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Dynamical zero modes and pure glue (1+1)-dimensional QCD in light-cone field theory

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We consider light-cone quantized (1+1)-dimensional QCD on a "cylinder" with periodic boundary conditions on the gluon fields. This is the framework of discretized light-cone quantization. We review the argument that the light-cone gauge $A^+ = 0$ is not attainable. The zero mode is a dynamical and gauge-invariant field. The attainable gauge has a Gribov ambiguity. We exactly solve the problem of pure glue theory coupled to some zero mode external sources. We verify the identity of the front and the more familiar instant form approaches. We obtain a discrete spectrum of vacuum states and their wave functions.

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I. INTRODUCTION

Recently the Hamiltonian approach to field theory has been tackled with renewed interest. The hope is that Dirac's "front form" Hamiltonian scheme [1] is useful for confronting quantum chromodynamics (QCD). Often in the literature this is called "light-cone," "null-plane," or "light-front" quantization. In what follows we shall persist with the original Dirac nomenclature. This formulation uses $x^+ = (1/\sqrt{2})(ct + z)$, called the light-cone time, as the "time" evolution parameter rather than the conventional $x^0 = ct$. For an extensive bibliography the reader is referred to Ref. [2]. One reason for the modern phase of this approach is the apparent simplicity of the vacuum in front form theory. In the more familiar "instant form" quantization the QCD vacuum contains an infinite number of soft particles. But then in front form field theory the following question arises: Where can long range phenomena of spontaneous symmetry breaking and perhaps even confinement appear in the apparent absence of any "infrared" vacuum structure?

The specific approach of *discretized* light-cone quantization (DLCQ) is one setting in which one can answer this question and hopefully pursue the program to a solution. Here the theory is defined in a finite "spatial volume" with periodic or antiperiodic boundary conditions imposed on bosonic or fermionic fields, respectively. There are two appealing reasons for such a formulation. One obtains an infrared regulated theory, and the discretization of momenta facilitates putting the many-body problem onto the computer. The price one has to pay, shown actually some time ago [3], is that Fourier zero modes of the fields are often not independent dynamical quanta. Rather, by a constraint equation, they are dependent on them. Recent work on such a constrained zero mode in scalar (1+1)-dimensional ϕ^4 (ϕ^4_{1+1}) theory has led to the insight that it gives rise to the phenomena of spontaneous symmetry breaking and field condensates [4], aspects normally attributed to a nontrivial vacuum structure.

Our concern in this paper, however, is with zero modes that are true dynamical independent fields. One way they can arise is as follows. Because of the boundary conditions in gauge theory, one cannot fully implement the traditional light-cone gauge $A^+ = 0$. The development of the understanding of this problem in DLCQ can be traced in Ref. [5]. The field A^+ turns out to have a zero mode which cannot be gauged away [6]. This mode is indeed dynamical, and is the object we study in this paper. It has its analogue in instant form approaches to gauge theory. For example, there exists a large body of work on Abelian and non-Abelian gauge theories in 1+1 dimensions quantized on a cylinder geometry [7]. There indeed this dynamical zero mode plays an important role.

We too shall concern ourselves in the present work with non-Abelian gauge theory in 1+1 dimensions, examining the model introduced by 't Hooft [8]. A DLCQ treatment of the theory, giving meson and baryon spectra and wave functions, was undertaken by Hornbostel *et al.* [9]. Apart from a modified approach by Lenz *et al.* [10], zero modes have been neglected in previous DLCQ studies of (1+1)-dimensional QCD (QCD_{1+1}). This we rectify to some extent in the present paper.

The specific task we undertake here is to understand the zero mode *subsector* of the pure glue theory, namely, where only zero mode external sources excite only zero mode gluons. We shall see that this is not an approximation but rather a consistent solution, a subregime within the complete theory. A similar framing of the problem lies behind the work of Lüscher [11] and van Baal [12] using the instant form Hamiltonian approach to pure glue gauge theory in 3+1 dimensions. The beauty of this reduction in the (1+1)-dimensional theory is twofold. First, it yields a theory which is exactly soluble. This is useful given the dearth of soluble models in field theory. Second, the zero mode theory represents a paring down

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to the point where the front and instant forms are manifestly *identical*, which is nice to know indeed. We solve the theory in this specific dynamical regime and find a discrete spectrum of states whose wave functions can be completely determined. These states have the quantum numbers of the vacuum. There is a summary and discussion of the results at the end of the paper. The Appendix explains the notation.

