

Pair production in the quantum Boltzmann equation

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A source term in the quantum Boltzmann equation, which accounts for the spontaneous creation of e^+e^- pairs in external electric fields, is derived from first principles and evaluated numerically. Careful analysis of time scales reveals that this source term is generally non-Markovian. This implies in particular that there may be temporary violations of the H theorem.

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I. INTRODUCTION

The evolution of the quark-gluon plasma, believed to be formed in the course of relativistic heavy-ion collisions, is commonly described by means of a transport equation [1–4]. It is well understood how a transport equation can account for acceleration in external fields, scattering, or (hadro)chemical reactions of the microscopic constituents. There is, however, another physical process which becomes increasingly important at high energies: regions of very large chromoelectric field strength may develop and subsequently decay by emitting quark-antiquark pairs [5,6]. This gives rise to the fragmentation of chromoelectric flux tubes (“strings”), a mechanism frequently invoked to model hadron production [7–10]. How such spontaneous creation of particles can be incorporated into a transport equation is still not fully understood.

Clearly, the transport equation has to be modified by a source term. What is this source term? How can it be derived from the underlying microscopic dynamics? These issues have recently been approached in a Wigner function formulation [11–15]. But, aside from the fact that it lacks an intuitive probabilistic interpretation, this approach suffers from several practical limitations. The source term cannot be determined completely: it is not known how the longitudinal momenta of the produced particles are distributed. It has been suggested that the distribution is a δ function [7]; but while such an ansatz may be useful for practical purposes [13–15], it is certainly not exact. Furthermore, an interplay of pair creation and collisions, possibly leading to a modification of the source term, has not yet been considered. And finally, the Wigner description is not suited for discussing the apparent irreversibility of the particle creation process or the associated generation of entropy. Since pair creation in an external field is merely a single-particle problem

(see below), the Wigner function retains complete information about the microscopic state of the system. Yet irreversibility never manifests itself on the microscopic level; it only emerges after a suitable coarse graining.

I choose a different approach. In collision experiments one usually measures the momentum distribution of the outgoing particles; i.e., one determines the occupation $n_{\mp}(\mathbf{p}, t) := \langle N_{\mp}(\mathbf{p}) \rangle(t)$ of the various momentum states, with the number operators given by

$$\begin{aligned} N_{-}(\mathbf{p}) &:= \sum_{m_z} a^{\dagger}(\mathbf{p}, m_z) a(\mathbf{p}, m_z), \\ N_{+}(\mathbf{p}) &:= \sum_{m_z} b^{\dagger}(\mathbf{p}, m_z) b(\mathbf{p}, m_z). \end{aligned} \quad (1)$$

(Here a and b denote particle and antiparticle field operators, respectively, \mathbf{p} the momentum, and m_z the spin component.) This suggests attempting to describe the evolution of the occupation numbers $\{n_{\mp}(\mathbf{p}, t)\}$ directly and to derive a kinetic equation for them, including the source term, from first principles. I will do so with the help of a very powerful and broadly applicable tool: the so-called projection method. This method, pioneered by Nakajima [16], Zwanzig [17–19], and others [20–25], is based on projecting the motion of the quantum system onto a low-dimensional subspace (the “level of description”) of the space of observables (Liouville space). It allows for a clear definition of crucial concepts such as the memory time or the coarse-grained entropy, making it especially suited for an investigation of the irreversible features of the dynamics.

