Self-dual SU(3) Chern-Simons Higgs systems

Hsien-Chung Kao and Kimyeong Lee
Physics Department, Columbia University, New York, New York, 10027
(Received 13 June 1994)

We explore self-dual Chern-Simons Higgs systems with the local SU(3) and global U(1) symmetries where the matter field lies in the adjoint representation. We show that there are three degenerate vacua of different symmetries and study the unbroken symmetry and the particle spectrum in each vacuum. We classify the self-dual configurations into three types and study their properties.

PACS number(s): 11.15.Kc, 11.10.Lm

I. INTRODUCTION

There has been some recent activity related to the various self-dual Chern-Simons Higgs systems. A bound on the energy functional of these systems is saturated by the solitonic configurations of the fractional spin. When the gauge symmetry is Abelian, the structure of these configurations has been studied quite thoroughly [1,2]. While the self-dual systems with an arbitrary gauge group and matter are shown to exist in the theories with a global U(1) symmetry, the self-dual configurations in these systems have been studied only in the cases where either the gauge symmetry or the matter is simple [3]. The nonrelativistic limit of these systems with matter in the adjoint representation has also been studied extensively, where the classification of finite energy soliton solutions has been found [4].

However, the soliton structure with non-Abelian symmetry turns out in general very rich and intricate. In this paper we investigate a somewhat tractable case: The theory with an SU(3) gauge group with matter made of a complex scalar field in the adjoint representation. This model is one of the simple models with nontrivial non-Abelian features and exhibits rich vacuum and solitonic structures. Solitons in this theory would carry fractional spins and non-Abelian charges. We hope our work will shed some light on the general structures of the self-dual systems.

First, we investigate the general consequences of the self-dual equations for the configurations saturating the energy bound. Then we show that there are three degenerate vacua of various unbroken symmetries and topologies, and analyze the particles spectrum at each vacuum. After that we study the characteristics of the self-dual configurations and classify them into three types. In general we expect these self-dual configurations describe topological and nontopological solitons dwelling on each phase. We study the topology of these solitons. Our analysis here provides a significant but not complete understanding of these classical soliton solutions of the self-dual equations in our model.

There is usually an underlying $N = 2$ supersymmetry behind every self-dual model [5]. In the similar line, there have been some studies of the underlying $N = 2$ supersymmetry in (Maxwell) Chern-Simons Higgs systems [6]. In addition, it is obvious that the maximal possible symmetry for three dimensions is $N = 3$ because a maximal vector multiplet can have spin 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$ up to sign. All such $N = 3$ supersymmetric Chern-Simons-Higgs theories have been constructed recently [7].

In Sec. II, we review briefly the self-dual model with the SU(3) gauge group and a complex scalar field in the adjoint representation. We then investigate in some detail the restrictions on the field configurations imposed by the self-dual equations. In Sec. III we study the ground states, their symmetric properties and elementary excitations. In Sec. IV, we classify the self-dual configurations into three types and study their properties. In Sec. V we conclude with some remarks.

II. MODEL

Let us consider a Chern-Simons-Higgs theory with local SU(3) and global U(1) symmetries. The generators of SU(3) in the fundamental representation are made of 3×3 Hermitian matrices, T^a, with $a = 1, 2, \ldots, 8$ and satisfy the commutation relations $[T^a, T^b] = if^{abc}T^c$ with f^{abc} as the structure constants of SU(3). The normalization is such that $\mathrm{tr}T^aT^b = \delta^{ab}/2$. The scalar matter field $\phi = (\phi_R^a + i\phi_I^a)T^a$ is made of a pair ϕ_R^a, ϕ_I^a in the adjoint representation of SU(3). The Lagrangian density for the theory is given by

$$\mathcal{L} = \kappa\epsilon^{\mu\nu\rho}\mathrm{tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + 2\,\mathrm{tr}\mid D_\mu\phi\mid^2 - \frac{2}{\kappa^2}\mathrm{tr}\mid[\phi,[\phi^\dagger,\phi]] - v^2\phi\mid^2 , \tag{2.1}$$

where $D_\mu\phi = \partial_\mu\phi - i[A_\mu, \phi]$ with $A_\mu = A_\mu^a T^a$. The gauge field strength is given by $F_{\mu\nu} \equiv F_{\mu\nu}^a T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. The theory is renormalizable because in 2+1 dimensions the sixth order term in the potential energy has a dimensionless coupling constant. For consistent quantum mechanics, the coefficient κ should be quantized so that $\kappa = k/4\pi$ with a nonzero integer, k [8].