II. GAUGE FIXING

We consider an SU(2) non-Abelian gauge theory in 1+1 dimensions with classical sources coupled to the gluons. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + 2\,\mathrm{Tr}\,(J_\mu A^\mu)\ , \tag{1}$$

where $F_{\mu\nu} = \partial_\nu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]$. With a finite interval in x^- from $-L$ to L, we impose periodic boundary conditions on all gauge potentials A_μ.

We now show that the light-cone gauge $A^+ = 0$ cannot be reached. A gauge transformation U bringing a gauge potential B^μ, itself in some arbitrary gauge configuration, to some other gauge configuration A^μ is

$$gA^\mu = \partial_\mu U U^{-1} + gUB^\mu U^{-1}\ . \tag{2}$$

Here g is the coupling constant and U is an element of the Lie algebra of SU(2). Clearly U given by

$$U = P\exp\left[-g\int_{-L}^{x^-} dy^- B^+(y^-)\right] \tag{3}$$

will bring us to the gauge $A^+ = 0$.

We appear to have been successful in getting the light-cone gauge. However, the element U through which we wish to achieve the gauge condition must satisfy Z_2-periodic boundary conditions, as in [13], namely, $U(x) = (\pm)U(x+2L)$. Clearly Eq. (3) does not satisfy these boundary conditions. So in fact the attempt has failed.

With the notation given in the Appendix, a modification of Eq. (3) is

$$U(x) = e^{gx^-\overset{0}{B}{}^+}\,Pe^{-g\int_{-L}^{x^-} dy^- B^+(y^-)}\ . \tag{4}$$

Since $\overset{0}{B}{}^+$ is the zero mode of B^+, this is an allowed gauge transformation but it does not completely bring us to the light-cone gauge. We find instead

$$A^+ = \overset{0}{B}{}^+\ . \tag{5}$$

In other words, we cannot eliminate the zero mode of the gauge potential. The reason is evident: It is *invariant* under periodic gauge transformations. But of course we can always perform a rotation in color space. In line with other authors [14], we choose this so that $\overset{0}{A}{}_3^+$ is the only nonzero element, since in our representation only σ^3 is diagonal.

In addition, we can impose the subsidiary gauge condition

$$\overset{0}{A}{}_3^- = 0\ . \tag{6}$$

The reason is that there still remains freedom to perform gauge transformations that depend only on light-cone time x^+ and the color matrix σ^3. The above condition Eq. (6) can be reached from the arbitrary configuration B^μ by the Lie algebra element

$$W = P\exp\left[-ig\int_{x_0^+}^{x^+} d\tilde{x}^+\,\overset{0}{B}{}_3^-(\tilde{x}^+)\frac{\sigma^3}{2}\right]\ , \tag{7}$$

where x_0^+ is some arbitrary but fixed light-cone time. It, moreover, does not "undo" the previous gauge condition.

The above procedure would appear to have enabled complete fixing of the gauge. This is still not so. Gauge transformations

$$V = \exp\left\{ix^-\left(\frac{n\pi}{2L}\right)\sigma^3\right\} \tag{8}$$

generate shifts, according to Eq. (2), in the zero mode component

$$\overset{0}{A}{}_3^+ \to \overset{0}{A}{}_3^+ + \frac{n\pi}{gL}\ . \tag{9}$$

All of these possibilities, labeled by the integer n, of course, still satisfy $\partial_- A^+ = 0$, but as one sees $n = 0$ should not really be included. One can verify that the transformations V also preserve the subsidiary condition, Eq. (6). One notes that the transformation is x^- dependent and Z_2 periodic. It is thus a simple example of a Gribov copy [15] in 1+1 dimensions. We follow the conventional procedure of restricting $\overset{0}{A}{}_3^+$ to a region free of copies, for example,

$$0 \le \overset{0}{A}{}_3^+ \le \frac{\pi}{gL}\ , \tag{10}$$

which defines a *fundamental modular region* [12].

III. EQUATIONS OF MOTION

Equations for pure glue theory. Ultimately, the argument that the vacuum in front form field theory is trivial rests on the *linearity* of the Euler-Lagrange equations of motion in the light-cone time x^+. This itself stems from the expression for the D'Alembertian in light-cone coordinates $\Box = \partial_+\partial_-$ in one space dimension. It is the very same fact that causes most zero modes to be constrained when there are transverse dimensions: The space derivative destroys the mode, thus eliminating the time derivative in the equation of motion. However, a careful examination of the equations can sometimes reveal double time derivatives ∂_+^2 due to the gauge structure. Thus there can still be dynamical zero mode degrees of freedom even in DLCQ which could, in principle, undermine the vacuum "triviality" argument. This is what we now explore for SU(2).