II. MODEL

As in the works cited above, my investigation is based on a simple model from quantum electrodynamics. I consider the creation of e^+e^- pairs in a homogeneous, time-independent electric field \mathbf{E} , a process often referred to as the Schwinger mechanism [26–28]. The starting point is the Dirac equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle, \quad (2)$$

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with

$$H = \hat{\mathbf{p}} \cdot \boldsymbol{\alpha} + m\beta + qA_0 \quad (3)$$

and $A_0(\mathbf{r}) = -\mathbf{E} \cdot \mathbf{r}$ ($q = -|e|$ for electrons). We further define $\mathbf{p}(t) := \mathbf{p} + q\mathbf{E}t$, the transverse energy $\epsilon_{\perp} := \sqrt{m^2 + p_{\perp}^2}$, the total kinetic energy $\epsilon[\mathbf{p}(t)] := \sqrt{\epsilon_{\perp}^2 + p_{\parallel}(t)^2}$, and the dynamical phase

$$\phi_{fi} := -\frac{1}{\hbar} \int_{t_i}^{t_f} dt' \epsilon[\mathbf{p}(t')] \quad (4)$$

“Longitudinal” and “transverse” refer to the direction of the electric field.

The eigenstates $|i, \pm\rangle \equiv |\mathbf{p}(t_i), m_z, \pm\rangle$, which correspond to momentum $\mathbf{p}(t_i)$, spin component m_z , and positive or negative energy $\pm\epsilon[\mathbf{p}(t_i)]$, evolve according to

$$\begin{aligned} U(t_f, t_i) & \begin{pmatrix} |i, +\rangle \\ |i, -\rangle \end{pmatrix} \\ & = \begin{pmatrix} \alpha_{fi} & \beta_{fi} \\ -\beta_{fi}^* & \alpha_{fi}^* \end{pmatrix} \begin{pmatrix} e^{+i\phi_{fi}} & 0 \\ 0 & e^{-i\phi_{fi}} \end{pmatrix} \begin{pmatrix} |f, +\rangle \\ |f, -\rangle \end{pmatrix} \quad (5) \end{aligned}$$

The evolution thus mixes positive and negative energy eigenstates, with respective amplitudes α_{fi} and β_{fi} ; $|\beta_{fi}|^2$ equals the probability for having created an e^+e^- pair with (final) momenta $\pm\mathbf{p}(t_f)$ during the time interval $[t_i, t_f]$. The amplitudes are determined by the differential equation

$$\begin{aligned} \begin{pmatrix} \dot{\alpha}_{fi} \\ \dot{\beta}_{fi} \end{pmatrix} & = \frac{qE}{2} \cdot \frac{\epsilon_{\perp}}{\epsilon[\mathbf{p}(t_f)]^2} \\ & \times \begin{pmatrix} 0 & -e^{-i2\phi_{fi}} \\ e^{+i2\phi_{fi}} & 0 \end{pmatrix} \begin{pmatrix} \alpha_{fi} \\ \beta_{fi} \end{pmatrix}, \quad (6) \end{aligned}$$

with initial conditions $\alpha_{ii} = 1$ and $\beta_{ii} = 0$, and the overdot indicating differentiation with respect to t_f .

In view of applying the projection method, the above results have to be translated into the language of field operators. To do so, I will use the formulation of quantum statistical mechanics in Liouville space [29]. There the evolution of (Heisenberg picture) operators is determined by the so-called superoperators \mathcal{L} (“Liouvillian”) and \mathcal{U} ; these superoperators play a role analogous to that of H and U in Hilbert space. Employing the shorthand notation $a_j \equiv a(\mathbf{p}(t_j), m_z)$ and $b_{-j} \equiv b(-\mathbf{p}(t_j), -m_z)$ for the particle and antiparticle field operators, and making use of the general rule $\mathcal{U}(t_2, t_1)a^\dagger(\psi) = a^\dagger(U(t_2, t_1)\psi)$, one finds

$$\begin{aligned} \mathcal{U}(t_2, t_1) & \begin{pmatrix} a_1^\dagger \\ b_{-1} \end{pmatrix} = \begin{pmatrix} \alpha_{21} & \beta_{21} \\ -\beta_{21}^* & \alpha_{21}^* \end{pmatrix} \\ & \times \begin{pmatrix} e^{+i\phi_{21}} & 0 \\ 0 & e^{-i\phi_{21}} \end{pmatrix} \begin{pmatrix} a_2^\dagger \\ b_{-2} \end{pmatrix}; \quad (7) \end{aligned}$$

the evolution law for (a, b^\dagger) follows by Hermitian conjugation. Thus pair creation can be described by a time-dependent Bogoliubov transformation [30].

