Conformal properties of Chern-Simons vortices in external fields

C. Duval*
Département de Physique, Université d'Aix-Marseille II,
and Centre de Physique Théorique, Centre National de la Recherche Scientifique–Luminy, Case 907,
F-13288 Marseille Cedex 09, France

P. A. Horváthy†
Département de Mathématiques, Université de Tours, Parc de Grandmont, F-37200 Tours, France

L. Palla‡
Institute for Theoretical Physics, Eötvös University, H-1088 Budapest, Puskin u. 5-7, Hungary

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The construction and the symmetries of Chern-Simons vortices in harmonic and uniform magnetic force backgrounds found by Ezawa, Hotta, and Iwazaki and by Jackiw and Pi are generalized using the nonrelativistic Kaluza-Klein-type framework presented in our previous paper. All Schrödinger-symmetric backgrounds are determined.

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The construction of static, nonrelativistic Chern-Simons solitons [1] was recently generalized to time-dependent solutions, yielding vortices in a constant external magnetic field $\mathcal{B}$ [2–4]. Putting $\omega = \mathcal{B}/2$, the equation to be solved is

$$i(D_\omega)_t\Psi_\omega = \{-\tfrac{1}{2}\mathbf{D}_\omega^2 - \Lambda\Psi_\omega^\star\Psi_\omega\}\Psi_\omega. \tag{1}$$

(We use units where $e = m = 1$.) Here the covariant derivative means

$$(D_\omega)_\alpha = \partial_\alpha - i(A_\omega)_\alpha - i\mathcal{A}_\alpha \tag{2}$$

(α=0,1,2), where $\mathcal{A}_\alpha$ is a vector potential for the constant magnetic field, $\mathcal{A} = 0, \mathcal{A}_i = \frac{1}{2}\epsilon_{ij}x^j\mathcal{B} \equiv \omega\epsilon_{ij}x^j$ (i,j=1,2), and $(A_\omega)_\alpha$ is the vector potential of Chern-Simons electromagnetism; i.e., its field strength is required to satisfy the field-current identity

$$B_\omega \equiv \epsilon^{ij}\partial_i A_\omega^j = -(1/\kappa)\rho_\omega \quad\text{and}\quad E_\omega^i \equiv -\partial_i A_\omega^0 - \partial_t A_\omega^i = (1/\kappa)\epsilon^{ij}J_\omega^j, \tag{3}$$

with $\rho_\omega = \Psi_\omega^\star\Psi_\omega$ and

$$\mathbf{J}_\omega = (1/2i)[\Psi^\star\mathbf{D}_\omega\Psi_\omega - \Psi_\omega(\mathbf{D}_\omega\Psi_\omega)^\star].$$

These equations can be solved [2–4] by applying a coordinate transformation to a solution Ψ of the problem with $\omega = 0$ studied in Ref. [1], according to

$$\Psi_\omega(t,\mathbf{x}) = \frac{1}{\cos\omega t}\exp\left\{-i\omega\frac{r^2}{2}\tan\omega t\right\} \times \exp\{i(\mathcal{N}/2\pi\kappa)\omega t\}\Psi(\mathbf{X},T),$$
$$(A_\omega)_\alpha = A_\beta\frac{\partial X^\beta}{\partial x^\alpha} - \partial_\alpha\left(\frac{\omega}{2\pi\kappa}\mathcal{N}t\right), \tag{4}$$

with

$$T = \tan\omega t/\omega, \quad \mathbf{X} = (1/\cos\omega t)R(\omega t)\mathbf{x}. \tag{5}$$

Here $\mathcal{N} = \int|\Psi|^2d^2\mathbf{x}$ is the vortex number and $R(\theta)$ is the matrix of a rotation by angle θ in the plane. (The prefactor $\exp[i\mathcal{N}\omega t/2\pi\kappa]$ and the extra term $-\partial_\alpha[(\omega/2\pi\kappa)\mathcal{N}t]$ are absent from the corresponding formula of Ezawa, Hotta, and Iwazaki [2].) A similar construction works in a harmonic background [4].