The Gauss law constraint obtained from the variation of A_0^a is

$$\mathcal{G} \equiv -\kappa F_{12} - i([D_0\phi^\dagger, \phi] - [\phi^\dagger, D_0\phi]) = 0 \; . \qquad (2.2)$$

The local gauge transformation generators are made of $\mathrm{tr}(\mathcal{G}T^a)$. The Lagrangian (2.1) is invariant under the global phase rotation of the scalar field. The charge density of the corresponding global U(1) symmetry is

$$\rho_Q = 2i\,\mathrm{tr}(D_0\phi^\dagger\phi - \phi^\dagger D_0\phi) \qquad (2.3)$$

and the charge is $Q = \int d^2r\,\rho_Q$.

We are interested in finding a bound on the energy functional. The energy functional for the Lagrangian (2.1) is

$$E = \int d^2r\,q\Big\{\mathrm{tr}\mid D_0\phi\mid^2 + \mathrm{tr}\mid D_i\phi\mid^2 + \frac{1}{\kappa^2}\mathrm{tr}\mid [\phi,[\phi^\dagger,\phi]] - v^2\phi\mid^2\Big\} \; . \qquad (2.4)$$

With Gauss's law (2.2), the second term in curly brackets becomes

$$\begin{aligned}\mathrm{tr}\mid D_i\phi^2\mid \; &= \mathrm{tr}\mid (D_1 \pm iD_2)\phi\mid^2 \\ &\pm \frac{i}{\kappa}\Big\{\mathrm{tr}[D_0\phi^\dagger([\phi,[\phi^\dagger,\phi] - v^2\phi])\\ &-\mathrm{H.c.}\Big\} \pm \frac{v^2}{2\kappa}\rho_Q \end{aligned} \qquad (2.5)$$

up to a total derivative. This allows us to put $\mid D_0\phi\mid^2$ and the potential energy density into a total square. After integrating by parts, the energy functional can be written as

$$\begin{aligned} E = \int d^2r\,2\Big\{&\mathrm{tr}\left|D_0\phi \pm \frac{i}{\kappa}([\phi,[\phi^\dagger,\phi]] - v^2\phi)\right|^2 \\ &+ \mathrm{tr}\mid D_1\phi \pm iD_2\phi\mid^2\Big\} \pm \frac{v^2}{\kappa}Q \; . \end{aligned} \qquad (2.6)$$

Since the integrand in Eq. (2.6) is non-negative, there is a bound on the energy functional

$$E \geq m\mid Q\mid \; , \qquad (2.7)$$

where $m \equiv v^2/\kappa$ is the mass of elementary particles in the symmetric phase. The bound is given by a total global charge, which is not *a priori* related to any topological quantity. The bound (2.7) is saturated by the configurations satisfying Gauss's law and the self-dual equations

$$D_0\phi \pm \frac{i}{\kappa}([\phi,[\phi^\dagger,\phi]] - v^2\phi) = 0 \; , \qquad (2.8)$$

$$D_1\phi \pm iD_2\phi = 0 \; , \qquad (2.9)$$

where the upper (lower) sign corresponds to the positive (negative) value of Q. In addition, Gauss's law and Eq. (2.8) can be combined to

$$F_{12} = \pm\frac{2}{\kappa^2}([\phi^\dagger,[\phi,[\phi^\dagger,\phi]]] - v^2[\phi^\dagger,\phi]) \; . \qquad (2.10)$$