The Liouvillian

$$\mathcal{L} = i \left. \frac{\partial}{\partial t_2} \right|_{t_2=t_1} \mathcal{U}(t_2, t_1) \quad (8)$$

may be written as the sum

$$\mathcal{L} = \mathcal{L}_{\text{diag}} + \delta\mathcal{L} \quad (9)$$

of a diagonal part, responsible for acceleration, and an off-diagonal part which is responsible for the mixture of particle and antiparticle states, i.e., for pair creation.

With the definition $\dot{\beta}_{11} := \dot{\beta}_{21} \Big|_{t_2=t_1}$, the latter is given by

$$\delta\mathcal{L} \begin{pmatrix} a_1^\dagger \\ b_{-1} \end{pmatrix} = i \begin{pmatrix} 0 & \dot{\beta}_{11} \\ -\dot{\beta}_{11}^* & 0 \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ b_{-1} \end{pmatrix} \quad (10)$$

III. SOURCE TERM

Starting from the above microscopic equations, we now want to derive a kinetic equation for the occupation numbers $n_{\mp}(\mathbf{p}, t)$. Let us assume that the homogeneous electric field is switched on at time t_0 , and that the initial state is the vacuum [31]

$$\rho(t_0) = |0\rangle\langle 0| \quad (11)$$

Since momentum and charge conservation dictate $n_+(\mathbf{p}, t) = n_-(-\mathbf{p}, t)$ for all later times t , it suffices to consider the evolution of only, say, the electron occupation numbers $n_-(\mathbf{p}, t)$. Their evolution equation must have the structure

$$\dot{n}_-(\mathbf{p}, t) + q\mathbf{E} \cdot \nabla_{\mathbf{p}} n_-(\mathbf{p}, t) = \dot{n}_-^{\text{sou}}(\mathbf{p}, t) \quad (12)$$

with some source term $\dot{n}_-^{\text{sou}}(\mathbf{p}, t)$. Since we know that this source term accounts for transitions between positive and negative energy eigenstates, it is tempting to write down a rate equation of the form

$$\dot{n}_-^{\text{sou}}(\mathbf{p}, +\epsilon, t) = \frac{1}{2} r(\mathbf{p}) \cdot [n(\mathbf{p}, -\epsilon, t) - n(\mathbf{p}, +\epsilon, t)] \quad (13)$$

$r(\mathbf{p})$ being the respective transition rate. If this were correct, the identifications $n(\mathbf{p}, +\epsilon, t) \equiv n_-(\mathbf{p}, t)$ and $n(\mathbf{p}, -\epsilon, t) \equiv 2 - n_+(-\mathbf{p}, t)$ would then lead to

$$\dot{n}_-^{\text{sou}}(\mathbf{p}, t) = r(\mathbf{p}) \cdot S(\mathbf{p}, t) \quad (14)$$

with

$$S(\mathbf{p}, t) := 1 - \frac{1}{2} n_-(\mathbf{p}, t) - \frac{1}{2} n_+(-\mathbf{p}, t) \quad (15)$$

Such a source term, however, can only be correct in the Markovian limit, an approximation which is *not* always justified. Careful investigation [32,33] reveals that the above ansatz for the source term has to be modified: assuming the quasistationary limit ($t_0 \rightarrow -\infty$) one finds

$$\dot{n}_-^{\text{sou}}(\mathbf{p}, t) = \int_0^\infty d\tau R(\mathbf{p}, \tau) \cdot S(\mathbf{p} - q\mathbf{E}\tau, t - \tau) \quad (16)$$

This source term involves an integration over the entire history of the system, thus accounting for finite memory effects and rendering the evolution of the occupation numbers generally *non*-Markovian.