The equations of motion for the theory are

$$[D^\mu, F_{\mu\nu}] = \partial^\mu F_{\mu\nu} - g[A^\mu, F_{\mu\nu}] = J_\nu\ . \tag{11}$$

For our purposes it is convenient to break this equation up into color components A_a^μ. Color will always be the lower index. Rather than the three color fields A_1^μ, A_2^μ, and A_3^μ, we will use chiral notation with $A_+^\mu = A_1^\mu + iA_2^\mu$ and $A_-^\mu = A_1^\mu - iA_2^\mu$. In terms of these components the equations of motion are

$$\partial_\mu\partial^\mu A_3^\nu - \partial^\nu\partial_\mu A_3^\mu + \frac{ig}{2}A_-^\mu \overleftrightarrow{\partial^\nu} A_{\mu,+} + \frac{ig}{2}(A_-^\nu\partial_\mu A_+^\mu - A_+^\nu\partial_\mu A_-^\mu) + ig(\partial_\mu A_-^\nu A_+^\mu - \partial_\mu A_+^\nu A_-^\mu)$$
$$+g^2[-A_{\mu,+}A_-^\mu A_3^\nu + \tfrac{1}{2}A_{\mu,3}(A_+^\nu A_-^\mu + A_-^\nu A_+^\mu)] = J_3^\nu \quad (12)$$

and

$$\partial^\mu\partial_\mu A_-^\nu - \partial^\nu\partial_\mu A_-^\mu + igA_3^\mu\overleftrightarrow{\partial^\nu}A_{\mu,-} + ig(A_3^\nu\partial_\mu A_-^\mu - A_-^\nu\partial_\mu A_3^\mu) + 2ig(\partial_\mu A_3^\nu A_-^\mu - \partial_\mu A_-^\nu A_3^\mu)$$
$$+g^2[A_{\mu,3}(A_-^\mu A_3^\nu - A_3^\mu A_-^\nu) + \tfrac{1}{2}A_{\mu,-}(A_+^\nu A_-^\mu - A_-^\nu A_+^\mu)] = J_-^\nu\,, \quad (13)$$

where we use the antisymmetric derivative $A\overleftrightarrow{\partial}B = A(\partial B) - (\partial A)B$. A third equation is the complex conjugate of Eq. (13).

Next we break these equations up into normal and zero mode components [6], and look at the equations for each Lorentz component $\nu = +,-$ and each color component $a = 3,+$. With the above gauge conditions the $\nu = +$ equations are

$$(i\partial^+)^2\overset{n}{A_3^-} = \overset{n}{J_3^+}, \quad (14)$$

$$0 = \overset{0}{J_3^+}, \quad (15)$$

$$(i\partial^+ + g\overset{0}{A_3^+})^2\overset{n}{A_-^-} = \overset{n}{J_-^+}, \quad (16)$$

and

$$g^2(\overset{0}{A_3^+})^2\overset{0}{A_-^-} = \overset{0}{J_-^+}\,. \quad (17)$$

Observe that these equations exhibit no time ∂_+ derivatives. Correspondingly, for $\nu = -$,

$$\partial^+\partial^-\overset{n}{A_3^-} - \frac{ig}{2}\langle A_-^-\overleftrightarrow{\partial^+}A_+^-\rangle_n + g^2\overset{0}{A_3^+}\langle A_+^-A_-^-\rangle_n = \overset{n}{J_3^-}, \quad (18)$$

$$-(\partial^-)^2\overset{0}{A_3^+} - \frac{ig}{2}\langle A_-^-\overleftrightarrow{\partial^+}A_+^-\rangle_0 + g^2\overset{0}{A_3^+}\langle A_+^-A_-^-\rangle_0 = \overset{0}{J_3^-}, \quad (19)$$

$$-\partial^+\partial^-\overset{n}{A_-^-} - ig\overset{0}{A_3^+}\partial^-\overset{n}{A_-^-}$$
$$-2ig\partial^-\overset{0}{A_3^+}\overset{n}{A_-^-} - ig\langle A_3^-\partial^+A_-^-\rangle_n \quad (20)$$
$$+ig\langle\partial^+A_3^-A_-^-\rangle_n - g^2\overset{0}{A_3^+}\langle A_3^-A_-^-\rangle_n = \overset{n}{J_-^-}, \quad (21)$$

$$-ig\overset{0}{A_3^+}\partial^-\overset{0}{A_-^-} - 2ig\partial^-\overset{0}{A_3^+}\overset{0}{A_-^-} - ig\langle A_3^-\partial^+A_-^-\rangle_0 \quad (22)$$
$$+ig\langle\partial^+A_3^-A_-^-\rangle_0 - g^2\overset{0}{A_3^+}\langle A_3^-A_-^-\rangle_0 = \overset{0}{J_-^-}\,. \quad (23)$$