When Eq. (2.8) is satisfied the U(1) charge density (2.3) becomes

$$\rho_Q = \mp\frac{4}{\kappa}(\mathrm{tr}[\phi^\dagger,\phi]^2 - v^2\mathrm{tr}\phi^\dagger\phi) \; . \qquad (2.11)$$

For the configurations satisfying Eqs. (2.8) and (2.9), the total angular momentum becomes

$$\begin{aligned} J &= -\int d^2r\,2\epsilon_{ij}r_i\mathrm{tr}(D_0\phi^\dagger D_j\phi + D_j\phi^\dagger D_0\phi) \\ &= \int d^2r\,r_i\partial_i\mathrm{tr}([\phi^\dagger,\phi]^2 - 2v^2\phi^\dagger\phi) \\ &= 2\int d^2r\{v^4C_{\mathrm{phase}} - \mathrm{tr}([\phi^\dagger,\phi]^2 - 2v^2\phi^\dagger\phi)\} \; , \end{aligned} \qquad (2.12)$$

where the non-negative C_{phase} is the spatial asymptotic value of $\mathrm{tr}([\phi^\dagger,\phi]^2 - 2v^2\phi^\dagger\phi)/v^4$. C_{phase} depends on the phase or vacuum the system resides on and will be calculated Sec. III for each phase. As the potential energy density is non-negative, the ground states are characterized by the zeros of the potential energy density, satisfying

$$[\phi,[\phi^\dagger,\phi]] - v^2\phi = 0 \qquad (2.13)$$

which implies $\mathrm{tr}\phi^n = 0$ for any natural number n.

Let us now explore some aspects of the self-duality equations Eqs. (2.9) and (2.10). One can easily see that these equations satisfy $\partial_0\mathrm{tr}\phi^n \mp n\mathrm{tr}\phi^n/kv^2 = 0$ and $(\partial_1 \pm i\partial_2)\mathrm{tr}\phi^n = 0$ for any natural number n, which implies that $\mathrm{tr}\phi^n$ is a (anti)holomorphic function. As the field configuration approaches one of vacua at spatial infinity where $\mathrm{tr}\phi^n = 0$ because of Eq. (2.13), the holomorphic function should vanish everywhere, i.e., $\mathrm{tr}\phi^n(x) = 0$. After triangularization with a similar transformation, the trace conditions imply that the diagonal elements of the triangularized matrix vanish, leading to $\phi^n = 0$ for $n \geq 3$.

The relation $\phi^3(x) = 0$ everywhere is an important property of the self-dual configurations. If $\phi^3 = 0$ and $\phi^2 \neq 0$, there is a three-dimensional complex vector $\mathbf{u}$ such that $\phi^2\mathbf{u} \neq 0$. Three vectors, $\mathbf{u}, \phi\mathbf{u}, \phi^2\mathbf{u}$ are linearly independent and form a basis of a three dimensional complex vector space. Starting from $\phi^2\mathbf{u}$, we can find an orthonormal basis by the standard procedure in linear algebra. In this orthonormal basis, ϕ becomes

$$\phi = v\begin{pmatrix} 0 & f & h \\ 0 & 0 & g \\ 0 & 0 & 0 \end{pmatrix} \; , \qquad (2.14)$$

where dimensionless f, g, h are in general complex. If $\phi^2 = 0$ and $\phi \neq 0$ then, one can show easily that ϕ is again given by Eq. (2.14) with either f or g being zero. As there is an orthonormal basis where ϕ is given by the above triangular matrix, one can see that there is always a special unitary transformation from any ϕ satisfying $\phi^3 = 0$ to this triangular matrix. We can use a local gauge transformation to put the ϕ field in the above form at each spacetime point. Here we will not consider the possibility of configurations, e.g., magnetic monopole instantons, for which there may be a topological obstruction to choose such a gauge globally.