The kernel $R(\mathbf{p}, \tau)$ can be obtained with the help of the projection method [32,33]. One key ingredient in the derivation is the introduction of a superoperator \mathcal{Q} which projects onto the *irrelevant* degrees of freedom; in our case,

$$\mathcal{Q}N_- = \mathcal{Q}N_+ = 0 \quad , \quad (17)$$

whereas other combinations of field operators are unaffected:

$$\mathcal{Q}(a^\dagger b^\dagger) = a^\dagger b^\dagger \quad , \quad \mathcal{Q}(ba) = ba \quad . \quad (18)$$

With this definition one finds

$$R(\mathbf{p}, \tau) = -\langle 0 | \delta\mathcal{L} \exp(i\mathcal{Q}\mathcal{L}\mathcal{Q}\tau) \delta\mathcal{L}N_-(\mathbf{p}) | 0 \rangle \quad , \quad (19)$$

a general result which holds for *arbitrary* field strengths.

The formal expression for the source term can be easily evaluated in the limit of weak fields. Provided $E \ll m^2/\hbar q$, then $|\dot{\beta}| \ll |\dot{\phi}|$ and hence $\delta\mathcal{L}$ may be regarded as a small perturbation. In this case it is legitimate to replace

$$\exp(i\mathcal{Q}\mathcal{L}\mathcal{Q}\tau) \rightarrow \exp(i\mathcal{Q}\mathcal{L}_{\text{diag}}\mathcal{Q}\tau) \quad , \quad (20)$$

leading to

$$\begin{aligned} \dot{n}_-^{\text{sou}}(\mathbf{p}, t) = & 4 \text{Re} \int_0^\infty d\tau \dot{\beta}^*(-\tau, -\tau) e^{-i2\phi(-\tau, 0)} \\ & \times \dot{\beta}(0, 0) S(\mathbf{p} - q\mathbf{E}\tau, t - \tau) \end{aligned} \quad (21)$$

[with $\dot{\beta}(t_2, t_1) \equiv \dot{\beta}_{21}$ and $\phi(t_2, t_1) \equiv \phi_{21}$]. This source

term is consistent with the Schwinger formula: assuming that the system is dilute ($S = 1$), using the differential equation (6) in the limit $\alpha_{fi} \approx 1$, and employing the Landau-Zener formula [34,35], one finds that

$$w := \frac{1}{\hbar^3} \int d^3p \dot{n}_-^{\text{sou}}(\mathbf{p}, t) = \frac{(qE)^2}{4\pi^3\hbar^2} \exp\left(-\frac{\pi m^2}{\hbar qE}\right) \quad , \quad (22)$$

a result which does indeed agree with the leading term in the Schwinger formula.

IV. MOMENTUM DISTRIBUTION

The source term (21) may be evaluated numerically. For simplicity I will take the system to be dilute, $S = 1$; the source term then no longer depends on t . It is convenient to introduce

$$a := \hbar qE/\epsilon_\perp^2 \quad (23)$$

and to consider, rather than the source term itself, the dimensionless quantity

$$\eta(a, p_\parallel/\epsilon_\perp) := \frac{\epsilon_\perp}{qE} \exp\left(\frac{\pi}{2a}\right) \dot{n}_-^{\text{sou}}(\mathbf{p}) \quad . \quad (24)$$

The prefactor in its definition has been chosen such that for $p_\parallel = 0$, η is of the order one. As the prefactor is independent of p_\parallel , η will correctly describe the distribution of the longitudinal momenta of the produced electrons. One can show that

$$\begin{aligned} \eta(a, p_\parallel/\epsilon_\perp) = & \exp\left(\frac{\pi}{2a}\right) \frac{1}{2[1 + (p_\parallel/\epsilon_\perp)^2]} \\ & \times \int_{-\infty}^0 dx \frac{1}{\cosh^3 \varphi(x, p_\parallel/\epsilon_\perp)} \cos\left(\frac{1}{a}x\right) \quad , \end{aligned} \quad (25)$$