In this paper we show that the above generalizations arise by reduction from suitable curved spaces. As explained in a previous paper [5], (2+1)-dimensional nonrelativistic Chern-Simons theory can in fact be lifted to "Bargmann space," i.e., to a four-dimensional Lorentz manifold (M,g) endowed with a covariantly constant null vector ξ [5]. Our theory is decribed by a massless nonlinear wave equation

$$\{D_\mu D^\mu - R/6 + \lambda|\psi|^2\}\psi = 0, \tag{6}$$

where $D_\mu = \nabla_\mu - ia_\mu$ (μ=0,1,2,3), ∇ is the metric-covariant derivative, and R denotes the scalar curvature. The scalar field ψ and the "electromagnetic" field strength $f_{\mu\nu} = 2\partial_{[\mu}a_{\nu]}$ are related by the field-current identity $\kappa f_{\mu\nu} = \sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\xi^\rho j^\sigma$, where $j^\mu = (1/2i)[\psi^\star(D^\mu\psi) - \psi(D^\mu\psi)^\star]$.

A Bargmann space admits local coordinates $(t,\mathbf{x},s)$ such that Q, the quotient of M by the integral curves of $\xi = \partial_s$, can be labeled by $(t,\mathbf{x})$. The field strength $f_{\mu\nu}$ is clearly the lift of a closed two-form $F_{\mu\nu}$ on Q. So the vector potential may be chosen as $a_\mu = (A_\alpha, 0)$ with A_α s independent. When supplemented by the equivariance condition

$$\xi^\mu D_\mu\psi = i\psi, \tag{7}$$

our theory projects to a nonrelativistic nonlinear Schrödinger–Chern-Simons theory on the (2+1)-dimensional manifold Q for $\Psi(t,\mathbf{x}) = e^{-is}\psi(t,\mathbf{x},s)$.

*Electronic address: duval@cpt.univ-mrs.fr.
†Electronic address: horvathy@univ-tours.fr
‡Electronic address: palla@ludens.elte.hu

A *symmetry* is a transformation of M which interchanges the solutions of the coupled system. Each ξ-preserving conformal transformation is a symmetry. In Ref. [5] we proved a version of Noether's theorem saying that, for any ξ-preserving conformal vectorfield (X^μ) on Bargmann space, the quantity

$$
\begin{aligned}
Q_X &= \int_{\Sigma_t} \vartheta_{\mu\nu} X^\mu \xi^\nu \sqrt{\gamma}\, d^2\mathbf{x}, \\
\vartheta_{\mu\nu}\xi^\nu &= (1/2i)[\psi^\star(D_\mu\psi) - \psi(D_\mu\psi)^\star] \\
&\quad -\frac{1}{6}\xi_\mu\left(\frac{R}{6}|\psi|^2 + (D^\nu\psi)^\star D_\nu\psi + \frac{\lambda}{2}|\psi|^4\right)
\end{aligned}
\tag{8}
$$

is a constant of the motion. (Here γ_{ij} is the metric induced on it by $g_{\mu\nu}$ on Σ_t, the "transverse space" t=const.)

For example, M can be flat Minkowski space with metric $d\mathbf{X}^2 + 2dT\, dS$, where $\mathbf{X} \in \mathbb{R}^2$ and S and T are light-cone coordinates. This is the Bargmann space of a free, nonrelativistic particle [6]. The above system of equations projects in this case to that of Ref. [1]; the ξ-preserving conformal transformations form the (extended) planar Schrödinger group, consisting of the Galilei group with generators $\mathcal{J}$ (rotation), $\mathcal{H}$ (time translation), $\mathcal{G}$ (boosts), $\mathcal{P}$ (space translations), augmented with the dilatation $\mathcal{D}$ and expansion $\mathcal{K}$, and centrally extended by the "vertical" translation $\mathcal{N}$ [7]. With a slight abuse of notation, the associated conserved quantities are denoted by the same symbols. (Explicit formulas are listed in [1] and [5].) Applying any symmetry transformation to a solution of the field equations yields another solution. For example, a boost or an expansion applied to the static solution $\Psi_0(\mathbf{X})$ of Jackiw and Pi produces time-dependent solutions. Using the formulas in [6], we recover the expression [1]

$$
\begin{aligned}
\Psi(T,\mathbf{X}) = \frac{1}{1-kT}\exp\Bigg\{-\frac{i}{2}\Bigg[&2\mathbf{X}\cdot\mathbf{b} + T\mathbf{b}^2 \\
&+k\frac{(\mathbf{X}+\mathbf{b}T)^2}{1-kT}\Bigg]\Bigg\}\Psi_0\left(\frac{\mathbf{X}+\mathbf{b}T}{1-kT}\right).
\end{aligned}
\tag{9}
$$

