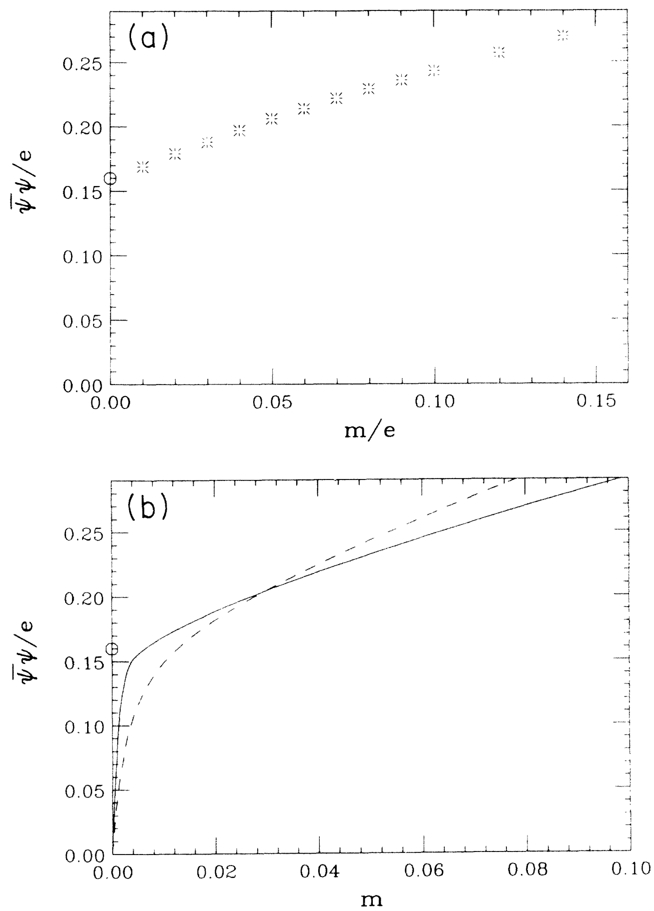

FIG. 3. (a) Chiral condensate vs $\frac{m}{e}, n_f = 1$; errors are smaller than symbols. (b) Chiral condensate vs fermion masses in lattice units ma at $\beta = 3.60$ (continuous line) and $\beta = 5.38$ (discontinuous one).

quire a separate simulation of the fermionic contribution for each pair of parameters (β, m).

In Fig. 2 we report the value of the chiral condensate for three values of the ratio $\frac{m_c}{e_c}$. For relatively large values of this ratio, scaling sets up already near $\beta \sim 1$, but even for a very small value (in this lattice) as $\frac{m_c}{e_c} = 0.01$, where finite spacing and volume effects appear in the small and large β regions, there is a clear scaling window.

We have repeated this procedure for 12 values of $\frac{m_c}{e_c}$, and the values of the chiral condensate in the scaling window so obtained have been reported in Fig. 3(a).

The behavior of the condensate is very clear towards the continuum value, indicated in the figure as a circle. By fitting the points at small $\frac{m_c}{e_c}$ with polynomials we have always obtained consistent results for the intercept:

$$\langle \bar{\psi}\psi \rangle = 0.160 \pm 0.002 \tag{13}$$

in perfect agreement with the theoretical value.

To put in evidence the potentiality of our method of computing the continuum chiral condensate, we have plotted in Fig. 3(b) the chiral condensate against the fermion mass in lattice units, for two typical values of the coupling β in the scaling window. One can conclude from this figure that the value of the chiral condensate computed by mean of mass extrapolation, will be strongly dependent on both the extrapolation function and the mass region used to extrapolate. Deriving the chiral condensate from constant physics lines has the advantage that we can unambiguously extract the value of the observable for each value of $\frac{m_c}{e_c}$, and that the minumum value of the continuum mass (before finite size effects set on) can be easily established looking at the (absence of) scaling window.

From a formal point of view the excellent result obtained for the chiral condensate is also important since it shows that, even with staggered fermions, where it cannot be proven rigorously, the usual introduction of the flavor number through powers of the fermionic determinant is correct; in fact the numerical value for the chiral condensate, which exactly matches the continuum value, is obtained here by taking the square root of the determinant in the partition function.

We have also analyzed the zero- and two-flavor cases. In the zero-flavor limit there is really no scaling region, with the chiral condensate increasing at large β, indicating that it diverges as expected [6]. On the contrary, in the two-flavor case, the behavior of the chiral condensate at finite mass indicates a vanishing value in the chiral limit, again in agreement with expectations [7].

In conclusion, the results we find agree completely with the analytical expectations of the continuum theory. This shows unambiguously the absence of systematic errors, at least for the lattice sizes explored in our numerical research, and hence the reliability of the MFA method. Another important conclusion which follows from this work is that the MFA approach can be also applied to lattice models where the continuum limit is approached at infinite inverse coupling, such as QCD.

It is particularly interesting, in view of more ambitious applications, the ease with which constant physics trajectories can be followed in this approach: in particular, since the mass dependence of the lattice Dirac operator has become trivial, it is possible to move in the β, m parameter space without extra computer cost. It is useful to remember that also the n_f dependence is trivial [2].

This potentiality has been fully exploited in the Schwinger model, where renormalization group amounts to simple dimensional analysis and constant physics trajectories can be exactly defined through the whole parameter space; as a consequence our numerical results for the chiral condensate are quite independent from the extrapolation to zero fermion mass and [as shown in Fig. 3(a)] are by far the best available in the literature. It would be interesting to have a similar detailed simulation of the Schwinger model with other standard methods, such as hybrid Monte Carlo algorithm, in order to compare both accuracy and efficiency.

All the above simulations have been performed on various Transputer networks at L'Aquila University, Zaragoza University (RTN), the bulk on the Transputer Networks of the Theory Group of the Frascati National Laboratories of the INFN. This work has been partly supported through a CICYT (Spain)–INFN (Italy) collaboration.

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