III. GROUND STATES AND SPECTRA

Let us now consider the ground states of the model and explore the unbroken symmetries and particle spectra. As the potential energy in the Lagrangian (2.1) is non-negative, the ground state configurations of zero energy satisfy Eq. (2.13). By identifying $J_z = [\phi^\dagger, \phi]/v^2$, $J_+ = \sqrt{2}\phi^\dagger/v$, $J_- = \sqrt{2}|,\phi/v$, one can see that $J_x = (J_+ + J_-)/2$, $J_y = (J + -J_-)/2i$ and J_z satisfy the angular momentum commutation relation. As ϕ is a 3×3 triangular matrix, "the total angular momentum" can be zero, one half and one, two, and three dimensional representations, respectively.

Alternatively, we notice that the vacuum configurations are the solutions of the self-dual equations. Thus a vacuum configuration can be chosen to the triangular form (2.14). We solve Eq. (2.13) with the triangular scalar field (2.14) to find the vacuum configurations. Since the vacuum energies of three phases are degenerate, there will be topological domain walls interpolating two different vacua.

A. Phase I

Let us consider first the case of the one-dimensional representation. The vacuum expectation value of ϕ becomes $\langle\phi\rangle = 0$. This is the symmetric phase or phase I where the global U(1) and local SU(3) symmetries are preserved. There is no propagating mode for the gauge field. The scalar field ϕ carries unit global charge and forms the adjoint representation of SU(3). The mass of the scalar field is $m \equiv v^2/\kappa$. C_{phase} is Eq. (2.12) vanishes.

B. Phase II

For the case of the two-dimensional representation the vacuum expectation value of ϕ can be chosen to be

$$\langle\phi\rangle_v = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.1}$$

In this case C_{phase} of Eq. (2.12) becomes $\frac{1}{2}$. In the unitary gauge, the scalar field becomes

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha/\sqrt{6} & 0 & v+\delta/\sqrt{2} \\ \beta_1 & -2\alpha/\sqrt{6} & 0 \\ \gamma & \beta_2 & \alpha/\sqrt{6} \end{pmatrix} . \tag{3.2}$$

All components except δ are complex. The fields are normalized to have the standard kinetic term. The masses of the fields are $m_\alpha = m, m_{\beta_1} = m_{\beta_2} = 3m/2$, and $m_\gamma = 2m$, and $m_\delta = 2m$. The gauge field in the unitary gauge bosons

$$A_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} a_\mu/\sqrt{6}+b_\mu/\sqrt{2} & c_{1\mu} & d_\mu \\ \bar{c}_{1\mu} & -2a_\mu/\sqrt{6} & c_{2\mu} \\ \bar{d}_\mu & \bar{c}_{2\mu} & a_\mu/\sqrt{6}-b_\mu/\sqrt{2} \end{pmatrix} \tag{3.3}$$

with masses $m_b = 2m$, $m_{c_1} = m_{c_2} = m/2$, $m_d = m$. The field a_μ is the gauge field for the leftover Abelian local gauge symmetry and there is no corresponding propagating degrees of freedom.

By examining the symmetry generators which leave Eq. (3.1) invariant we can find the unbroken generators. The SU(3)$\times$U(1) group is spontaneously broken to the global U$(1)_R$ and local U$(1)_S$ symmetry group. With the definition, $\tilde{T}^3 \equiv \mathrm{diag}(1,0,-1)$ and $\tilde{T}^8 \equiv (1,-2,1)/\sqrt{3}$, the generators of the unbroken symmetries are given as $R = \int d^2r(\rho_Q - \mathrm{tr}\tilde{T}^3\mathcal{G})$ and $S = \int d^2r\, \mathrm{tr}\tilde{T}^8\mathcal{G}$.

Since the group gauge acting on the adjoint representation is really SU(2)$/Z_3$ where Z_3 is the center of SU(3), the vacuum manifold of phase II would be $[\mathrm{SU}(3)/Z_3\times \mathrm{U}(1)_Q]/[\mathrm{U}(1)_R\times\mathrm{U}(1)_S]$ and 7 dimensional. We argue in the next section that the first fundamental homotopy group of this vacuum manifold is Z_2.