Now we present some new results. The most general "Bargmann" metric was found long ago by Brinkmann [8]:

$$
g_{ij}dx^i dx^j + 2dt[ds + \boldsymbol{\mathcal{A}}\cdot d\mathbf{x}] + 2\mathcal{A}_0 dt^2, \quad \mathcal{A}_0 = -U, \tag{10}
$$

where the "transverse" metric g_{ij} as well as the "vector potential" $\boldsymbol{\mathcal{A}}$ and the "scalar potential" U are functions of t and $\mathbf{x}$ only. Clearly, $\xi = \partial_s$ is a covariantly constant null vector. The null geodesics of this metric describe particle motion in curved transverse space in external electromagnetic fields $\boldsymbol{\mathcal{E}} = -\partial_t\boldsymbol{\mathcal{A}} - \boldsymbol{\nabla}U$ and $\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}$ [6].

Consider now a Chern-Simons vector potential $(a_\omega)_\mu = ((A_\omega)_\alpha, 0)$ in the background (10). [The subscript $(\cdot)_\omega$ refers to an external-field problem.] Using that the only nonvanishing components of the inverse of the metric (10) are g^{ij}, $g^{is} = -A^i$, $g^{ss} = 2U + A_iA^i$, g^{st}=1, we find that including the "vector-potential" components into the metric (10) results, after reduction, simply in modifying the covariant derivative D_α in "empty" space ($\mathcal{A}_\alpha = 0$) according to $D_\alpha \to (D_\omega)_\alpha$. The associated equation of motion is hence the obvious curved-space generalization of Eq. (1).

Let now φ denote a conformal Bargmann diffeomorphism between two Bargmann spaces; i.e., let φ: $(M,g,\xi) \to (M',g',\xi')$ be such that $\varphi^\star g' = \Omega^2 g$ and $\xi' = \varphi_\star\xi$. Such a mapping projects to a diffeomorphism of the quotients Q and Q' we denote by Φ. Then the same proof as in Ref. [5] allows one to show that if (a'_μ, ψ') is a solution of the field equations on M', then

$$
a_\mu = (\varphi^\star a')_\mu, \quad \psi = \Omega\varphi^\star\psi' \tag{11}
$$

is a solution of the analogous equations on M. Locally, we have $\varphi(t,\mathbf{x},s) = (t',\mathbf{x}',s')$, with $(t',\mathbf{x}') = \Phi(t,\mathbf{x})$ and $s' = s + \Sigma(t,\mathbf{x})$, so that $\psi = \Omega\varphi^\star\psi'$ reduces to

$$
\begin{aligned}
\Psi(t,\mathbf{x}) &= \Omega(t)e^{i\Sigma(t,\mathbf{x})}\Psi'(t',\mathbf{x}'), \\
A_\alpha &= \Phi^\star A'_\alpha \quad (\alpha = 0,1,2).
\end{aligned}
\tag{12}
$$

Note that φ takes a ξ-preserving conformal transformation of (M,g,ξ) into a ξ'-preserving conformal transformation of (M',g',ξ'). Conformally related Bargmann spaces therefore have isomorphic symmetry groups.

The associated conserved quantities can be related by comparing the expressions in Eq. 8. Using the transformation properties of the scalar curvature R, a short calculation shows that the conserved quantities associated with $X = (X^\mu)$ on (M,g,ξ) and to $X' = \varphi_\star X$ on (M',g',ξ') coincide:

$$
Q_X = \varphi^\star Q'_{X'}. \tag{13}
$$

The labels of the generators are, however, different (see the examples below).