Note the presence of both constraint and evolution equations.

The constrained nature of the first set of equations is not so much a property of the front form, but is rather the Gauss law exhibiting itself. The equations correspond to the fact that, in noncovariant gauges, the field A^- is generally a nondynamical field. In a Hamiltonian approach it plays the role of a Lagrange multiplier to the Gauss law. In the approach we shall take to quantum theory, we shall implement these as "strong," namely, operator constraints. However, special comment must be reserved for Eq. (15). It actually does not even occur since we have gauged away $\overset{0}{A_3^-}$. If the sources themselves were part of the dynamical problem, then this equation would have to be reintroduced as a "weak" constraint, namely, applied to physical states of the quantum Hilbert space. It has its analogue in the instant form approach [16], where the diagonal part of the color-charge operator must annihilate physical states,

$$Q_{\rm diag}|{\rm phys}\rangle = 0\,. \quad (24)$$

In the model we consider below, the sources are merely external classical fields, essentially just parameters, and so the specific theory we consider there is only meaningful if $\overset{0}{J_3^+}$ as a parameter vanishes.

IV. THE ZERO MODE SOURCE PROBLEM

The classical solution. We now consider a regime of the theory excited by sources that are purely time dependent. Thus our theory differs from that studied by other authors [16] in that the sources are classical, only their zero mode parts are retained, and they are part of a light-cone quantized theory. The reader is referred to the final section for more discussion on these sources for this problem. Vanishing normal mode gluons are then a consistent solution to the above equations of motion in the normal mode sector. Only zero mode gluons occur. From the zero mode equations of motion there are then only two equations with nontrivial content. The last of the Gauss law ($\nu = +$) equations is simply solved to give

$$\overset{0}{A_\pm^-} = \frac{\overset{0}{J_\pm^+}}{g^2(\overset{0}{A_3^+})^2}\,. \quad (25)$$

Once again, the other Gauss law in the zero mode sector constrains $\overset{0}{J}{}^+_3$ to vanish as a parameter. From the $\nu = -$ equations we extract only one relevant equation

$$-(\partial^-)^2 \overset{0}{A}{}^+_3 + g^2 \overset{0}{A}{}^+_3 \overset{0}{A}{}^-_+ \overset{0}{A}{}^-_- = \overset{0}{J}{}^-_3 \ . \tag{26}$$

We observe that the pure glue theory in 1+1 dimensions involves only a single genuine degree of freedom, the field $\overset{0}{A}{}^+_3$. Substituting our solutions Eq. (25) into the dynamical equation (26) we obtain

$$-(\partial^-)^2 \overset{0}{A}{}^+_3 + \frac{\overset{0}{J}{}^+_+ \overset{0}{J}{}^+_-}{g^2(\overset{0}{A}{}^+_3)^3} = \overset{0}{J}{}^-_3 \ . \tag{27}$$

From this we can see that this reduction of the theory is not equivalent to a perturbation around the free ($g = 0$) theory. For convenience we henceforth use the notation

$$\overset{0}{A}{}^+_3 = v \ , \quad x^+ = t \ , \quad w^2 = \frac{\overset{0}{J}{}^+_+ \overset{0}{J}{}^+_-}{g^2}, \overset{0}{J}{}^-_3 = \frac{B}{2} \ . \tag{28}$$

The dynamical equation can then be compactly written as

$$-\frac{\partial^2}{\partial t^2} v + \frac{w^2}{v^3} = \frac{B}{2} \ . \tag{29}$$

It can be solved by easy reduction to quadrature with the solution

$$\pm it = \int^v \frac{y\,dy}{\sqrt{By^3 + 2w^2Gy^2 + w^2}} \ , \tag{30}$$

where G is an integration constant.