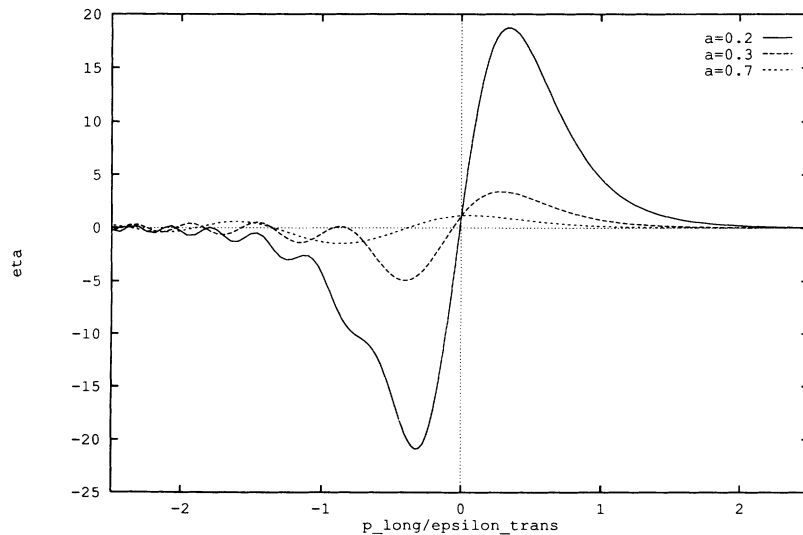


FIG. 1. The rescaled production rate η as a function of $p_\parallel/\epsilon_\perp$ for weak fields ($a = 0.2, 0.3, 0.7$).

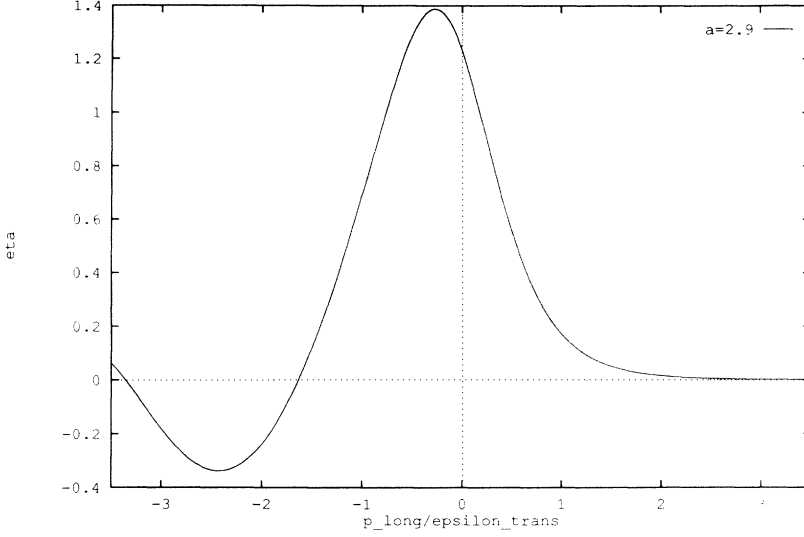


FIG. 2. The rescaled production rate η as a function of $p_{\parallel}/\epsilon_{\perp}$ for a strong field ($a = 2.9$).

with φ defined implicitly as the solution of the equation

$$\sinh \varphi \cosh \varphi + \varphi = x + (p_{\parallel}/\epsilon_{\perp}) \sqrt{1 + (p_{\parallel}/\epsilon_{\perp})^2} + \operatorname{arcsinh}(p_{\parallel}/\epsilon_{\perp}) . \quad (26)$$