Consider, for example, the Lorentz metric

$$
d\mathbf{x}_{\rm osc}^2 + 2dt_{\rm osc}ds_{\rm osc} - \omega^2 r_{\rm osc}^2 dt_{\rm osc}^2, \tag{14}
$$

where $\mathbf{x}_{\rm osc} \in \mathbf{R}^2$, $r_{\rm osc} = |\mathbf{x}_{\rm osc}|$, and ω is a constant. Its null geodesics correspond to a nonrelativistic, spinless particle in an oscillator background [6,9]. Requiring equivariance (7), the wave equation (11) reduces to

$$
i\partial_{t_{\rm osc}}\Psi_{\rm osc} = \left\{-\frac{\mathbf{D}^2}{2} + \frac{\omega^2}{2}r_{\rm osc}^2 - \Lambda\Phi_{\rm osc}\Psi_{\rm osc}^\star\right\}\Psi_{\rm osc} \tag{15}
$$

($\mathbf{D} = \partial - i\mathbf{A}$, $\Lambda = \lambda/2$), which describes Chern-Simons vortices in a harmonic force background, studied in Ref. [3]. The clue is that the mapping $\varphi(t_{\rm osc}, \mathbf{x}_{\rm osc}, s_{\rm osc}) = (T, \mathbf{X}, S)$ [9], where

$$
T = \frac{\tan\omega t_{\rm osc}}{\omega}, \quad \mathbf{X} = \frac{\mathbf{x}_{\rm osc}}{\cos\omega t_{\rm osc}}, \quad S = s_{\rm osc} - \frac{\omega r_{\rm osc}^2}{2}\tan\omega t_{\rm osc}, \tag{16}
$$

carries the oscillator metric (14) conformally into the free form $d\mathbf{X}^2 + 2dT\,dS$, with conformal factor $\Omega(t_{\rm osc} = |\cos\omega t_{\rm osc}|^{-1}$ such that $\varphi_\star\partial_{s_{\rm osc}} = \partial_S$. Our formula lifts the coordinate transformation of Ref. [4] to Bargmann space.

A solution in the harmonic background can be obtained by Eq. (11). A subtlety arises, though. The mapping (16) is many to one: It maps each "open strip"

$$I_j = \{(\mathbf{x}_{\rm osc}, t_{\rm osc}, s_{\rm osc}) | (j - \tfrac{1}{2})\pi < \omega t_{\rm osc} < (j + \tfrac{1}{2})\pi\}, \tag{17}$$

(where $j = 0, \pm 1, \ldots$), corresponding to a half oscillator period, onto the full Minkowski space. Application of (11) with Ψ an "empty-space" solution yields, in each I_j, a solution, $\Psi^{(j)}_{\rm osc}$. However, at the contact points $t_j \equiv (j + \frac{1}{2})(\pi\omega)$, these fields may not match. For example, for the "empty-space" solution obtained by an expansion, Eq. (9) with $\mathbf{b} = 0$, $k \neq 0$,

$$\begin{aligned}\lim_{t_{\rm osc}\to t_j - 0} \Psi^{(j)}_{\rm osc} &= (-1)^{j+1}\frac{\omega}{k} e^{i/(\omega^2/2k) r^2_{\rm osc}} \Psi_0\left(-\frac{\omega}{k}\mathbf{x}\right) \\ &= -\lim_{t_{\rm osc}\to t_j + 0} \Psi^{(j+1)}_{\rm osc}.\end{aligned} \tag{18}$$