The solution to the quantum problem. We pursue a Hamiltonian formulation where, in the front form, the generator of x^+ translations P^- or light-cone energy operator is taken as the Hamiltonian. The only conjugate momentum is

$$p \equiv \overset{0}{\Pi}{}^-_3 = \partial^- \overset{0}{A}{}^+_3 = \partial^- v \ . \tag{31}$$

The Hamiltonian density $T^{+-} = \partial^- \overset{0}{A}{}^+_3 \Pi^-_3 - \mathcal{L}$ leads to the Hamiltonian

$$H = \frac{1}{2}\left[p^2 + \frac{w^2}{v^2} + Bv\right](2L) \ . \tag{32}$$

Of course, Hamilton's equations of motion agree with Eq. (25) and Eq. (26). Quantization is achieved by imposing a commutation relation at equal light-cone time on the dynamical degree of freedom. Introducing the variable $q = 2Lv$, the appropriate commutation relation is

$$[q(x^+), p(x^+)] = i \ . \tag{33}$$

Note that the zero mode v or q satisfies a field theory of one dimension less than the original field theory. In 1+1 dimensions the field theoretic problem reduces to quantum mechanics of a single particle as in Manton's treatment of the Schwinger model in Ref. [7]. One thus has to solve the Schrödinger equation

$$\frac{1}{2}\left(-\frac{d^2}{dq^2} + \frac{(2Lw)^2}{q^2} + \frac{Bq}{2L}\right)\psi = \mathcal{E}\psi, \tag{34}$$

with the eigenvalue $\mathcal{E} = E/(2L)$ actually being an energy density.

Before proceeding with the solution let us briefly show that exactly the same structure is obtained beginning in the instant form. Here we introduce the periodic boundary conditions on a finite interval of length $2L$ in x^3. The appropriate gauge choice is $\partial_3 A^3_a = 0$ and then a color rotation can single out the diagonal color component of $v = \overset{0}{A}{}^3_3$. Zero modes are of course now defined with respect to the x^3 direction. After the color diagonalization, one can gauge away $\overset{0}{A}{}^0_3$ and, by analogy to the above, set all normal mode sources to zero. With

$$F^a_{03} = \partial_0 v \delta_{a3} + g\epsilon_{ab3} A^0_b v, \tag{35}$$

one gets $p = -\partial_0 v$ as the only conjugate momentum. The Hamiltonian is now taken as the generator of translations in x^0. Thus

$$H = \frac{1}{2}[p^2 - g^2(\overset{0}{A}{}^0_\alpha)^2 v^2 + 2\overset{0}{J}{}^0_\alpha \overset{0}{A}{}^0_\alpha + 2v\overset{0}{J}{}^3_3](2L) \ , \quad \alpha = 1,2. \tag{36}$$

The Gauss law is

$$\overset{0}{A}{}^0_\alpha = \frac{\overset{0}{J}{}^0_\alpha}{g^2 v^2}, \tag{37}$$

which upon substitution into the Hamiltonian yields

$$H = \frac{1}{2}\left[p^2 + \frac{(\overset{0}{J}{}^0_\alpha)^2}{g^2v^2} + 2v\overset{0}{J}{}^3_3\right](2L) \ . \tag{38}$$

With the same chiral color convention one has $(\overset{0}{J}{}^0_\alpha)^2 = \overset{0}{J}{}^0_+ \overset{0}{J}{}^0_-$ and thus obviously the same Hamiltonian as in Eq. (32).

Let us return to solving the Schrödinger equation (34). All eigenstates ψ have the quantum numbers of the naive vacuum adopted in standard front form field theory: All of them are eigenstates of the light-cone momentum operator P^+ with zero eigenvalue. The true vacuum is now that state with lowest P^- eigenvalue. In order to get an exactly soluble system we perform one more simplification. We eliminate the source $2B = \overset{0}{J}{}^-_3$. One of the solutions to Eq. (34) is then $\psi(q) = \sqrt{q}\, Z_\nu(\sqrt{2\mathcal{E}}q)$ where, in the notation of [17], Z_ν is the Bessel function with $\nu^2 \equiv (2Lw)^2 + 1/4$. Note that wL is independent of L if w, which is proportional to the external source, scales in L like a dynamical source [18]. The general solution is a superposition of the regular and irregular Bessel functions, that is,

$$\psi(q) = R\sqrt{q}J_\nu(\sqrt{2\mathcal{E}}q) + S\sqrt{v}J_{-\nu}(\sqrt{2\mathcal{E}}q) \ . \tag{39}$$