I calculated η numerically, using a combination of Filon's integration formula [36] with an efficient root-finding algorithm. The results for weak fields ($a < 1$; Fig. 1) may at first seem surprising. Clearly the momentum distribution of the produced electrons is *not* narrowly peaked around $p_{\parallel} = 0$; it is neither a δ function nor a thermal distribution. Rather, electrons are being produced predominantly in the direction of the external field ($p_{\parallel} > 0$). Electrons moving in the opposite direction ($p_{\parallel} < 0$) are being annihilated: for them, the production rate is negative. Of course, such negative production rates are sensible only if there are electrons available for annihilation. In the quasistationary limit this is the case: electrons which have been emitted with positive momentum are subsequently being decelerated and may then, as soon as $p_{\parallel} < 0$, be (partly) annihilated again; there remains a small surplus which manifests itself as a positive total production rate. As another surprising feature, η displays (approximately) periodic oscillations whose period scales with a . This may be understood qualitatively if one views pair creation as a tunneling process from the negative to the positive energy continuum [7]. The barrier between these continua has a spatial width of the order ϵ_{\perp}/qE , inducing a "momentum quantization" $\Delta p_{\parallel} \sim \hbar qE/\epsilon_{\perp}$ and thus $\Delta(p_{\parallel}/\epsilon_{\perp}) \sim a$. Interference of multiply reflected electron wave functions then leads to the observed oscillations. For a strong field ($a > 1$; Fig. 2), the naive tunneling picture breaks down; both the oscillations and the annihilation of particles (negative rates) become less pronounced.

V. TIME SCALES

As we discussed previously, the source term is generally non-Markovian. It exhibits two characteristic time scales: (i) the memory time $\tau_{\text{mem}}(\mathbf{p})$, which corresponds to the temporal extent of each individual creation process and which indicates how far back into the past one has to reach in order to predict future occupation numbers; and (ii) the production interval $\tau_{\text{prod}}(\mathbf{p})$, the inverse of the production rate, which corresponds to the average time that elapses *between* creation processes and thus constitutes the typical time scale on which the occupation numbers change. Only if $\tau_{\text{mem}} \ll \tau_{\text{prod}}$ can memory effects be neglected and the evolution be considered approximately Markovian.

In the weak-field limit both time scales can be extracted from the source term (21). First the *memory time*: The factor

$$\dot{\beta}^*(-\tau, -\tau) \propto \frac{(\epsilon_{\perp}/qE)}{(\tau - p_{\parallel}/qE)^2 + (\epsilon_{\perp}/qE)^2} \quad (27)$$

constitutes a Lorentz distribution in τ , centered around p_{\parallel}/qE with width ϵ_{\perp}/qE . Significant contributions to the source term thus come from times τ which are smaller than $(p_{\parallel} + \epsilon_{\perp})/qE$. As the typical momentum scale is set by $\Delta p_{\parallel} \sim \hbar qE/\epsilon_{\perp}$, we may conclude

$$\tau_{\text{mem}} \sim \frac{\hbar}{\epsilon_{\perp}} + \frac{\epsilon_{\perp}}{qE} . \quad (28)$$

The memory time combines two time scales of different origin. (i) The time \hbar/ϵ_{\perp} is proportional to \hbar and therefore of quantum mechanical origin. It corresponds (via the time-energy uncertainty relation) to the time needed to create a *virtual* particle-antiparticle pair, and may thus be regarded as the "time between two production attempts." (ii) The time ϵ_{\perp}/qE , on the other

hand, is independent of \hbar and therefore classical. It can be interpreted in various ways, depending on the picture employed to visualize the pair creation process. If pair creation is viewed as a tunneling process, the classical memory time coincides with the time needed for the wave function to traverse the barrier with the speed of light [37]. Alternatively, pair creation may be viewed as a nonadiabatic transition between the two time-dependent energy levels $\pm\epsilon[\mathbf{p}(t)]$. In that case the classical memory time corresponds to the width of the transition region, i.e., the region of closest approach of the two levels. Finding the *production interval* is less straightforward.