We can write the Lagrangian in terms of the fields (3.2) and (3.3), which would be invariant under these unbroken symmetries. One can calculate the charge density for these generators. For the global U$(1)_R$ symmetry, the charge density is given by

$$\begin{aligned}\rho_Q - \mathrm{tr}\mathcal{G}\tilde{T}^3 = {} & i\{(\alpha\pi_\alpha - \bar{\alpha}\pi_{\bar{\alpha}}) + \tfrac{3}{2}(\beta_i\pi_{\beta_i} - \bar{\beta}_i\pi_{\bar{\beta}_i}) + 2(\gamma\pi_\gamma - \bar{\gamma}\pi_{\bar{\gamma}})\} \\ & +\frac{\kappa}{2}\{2\nabla\times\mathbf{b} - i\mathbf{c}_1\times\bar{\mathbf{c}}_1 - i\mathbf{c}_2\times\bar{\mathbf{c}}_2 - 2i\mathbf{d}\times\bar{\mathbf{d}}\}.\end{aligned} \tag{3.4}$$

For the local U$(1)_S$ symmetry, the charge density is

$$\mathrm{tr}\mathcal{G}\tilde{T}^8 = 3\{-(\beta_1\pi_{\beta_1} - \bar{\beta}_1\pi_{\bar{\beta}_1}) + (\beta_2\pi_{\beta_2} - \bar{\beta}\pi_{\bar{\beta}_2})\} + \kappa\{\nabla\times\mathbf{a} + 3i\mathbf{c}_1\times\bar{\mathbf{c}}_1 - 3i\mathbf{c}_2\times\bar{\mathbf{c}}_2\} . \tag{3.5}$$

In phase II the energy bound (2.7) becomes

$$\mathcal{E} \geq \pm m\int d^2r(\rho_Q - \mathrm{tr}\tilde{T}^3\mathcal{G}) \tag{3.6}$$

because of Gauss's law on the physical configurations. Note that all particles of global U$(1)_R$ charge saturate the above bound.

Here we should note that there could be magnetic monopole instantons in this phase leading to the violation of the charge (3.5) for the local gauge symmetry [9]. The gauge charge would be conserved modulo an integer which depends on the coefficient of the Chern-Simons term and the minimum monopole magnetic flux. Further investigation is necessary to settle this interesting possibility.

C. Phase III

The three-dimensional representation would lead to the ground configuration

$$\langle\phi\rangle_v = v\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.7}$$

In this phase $C_{\rm phase}$ in Eq. (2.12) becomes 2. In the unitary gauge we can choose the scalar field to be

$$\phi = \begin{pmatrix} \chi/\sqrt{30} & v+(\zeta+\xi)/2\sqrt{2} & -3\bar{\chi}/\sqrt{30} \\ \eta/2 & -2\chi/\sqrt{30} & v+(\zeta-\xi)/2\sqrt{2} \\ \psi/\sqrt{2} & \eta/2 & \chi/\sqrt{30} \end{pmatrix} . \tag{3.8}$$

The mass spectrum is given by $m_\chi = 5m$, $m_\eta = 2m$, $m_\psi = 3m$, $m_\zeta = 2m$ and $m_\xi = 6m$. The gauge field in the unitary gauge is given by

$$A_\mu = \begin{pmatrix} p_\mu/2 + q_\mu/2\sqrt{3} & (r_\mu + s_\mu)/2 & t_\mu/\sqrt{2} \\ (\bar{r}_\mu + \bar{s}_\mu)/2 & -q_\mu/\sqrt{3} & (r_\mu - s_\mu)/2 \\ \bar{t}_\mu/\sqrt{2} & (\bar{r}_\mu - \bar{s}_\mu)/2 & -p_\mu/2 + q_\mu/2\sqrt{3} \end{pmatrix} \tag{3.9}$$

with masses $m_p = 2m$, $m_q = 6m$, $m_r = m$, $m_s = 5m$, and $m_t = 2m$.