Then continuity is restored by including the "Maslov" phase correction [10]

$$\begin{aligned}\Psi_{\rm osc}(t_{\rm osc}, \mathbf{x}_{\rm osc}) &= (-1)^j (1/\cos\omega t_{\rm osc}) \\ &\quad\times \exp\{-(i\omega/2) r^2_{\rm osc} \tan\omega t_{\rm osc}\} \Psi(T, \mathbf{X}), \\ (A_{\rm osc})_0(t_{\rm osc}, \mathbf{x}_{\rm osc}) &\\ &= \frac{1}{\cos\omega t_{\rm osc}}[A_0(T, \mathbf{X}) - \omega \sin\omega t_{\rm osc} \mathbf{x}_{\rm osc} \cdot \mathbf{A}(T, \mathbf{X})], \\ \mathbf{A}_{\rm osc}(t_{\rm osc}, \mathbf{x}_{\rm osc}) &= (1/\cos\omega t_{\rm osc}) \mathbf{A}(T, \mathbf{X}),\end{aligned} \tag{19}$$

where j is as in (17). Equation (19) extends the result in Ref. [4] from $|t_{\rm osc}| < \pi 2\omega$ to any $t_{\rm osc}$. For the static solution in [1] or for that obtained from it by a boost, $\lim_{t_{\rm osc}\to t_j} \Psi^{(j)}_{\rm osc} = 0$, and the inclusion of the correction factor is not mandatory.

Since the oscillator metric (14) is Bargmann-conformally related to Minkowski space, Chern-Simons theory in the oscillator background has again a Schrödinger symmetry. The generators of this symmetry are $J_{\rm osc} = \mathcal{J}$, $H_{\rm osc} = \mathcal{H} + \omega^2\mathcal{K}$, and $N_{\rm osc} = \mathcal{N}$, already found in Ref. [3], completed by

$$\begin{aligned}&(C_{\rm osc})_\pm = (\mathcal{H} - \omega^2\mathcal{K} \pm 2i\omega\mathcal{D}) \quad \text{and} \\ &(\mathcal{P}_{\rm osc})_\pm = (\mathcal{P} \pm i\omega\mathcal{G}).\end{aligned} \tag{20}$$

Consider next the metric

$$d\mathbf{x}^2 + 2dt[ds + \tfrac{1}{2}\epsilon_{ij}\mathcal{B}x^j dx^i], \tag{21}$$

where $\mathbf{x} \in \mathbf{R}^2$ and $\mathcal{B}$ is a constant. Its null geodesics describe a charged particle in a uniform magnetic field in the plane [6]. Again, when imposing equivariance, Eq. (11) reduces precisely to Eq. (1) with $\Lambda = \lambda/2$ and covariant derivative D_ω given as in Eq. (2). The metric (21) is readily transformed into an oscillator metric (14): The mapping $\varphi(t, \mathbf{x}, s) = (t_{\rm osc}, \mathbf{x}_{\rm osc}, s_{\rm osc})$ given by

$$t_{\rm osc} = t, \quad x^i_{\rm osc} = x^i\cos\omega t + \epsilon^i_j x^j \sin\omega t, \quad s_{\rm osc} = 2, \tag{22}$$

(which amounts to switching to a rotating frame with angular velocity $\omega = \mathcal{B}/2$) takes the "constant-$\mathcal{B}$ metric" (21) into the oscillator metric (14). The vertical vectors $\partial_{s_{\rm osc}}$ and ∂_s are permuted. Thus the time-dependent rotation (22) followed by the transformation (16), which projects to the coordinate transformation (5) of Refs. [2] and [3], carries conformally the constant-$\mathcal{B}$ metric (21) into the $\omega = 0$ metric. It carries therefore the "empty-space" solution $e^{is}\Psi$ with Ψ as in (9) into that in a uniform magnetic field background according to Eq. (11). Taking into account the equivariance, we get the formulas of [2], i.e., (4) without the $\mathcal{N}$ terms, but multiplied with the Maslov factor $(-1)^j$. (The $\mathcal{N}$ term arises due to a subsequent gauge transformation required by the gauge fixing in [3]).

It also allows one to "export" the Schrödinger symmetry to nonrelativistic Chern-Simons theory in the constant magnetic field background. The (rather complicated) generators, listed in Ref. [11], are readily obtained using Eq. (13). For example, time translation $t \to t + \tau$ in the $\mathcal{B}$ background amounts to a time translation for the oscillator with parameter τ plus a rotation with angle $\omega\tau$. Hence $H_\mathcal{B} = H_{\rm osc} - \omega\mathcal{J} = \mathcal{H} + \omega^2\mathcal{K} - \omega\mathcal{J}$. Similarly, a space translation for $\mathcal{B}$ amounts, in "empty" space, to a space translation and a boost, followed by a rotation, yielding $P^i_\mathcal{B} = \mathcal{P}^i + \omega\epsilon^{ij}\mathcal{G}^j$, etc.