The constants R and S need to be specified by boundary conditions, square integrability, and continuity of the first derivative. When $\nu > 1/2$ square integrability leads to $S = 0$. The boundary condition that is to be imposed comes from the restriction to the fundamental modular domain. Since the wave function vanishes at $q = 0$, we must demand that the wave functions vanish at $q = \pm 2\pi/g$. The overall constant R is then fixed by normalization. Note that this requirement does not automatically ensure that the wave function vanishes at $\pm 2\pi n/g$ for all n, when arbitrary sources are present. Therefore the pieces of the wave function for each fundamental modular region will not be exact copies of each other. For the source-free case the wave functions for the different regions are indeed exact copies [13]. The boundary condition leads to the energy density only assuming the discrete values

$$\mathcal{E}_m^{(\nu)} = \frac{g^2}{8\pi^2}[(X_m^{(\nu)})^2 - (X_1^{(\nu)})^2], \quad m = 1, 2, \ldots, \tag{40}$$

where $X_m^{(\nu)}$ denotes the mth zero of the νth Bessel function J_ν. We have shifted the lowest eigenvalue to zero. In general, these zeros can only be obtained numerically. Thus

$$\psi_m(q) = R\sqrt{q}J_\nu(\sqrt{2\mathcal{E}_m^{(\nu)}}q) \tag{41}$$

is the complete solution. The true vacuum is the state of lowest energy, namely, with $m = 1$.

V. DISCUSSION AND PERSPECTIVES

Let us first summarize the essential points. We analyzed pure glue non-Abelian gauge theory in a compact spatial volume and periodic boundary conditions on the gauge potentials. Working in the front form Hamiltonian approach, we demonstrated how one carefully fixes the gauge. The equations of motion enabled identification of dynamical and constrained zero mode variables. We solved the quantum theory consisting of gluons excited only by pure time-dependent external sources. This reduction uncovered a basic regime of non-Abelian gauge theory where the front and the instant form approaches were seen to be identical. It also reduced a quantum field theory problem to a quantum mechanical one which could be solved for the Schrödinger representation wave function. With the explicit interaction term for the dynamical zero mode switched off, we exactly solved the theory in a fundamental modular domain.

The exact solution we obtained is genuinely nonperturbative in character. It describes vacuumlike states since for all of these states $P^+ = 0$. Consequently, they all have zero invariant mass $M^2 = P^+P^-$. The states are labeled by the eigenvalues of the operator P^-. We explain below why the nonzero sources are useful. But with them being nonzero we have obtained a generalization of the result of Hetrick [13]. The linear dependence on L in the result for the discrete *energy* levels is also consistent with what one would expect from a loop of color flux running around the cylinder. In the source-free case Hetrick [13] uses a wave function that is symmetric about $q = 0$. For our problem this corresponds to

$$\psi_m(q) = N\cos(\sqrt{2\epsilon_m}q) , \tag{42}$$

where N is fixed by normalization. At the boundary of the fundamental modular region $q = 2\pi/g$ and $\psi_m = (-1)^m N$; thus $\sqrt{2\epsilon_m}2\pi/g = m\pi$ and

$$\epsilon_m = \frac{g^2(m^2-1)}{8} . \tag{43}$$

Note that $m = 1$ is the lowest energy state and has as expected one node in the allowed region $0 \leq g \leq 2\pi/g$. Hetrick [13] discusses the connection to the results of Rajeev [7] but it amounts to a shift in ϵ and redefining $m \to m/2$. It has been argued by van Baal [19] that the correct boundary condition at $q = 0$ is $\psi(0) = 0$. This would give a sine in Eq. (42) and would match smoothly with our result Eq. (41). For the cosine solution there is a discontinuous transition from the source-free to the nonfree case. The *manifest* equivalence of the front and instant form treatments of this problem is a consequence of the elimination of all but topological features and in this respect the topology is identical in the two forms. We speculate that a point form [1] calculation would lead also to the identical Hamiltonian. In our picture, the two forms will begin to look different with the introduction of genuine dynamical content and in higher dimensions. However, the same physical content should be present.