Assuming $p_{\parallel} = 0$ for simplicity, again invoking the weak-field limit, and exploiting the fact that the source term must be consistent with the Schwinger formula, one can show that

$$\tau_{\text{prod}}(0, \mathbf{p}_{\perp}) \sim \frac{\epsilon_{\perp}}{qE} \exp\left(\frac{\pi\epsilon_{\perp}^2}{2\hbar qE}\right) . \quad (29)$$

As long as $E \ll m^2/\hbar q \leq \epsilon_{\perp}^2/\hbar q$, the particle creation process is Markovian: $\tau_{\text{mem}} \ll \tau_{\text{prod}}$. In the weak-field limit, therefore, the entropy associated with the coarse-grained level of description spanned by $\{N_{\pm}(\mathbf{p})\}$,

$$S_{\text{CG}}(t) := -2k \frac{\Omega}{h^3} \int d^3p \left[\frac{n_{-}(\mathbf{p}, t)}{2} \ln \frac{n_{-}(\mathbf{p}, t)}{2} + \left(1 - \frac{n_{-}(\mathbf{p}, t)}{2}\right) \ln \left(1 - \frac{n_{-}(\mathbf{p}, t)}{2}\right) \right] + (n_{-} \leftrightarrow n_{+}) , \quad (30)$$

obeys an H theorem. (Ω denotes the volume.) The monotonous increase of the coarse-grained entropy explains why spontaneous pair creation is perceived as irreversible. This apparent irreversibility is, of course, a consequence of the coarse graining: information is being transferred from accessible (slow) to inaccessible (fast) degrees of freedom. The slow degrees of freedom are the occupation numbers of the various momentum states. From these, information gradually “leaks” into unobserved degrees of freedom: correlations and rapidly oscillating phases which entangle the respective wave functions of the members of a particle-antiparticle pair.

As soon as $E \geq m^2/\hbar q$, the situation changes. The source term (21), which was derived in the weak-field limit, is then only a rough estimate. Already this weak-field estimate becomes non-Markovian: at $E = m^2/\hbar q$ the production interval and the memory time are of the same order $\tau \sim \hbar/m$. This is a clear indication that at this point conventional Markovian transport theories must break down. There may be temporary violations of the H theorem: the coarse-grained entropy, while still increasing on average, may now oscillate (on the same scale $\tau \sim \hbar/m$). Such oscillations have indeed been observed in numerical simulations [15]. A systematic study of these memory effects should proceed from the general equations (16) and (19). Although such an enterprise is beyond the scope of this paper, we can already say that (i) memory effects become significant at large-field strengths; and (ii) the projection method can account for these memory effects and thus appears to be a suitable tool for their investigation.

The above analysis can be extended to include binary collisions of the produced particles. This is done by replacing $\delta\mathcal{L} \rightarrow \delta\mathcal{L} + \mathcal{V}$, where \mathcal{V} contains the two-body interaction. To lowest order perturbation theory, pair creation and collisions do not interfere [32]; the additional interaction gives rise to a separate collision term. Like the source term, this collision term is generally

non-Markovian and must be subjected to a time-scale analysis, leading again to a criterion for the validity of the Markovian approximation. One finds that there are two contributions to the memory time: the average time needed for a particle to pass through an interaction range, and the typical “off-shell” time given by the time-energy uncertainty relation. For the Markovian approximation to be valid, these have to be smaller than the average time that elapses between two successive collisions [32,38].

VI. SUMMARY

Let me summarize the main conclusions. (i) The source term in the quantum Boltzmann equation can be derived in an unambiguous fashion by employing the projection method. (ii) To lowest order, the source term is not altered by the presence of collisions. (iii) In the weak-field limit, $E \ll m^2/\hbar q$, the source term is given by (21) or (25), respectively. It is then Markovian, and the coarse-grained entropy increases monotonically. As information is continuously being transferred to inaccessible degrees of freedom, spontaneous pair creation appears irreversible. (iv) But as soon as $E \geq m^2/\hbar q$, there may be sizable memory effects, leading to temporary violations of the H theorem. Their description is beyond the scope of conventional Markovian transport theories. A more suitable starting point appears to be the projection method, in particular Eqs. (16) and (19).

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