All our preceding results apply to any Bargmann space which can be conformally mapped into Minkowski space in a ξ-preserving way. Now we describe these "Schrödinger-conformally flat" spaces. In $D = n + 2 > 3$ dimensions, conformal flatness is guaranteed by the vanishing of the conformal Weyl tensor

$$\begin{aligned}C^{\mu\nu}{}_{\rho\sigma} &= R^{\mu\nu}{}_{\rho\sigma} - \frac{4}{D-2}\delta^{[\mu}{}_{[\rho}R^{\nu]}{}_{\sigma]} \\ &\quad + \frac{2}{(D-1)(D-2)}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}R.\end{aligned} \tag{23}$$

Now $R_{\mu\nu\rho\sigma}\xi^\mu \equiv 0$ for a Bargmann space, which implies some extra conditions on the curvature. Inserting the identity $\xi_\mu R^{\mu\nu}{}_{\rho\sigma} = 0$ into $C^{\mu\nu}{}_{\rho\sigma} = 0$ and using the identity $\xi_\mu R^\mu_\nu \equiv 0$ ($R^\nu_\sigma \equiv R^{\mu\nu}{}_{\mu\sigma}$), we find $0 = -[\xi_\rho R^\nu_\sigma - \xi_\sigma R^\nu_\rho] + R/(D-1)[\xi_\rho\delta^\nu_\sigma - \xi_\sigma\delta^\nu_\rho]$. Contracting again with ξ^σ and using that ξ is null, we end up with $R\xi_\rho\xi^\nu = 0$. Hence the scalar curvature vanishes, $R = 0$. Then the previous equation yields $\xi_{[\rho}R^\nu_{\sigma]} = 0$ and thus $R^\nu_\sigma = \xi_\sigma\eta^\nu$ for some vector field η. Using the symmetry of the Ricci tensor, $R_{[\mu\nu]} = 0$, we find that $\eta = \varrho\xi$ for some function ϱ. We finally get the *consistency relation*

$$R_{\mu\nu} = \varrho\xi_\mu\xi_\nu. \tag{24}$$

The Bianchi identities ($\nabla_\mu R^\mu_\nu = 0$ since $R = 0$) yield $\xi^\mu\partial_\mu\varrho = 0$; i.e., ϱ is a function on spacetime Q. The conformal Schrödinger-Weyl tensor is hence of the form

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - [4/(D-2)]\varrho\delta^{[\mu}{}_{[\rho}\xi^{[\nu}\xi_{\sigma]}. \tag{25}$$

It is noteworthy that Eq. (24) is the Newton-Cartan field equation with $\varrho/(4\pi G)$ as the matter density of the sources.

It follows from Eq. (24) that the transverse Ricci tensor of a Schrödinger-conformal flat Bargmann metric necessarily vanishes, $R_{ij} = 0$, for each t. The transverse space is hence (locally) flat, and we can choose $g_{ij} = g_{ij}(t)$. Then a change of coordinates $(t, \mathbf{x}, s) \to (t, G(t)\mathbf{x}, s)$, where $G = (G_{ij})$ is the square-root matrix $\delta_{ab}G^a_iG^b_j = g_{ij}$, casts our Bargmann metric into the form (10) with $g_{ij} = \delta_{ij}$. Note that this transformation brings in a uniform magnetic field and/or an oscillator into the metric, while ξ remains unchanged. In this case, the nonzero components of the Weyl tensor of the general $D = 4$ Brinkmann metric (10) are found as

$$C_{xyxt} = -C_{ytts} = -\tfrac{1}{4}\partial_x\mathcal{B},$$

$$C_{xyyt} = +C_{xtts} = -\tfrac{1}{4}\partial_y\mathcal{B},$$

$$C_{xtxt} = -\tfrac{1}{2}[\partial_t(\partial_y\mathcal{A}_y - \partial_x\mathcal{A}_x) - \mathcal{A}_x\partial_y\mathcal{B},] + \tfrac{1}{2}[\partial_x^2 - \partial_y^2]U, \tag{26}$$