This calculation offers the lesson that even in a front form approach, the vacuum might not be just the simple Fock vacuum. Dynamical zero modes do imbue the vacuum with a rich structure. However, the advantage of the front form is not severely lost. In higher dimensions we expect that the transverse gluon components are not dynamical but rather are constrained. If these constraints can be solved, the vacuum will not be inordinately beyond control. This is in sharp distinction to the instant form approach. There is nonetheless one possible scenario in which a simple vacuum could be restored, at least in 1+1 dimensions. The inclusion of normal mode dynamics via the sources will build additional states on top of the vacua of the present work. When the naive continuum limit $L \to \infty$ is taken only the states built on the lowest level might remain. When one goes to higher dimensions issues of how one takes the continuum limit and range of validity will become much more difficult. The issue is how much resolution is necessary to understand the physics one is interested in and as one moves to higher resolution what new information must be added so that one accurately reflects the low energy regime of QCD. Some discussion of these issues can be found in the work of van Baal [12]. We finish by briefly addressing the program for tackling the higher dimensional theory, and how our result will actually be valuable for the problem in 3+1 dimensions. A crucial observation is that as zero modes are independent of at least one space coordinate they satisfy a field theory in a fewer number of space dimensions than the original. One can thus envisage undertaking a hierarchy of projections from $3 \to 2 \to 1 \to 0$ space dimensions, at each level extracting a zero mode theory within the previous higher dimensional theory. A similar idea lies behind the recent work of [20]. In our approach one arrives at a quantum mechanical problem of similar structure to the one we have solved in the present work. The difference would be that the dynamical quanta of the higher dimensional theory, both fermions *and* gluons, will be the sources for the lower dimensional theory.

Our exact solution with nonvanishing sources provides for eventual understanding of how constrained and other dynamical zero mode quanta come in at higher dimensions, and how they generate QCD spectroscopy in the real world of 3+1 dimensions.

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APPENDIX: NOTATION AND CONVENTIONS

The convention for light-cone coordinates we employ is that of [21], $x^\pm = (x^0 \pm x^3)/\sqrt{2}$. The dot product decomposes as $A\cdot B = A^+B^- + A^-B^+$. Following Dirac [1], x^+ is taken as the time parameter. The time derivative is thus $\partial_+ \equiv \partial/\partial x^+$ and, implied, the metric tensor $g^{\mu\nu}$ leads to $\partial_+ = \partial^-$. Correspondingly, $\partial_- = \partial/\partial x^- = \partial^+$ is the space derivative. We consider the theory "compactified" in the space dimension: the light-cone space coordinate $x^- \in [-L, +L]$. Periodic boundary conditions are imposed. Thus a given field ϕ can be expanded in Fourier modes where the discrete momenta take values

$$k^+ = n\frac{\pi}{L}, \quad n = 1, 2, \dots \ . \tag{A1}$$

The missing zero mode $n = 0$ is projected out by

$$\overset{0}{\phi} \equiv \langle\phi\rangle_0 \equiv \frac{1}{2L}\int_{-L}^{+L} dx^- \phi(x^-) \ , \tag{A2}$$

while the sum of the remaining nonzero modes is the normal mode

$$\overset{n}{\phi} \equiv \langle\phi(x^-)\rangle_n \equiv \phi(x^-) - \langle\phi\rangle_0 \ . \tag{A3}$$

We use the notation of Itzykson and Zuber [22] for writing the SU(2) gauge theory. The gauge potentials are represented by

$$A^\mu = A^\mu_a t^a \ , \quad t^a = \frac{i\sigma^a}{2} \ , \quad a = 1, 2, 3 \ , \tag{A4}$$

where t^a are representation matrices satisfying the Lie algebra

$$\left[t^a, t^b\right] = -\epsilon^{abc} t^c \tag{A5}$$

and σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A6}$$

The following identities are useful:

$$\sigma^a\sigma^b = i\epsilon^{abc}\sigma^c + \delta^{ab} \ , \tag{A7}$$

$$\mathrm{tr}(t^a t^b) = -1/2\delta^{ab} \ . \tag{A8}$$

In component form, the field strength tensor can be written

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu \tag{A9}$$

and

$$D^\mu_{ab} = \partial^\nu \delta_{ab} - g\epsilon_{abc} A^\mu_c \tag{A10}$$

is the covariant derivative in the adjoint representation.