The original symmetry is then spontaneously broken to a global $\mathrm{U}(1)_U$ symmetry. The generator of this symmetry is $U = \int d^2r(\rho_Q - 2\,\mathrm{tr}\tilde{T}^3\mathcal{G})$, where

$$\begin{aligned} \rho_Q - 2\,\mathrm{tr}\mathcal{G}\tilde{T}^3 = {}& i\{(\chi\pi_\chi - \bar{\chi}\pi_{\bar{\chi}}) + 2(\eta\pi_\eta - \bar{\eta}\pi_{\bar{\eta}}) + 3(\psi\pi_\psi - \bar{\psi}\pi_{\bar{\psi}})\} \\ &+\kappa\{2\nabla\times\mathbf{p} + i\mathbf{r}\times\bar{\mathbf{r}} + i\mathbf{s}\times\bar{\mathbf{s}} + 2i\mathbf{t}\times\bar{\mathbf{t}}\} . \end{aligned} \tag{3.10}$$

In phase III, the energy bound (2.7) becomes

$$E \geq \pm m\int d^2r(\rho_Q - 2\,\mathrm{tr}\mathcal{G}\tilde{T}^3) \tag{3.11}$$

due to Gauss's law. The masses of all charged field except those of χ and s_μ saturate the energy bound. Usually the masses of charged particles in self-dual models saturate the energy bound. This seems to be the first example where the bound is not saturated by charged particles. If our theory is a part of $N = 2$ supersymmetric theory, χ and s_μ would be the bosonic part of a vector supermultiplet. The vacuum manifold of phase III would be given by an 8-dimensional space $[\mathrm{SU}(3)/Z_3\times\mathrm{U}(1)_Q]/\mathrm{U}(1)_U$. We argue in the next section that the first homotopy group of this manifold is Z_3.

IV. SELF-DUAL CONFIGURATIONS

In this section we study some properties of the self-dual configurations which satisfy Eqs. (2.9) and (2.10). Let us first try to classify the possible configurations. There is always a gauge where the scalar field is given by Eq. (2.14) as argued before. For convenience we introduce three dimensionless quantities F, G, H such that

$$F = |f|^2,\quad G = |g|^2,\quad H = |h|^2 , \tag{4.1}$$

where f, g, h are given in Eq. (2.14).

From gauge invariant combinations $\mathrm{tr}\phi^\dagger\phi$, $\mathrm{tr}\phi^\dagger\phi^2$, $\mathrm{tr}(\phi^\dagger\phi)^2$, and $\det[\phi^\dagger,\phi]$ of the scalar field, one can obtain some dimensionless gauge-invariant quantities:

$$\begin{gathered} K = F + G + H,\quad L = FG,\quad M = fg\bar{h} , \\ N = (F - G)L . \end{gathered} \tag{4.2}$$

We classify the nontrivial ($K \neq 0$) solutions of the self-dual equations into three types: type A with $M = L = 0$, type B with $M = 0, L \neq 0$, and type C with $M \neq 0$. As we will see, type A is the simples and type C is the most complicated and interesting. For each phase studied in the previous section, the above three types of self-dual solutions might exist. Some of them would be topological and others would nontopological.

In terms of the above gauge invariant quantities, the global charge density (2.11) for the self-dual configurations becomes

$$\rho_Q = \pm 4mv^2[K(1 - 2K) + 6L] . \tag{4.3}$$

The total angular momentum (2.12) becomes

$$J = 2v^4 \int d^2r\{C_{\mathrm{phase}} + 2(K - K^2 + 3L)\} . \tag{4.4}$$

To understand further implications of the self-dual equations we define $\partial = \partial_1 + i\partial_2$, $\bar{\partial} = \partial_1 - i\partial_2$, $A = A_1 + iA_2$, and $\bar{A} = A_1 - iA_2$. The magnetic field becomes $F_{12} = (\bar{\partial}A - \partial\bar{A} - i[\bar{A}, A])/2i$. From now on we will be only interested in positive Q configurations. Equation (2.9) can be written as

$$\partial\phi - i[A, \phi] = 0 . \tag{4.5}$$

With ϕ given in Eq. (2.14), the above equation implies that A should be a traceless triangular matrix:

$$A = \begin{pmatrix} a_1 & b_1 & c \\ 0 & -(a_1 + a_2) & b_2 \\ 0 & 0 & a_2 \end{pmatrix} . \tag{4.6}$$