$$C_{ytyt} = +\tfrac{1}{2}[\partial_t(\partial_y\mathcal{A}_y - \partial_x\mathcal{A}_x) - \mathcal{A}_y\partial_x\mathcal{B}] - \tfrac{1}{2}[\partial_x^2 - \partial_y^2]U,$$

$$C_{xtyt} = +\tfrac{1}{2}[\partial_t(\partial_x\mathcal{A}_y + \partial_y\mathcal{A}_x) + 2\partial_x\partial_y u] - \tfrac{1}{4}(\mathcal{A}_x\partial_x - \mathcal{A}_y\partial_y)\mathcal{B}.$$

Then Schrödinger-conformal flatness requires

$$\mathcal{A}_i = \tfrac{1}{2}\epsilon_{ij}\mathcal{B}(t)x^j + a_i, \quad \nabla \times \mathbf{a} = 0, \quad \partial_t\mathbf{a} = 0, \tag{27}$$

$$U(t, \mathbf{x}) = \tfrac{1}{2}C(t)R^2 + \mathbf{F}(t)\cdot\mathbf{x} + K(t).$$

[Note, in passing, that (24) automatically holds: The only nonvanishing component of the Ricci tensor is $R_{tt} = -\partial_t(\nabla\cdot\mathcal{A}) - \frac{1}{2}\mathcal{B}_2 - \Delta U$.]

The metric (10)–(27) describes a uniform magnetic field $\mathcal{B}(t)$, an attractive $[C(t) = \omega^2(t)]$ or repulsive $[C(t) = -\omega^2(t)]$ isotropic oscillator, and a uniform force field $\mathcal{F}(t)$ in the plane which may all depend arbitrarily on time. It also includes a curl-free vector potential $\mathbf{a}(\mathbf{x})$ that can be gauged away if the transverse space is simply connected: $a_i = \partial_i f$, and the coordinate transformation $(t, \mathbf{x}, s) \to (t, \mathbf{x}, s + f)$ results in the "gauge" transformation $\mathcal{A}_i \to \mathcal{A}_i - \partial_i f = -\frac{1}{2}\mathcal{B}\epsilon_{ij}x^j$. If, however, space is not simply connected, we can also include an external Aharonov-Bohm-type vector potential.

Being conformally related, all these metrics share the symmetries of flat Bargmann space: For example, if the transverse space is $\mathbf{R}^2$, we get the full Schrödinger symmetry; for $\mathbf{R}^2\backslash\{0\}$, the symmetry is reduced rather to $o(2)\times o(2,1)\times\mathbf{R}$, just like for a magnetic vortex [12].

The case of a constant electric field, which went unnoticed so far, is quite amusing. Its metric $d\mathbf{x}^2 + 2dt\,ds - 2\mathbf{F}\cdot\mathbf{x}\,dt^2$ can be brought to the free form by switching to an accelerated coordinate system:

$$\mathbf{X} = \mathbf{x} + \tfrac{1}{2}\mathbf{F}t^2, \quad T = t, \quad S = s - \mathbf{F}\cdot\mathbf{x}t, -\tfrac{1}{6}\mathbf{F}^2t^3. \tag{28}$$

This example also shows that the action of the Schrödinger group, e.g., a rotation, looks quite different in the inertial and in the moving frames.

Let us finally mention that Eqs. (24) and (25) are equivalent to the condition [13]

$$C_{\mu\nu\rho\sigma}\xi^\mu = 0. \tag{29}$$

In conclusion, our "nonrelativistic Kaluza-Klein" approach provides a unified view on the various known constructions and explains the common origin of the large symmetries. We described all such spaces, extending the set of generators in an oscillator background given by Jackiw and Pi in [3], presented one more example, and pointed out a possible time dependence as well as the possibility of adding an Aharonov-Bohm-type potential. We have also shown that formula (4) may require a phase correction for times larger than a half oscillator period.

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