[1] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[2] S. J. Brodsky, G. McCartor, H. C. Pauli, and S. S. Pinsky, Part. World **3**, 109 (1993); S. J. Brodsky and H. C. Pauli, in *Recent Aspects of Quantum Fields,* edited H. Mitter and H. Gausterer, Lecture Notes in Physics, Vol. 396 (Springer, Berlin, 1991).
[3] T. Maskawa and K. Yamawaki, Prog. Theor. Phys. **56**, 270 (1976); R. S. Wittman, in *Nuclear and Particle Physics on the Light Cone*, edited by M. B. Johnson and L. S. Kisslinger (World Scientific, Singapore, 1989).
[4] T. Heinzl, S. Krusche, S. Simbürger, and E. Werner, Z. Phys. C. **56**, 415 (1992); T. Heinzl, S. Krusche, E. Werner, and B. Zellerman, University of Regensburg Report No. TPR 92-17 1992 (unpublished); Suzhou Huang and Wei Lin, Ann. Phys. (N.Y.) **226**, 248 (1993); D. Robertson, Phys. Rev. D **47**, 2549 (1993); C. M. Bender, S. S. Pinsky, and B. van de Sande, *ibid.* **48**, 816 (1993); S. Pinsky, Ohio State University Report No. OHSTPY-HEP-TH-93-15, (1993) (unpublished); S. S. Pinsky and B. van de Sande, Phys. Rev. D **49**, 2001 (1994).
[5] G. McCartor, Z. Phys. C **41**, 271 (1988); **52**, 611 (1991); T. Heinzl, S. Krusche, and E. Werner, Phys. Lett. B **272**, 54 (1991); **275**, 410 (1992); Nucl. Phys. **A532**, 4290 (1991).
[6] A. C. Kalloniatis and H. C. Pauli, Z. Phys. C **60**, 255 (1993); **63**, 161 (1994).
[7] See, for example, N. Manton, Ann. Phys. (N.Y.) **159**, 220 (1985); J.E. Hetrick and Y. Hosotani, Phys. Rev. D **38**, 2621 (1988); Phys. Lett. B **230**, 88 (1989); F. Palumbo, Phys. Lett. B **243**, 109 (1990); D. G. Gross, Nucl. Phys.

B400, 161 (1993); S. Guruswamy and S. G. Rajeev, Mod. Phys. Lett. A **7**, 3783 (1992); S. G. Rajeev, Phys. Lett. B **212**, 203 (1988); E. Langmann and G. W. Semenoff, *ibid.* **303**, 303 (1993); J. Hallin, University of Göteborg Report No. ITP 93-8, 1993 (unpublished).

[8] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).

[9] K. Hornbostel, S. J. Brodsky, and H. C. Pauli, Phys. Rev. D **41**, 3814 (1990).

[10] F. Lenz, M. Thies, S. Levit, and K. Yazaki, Ann. Phys. (N. Y.) **208**, 1 (1991).

[11] M. Lüscher Nucl. Phys. B **219**, 233 (1983); M. Lüscher and G. Münster, *ibid.* **B232**, 445 (1984).

[12] P. van Baal, Nucl. Phys. **B369**, 259 (1992), and references therein.

[13] J.E. Hetrick, in *Lattice '92*, Proceedings of the International Symposium, Amsterdam, the Netherlands, edited by J. Smit and P. van Baal [Nucl. Phys. B (Proc. Suppl.) **30**, 228 (1993)]; Int. J. Mod. Phys. A **9**, 3153 (1994).

[14] V. A. Franke, Y. V. Novozhilov, E. V. Prokhvatilov, Lett. Math. Phys. **5**, 437 (1981); F. Lenz, H. W. L. Naus, and M. Thies, Ann. Phys. (N.Y.) (to be published).

[15] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978); H. Yabuki, Phys. Lett. B **231**, 271 (1989).

[16] E. Langmann and G. W. Semenoff, Phys. Lett. B **296**, 117 (1992); E. Langmann, M. Salmhofer, and A. Kovner, Report No. UBCTP 93-13/LA-UR-93-2991 (unpublished); J.A. Minahan and A. P. Polychronakos, Phys. Lett. B **326**, 288 (1994); M. Engelhardt and B. Schreiber, Erlangen Report 1994 (unpublished).

[17] I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1965).

[18] Note that $w \propto J^+$ and J^+ has dimension 1/length. It therefore must scale like $1/L$ since g and $1/L$ are the only dimensionful parameters and J^+ is, by construction, independent of the coupling g.

[19] P. van Baal (private communication).

[20] K. Demeterfi, I. R. Klebanov, and G. Bhanot, Nucl. Phys. **B418**, 15 (1994), and references therein.

[21] J. B. Kogut and D. E. Soper, Phys. Rev. D **1**, 2901 (1970).

[22] C. Itzykson and J-B. Zuber, *Quantum Field Theory* (McGraw-Hill, Singapore, 1985).