Furthermore Eq. (4.5) in components becomes

$$\begin{aligned} \partial f - i(2a_1 + a_2)f &= 0, \\ \partial g + i(a_1 + 2a_2)g &= 0, \\ \partial h - i(a_1 - a_2)h + ib_2 f - ib_1 g &= 0 . \end{aligned} \tag{4.7}$$

With the gauge field (4.6), the off-diagonal components of the self-dual equation (2.10) become

$$\begin{aligned} \partial\bar{b}_1 + i(2a_1 + a_2)\bar{b}_1 - ib_2\bar{c} - 4im^2(2K - 1)g\bar{h} &= 0, \\ \partial\bar{b}_2 - i(a_1 + 2a_2)\bar{b}_2 + ib_1\bar{c} - 4im^2(2K - 1)f\bar{h} &= 0, \\ \partial\bar{c} + i(a_1 - a_2)\bar{c} &= 0 . \end{aligned} \tag{4.8}$$

The diagonal components of Eq. (2.10) become

$$\begin{aligned} &\bar{\partial}a_1 - \partial\bar{a}_1 + i\mid b_1\mid^2 + i\mid c\mid^2 + 4im^2[(2K - 1)(K - G) - 3L] = 0 , \\ &\bar{\partial}a_2 - \partial\bar{a}_2 - i\mid b_2\mid^2 - i\mid c\mid^2 - 4im^2[(2K - 1)(K - F) - 3L] = 0. \end{aligned} \tag{4.9}$$

The self-dual equations in components are then given by Eqs. (4.7), (4.8), and (4.9). Let us now examine more closely what the self-dual equations imply for each type of solutions.

A. Type A solutions

For type A solutions we can easily see that there is a local gauge transformation where $f \neq 0$ and $g = h = 0$ everywhere. (This is a gauge equivalent to the case only h is not vanishing.) Thus, this type of configuration can exist only in phases I and II. As there is no contribution to the energy from a_2, b_2, c, we can regard a_2, b_2, c to be zero. (In the Lagrangian equation, the field strength is zero and so the vector potential can be chosen to be zero.) Then, Eqs. (4.7) and (4.8) lead to

$$\partial(f\bar{b}_1) = 0 . \tag{4.10}$$

As $f\bar{b}_1$ is holomorphic function and the gauge field goes to zero at the spatial infinity, $\bar{b}_1$ should be zero everywhere.

Thus Eqs. (4.7) and (4.8) become identical to the self-dual equations studied in Refs. [1,2] with different numerical factors:

$$\begin{aligned} \partial f - 2ia_1 f &= 0 , \\ \frac{1}{2i}(\bar{\partial}a_1 - \partial\bar{a}_1) + 2m^2F(2F - 1) &= 0 . \end{aligned} \tag{4.11}$$

With $\ln f \equiv \frac{1}{2}\ln F + i\sum_\alpha \mathrm{Arg}(\mathbf{r} - \mathbf{q}_\alpha)$ with vortices at $\mathbf{q}_\alpha$, we can combine the above two equations to

$$\nabla^2 \ln F + 8m^2F(1 - 2F) = 4\pi\sum_\alpha \delta(\mathbf{r} - \mathbf{q}_\alpha) , \tag{4.12}$$

where $\epsilon_{ij}\partial_i\partial_j\mathrm{Arg}(\mathbf{r} - \mathbf{q}) = 2\pi\delta(\mathbf{r} - \mathbf{q})$ is used. The solutions of this equation are made of Q balls in the symmetric phase I and vortices in the asymmetric phase II.

The topology of vortices in the asymmetric phase is interesting. In the SU(2) case, vortices are shown to have the Z_2 topology [3]. In our case, the f field of an elementary vortex in phase II would be given as $f \approx e^{i\varphi}/\sqrt{2}$ at large distance, which is equivalent to applying a gauge transformation $\exp[i\varphi(\lambda_3/2 \pm \lambda_8/\sqrt{3})]$ on the vacuum expectation value $f = 1/\sqrt{2}$. Since both of these mappings lead to the same vortex and are nontrivial elements of the first homotopy group $\pi_1[\mathrm{SU}(1)/Z_3] = Z_3$, the topology of elementary vortices in phase II should be Z_2. This is also supported by the fact that the asymptotic $e^{i2\varphi}/\sqrt{2}$ of the f field for vortices of vorticity 2 is represented by the gauge transformation $\exp[i\varphi\lambda_3]$ which is a trivial element of $\pi_1[\mathrm{SU}(3)/Z_3]$.

There is a simplification of the global charge and the total angular momentum. Equations (4.3) and (4.11) leads to the charge density as a total derivative

$$\rho_Q = 2\kappa\nabla \times \mathbf{a}_1 . \tag{4.13}$$

The total global charge would then get a contribution only from the spatial infinity. To simplify the angular momentum (2.12) we introduce a transverse vector $\tilde{\mathbf{a}}_1 = \mathbf{a}_1 - \frac{1}{2}\sum_\alpha \nabla\mathrm{Arg}(\mathbf{r} - \mathbf{q}_\alpha)$, which is not well defined at the vortex position. Since the angular momentum density is finite everywhere, there is no finite contribution from the vortex positions to the angular momentum (2.12) and the integration region may be reduced from R^2 to $R^2_* =$

$R^2 - \{\mathbf{q}_\alpha\}$. The angular momentum for type A can then be written as

$$\begin{aligned} J &= -8\kappa m^2 \int_{R_*^2} d^2r\, \mathbf{r} \times \tilde{\mathbf{a}}_1 (2F-1)F \\ &= 4\kappa \int_{R_*^2} d^2r\, \mathbf{r} \times \tilde{\mathbf{a}} \nabla \times \tilde{\mathbf{a}}_1 \\ &= 4\kappa \int_{R_*^2} d^2r\, \nabla \cdot \left\{ \frac{1}{2} \mathbf{r} \tilde{\mathbf{a}}_1^2 - \tilde{\mathbf{a}}_1 \mathbf{r} \cdot \tilde{\mathbf{a}}_1 \right\} . \end{aligned} \tag{4.14}$$

For a given type A self-dual configuration, the angular momentum can be evaluated as the sum of the boundary contributions from the vortex positions and spatial infinity. Equation (4.14) was used extensively in the second paper of Ref. [2] to study the vortex dynamics. Especially, the statistical phase of vortices is argued to be originated from both the Aharonov and Bohm phase and the quantum Magnus phase. Similar arguments would apply to our case under the study.

B. Type B solutions

Let us here start by considering types B and C in general terms. For types B and C solutions, $fg \neq 0$ and from Eqs. (4.7) and (4.8) we get

$$\begin{aligned} &\partial(fg\bar{c}) = 0, \\ &\partial(f\bar{b}_1 + g\bar{b}_2) - i(fb_2 - gb_1)\bar{c} = 0 . \end{aligned} \tag{4.15}$$

As the gauge fields vanish at spatial infinity, the first part of Eq. (4.15) implies that $fg\bar{c} = 0$ everywhere, which in turn implies $c = 0$ everywhere. The second part of the above equation implies $f\bar{b}_1 + g\bar{b}_2 = 0$, which can be satisfied by introducing a new variable u such that

$$b_1 = -i\bar{g}u, \quad b_2 = i\bar{f}u . \tag{4.16}$$

Equations (4.7) and (4.16) lead to an equation for the field h:

$$\partial h - i(a_1 - a_2)h - (F+G)u = 0 . \tag{4.17}$$

For type B solutions where $h = 0$, the off-diagonal elements b_i's of the gauge field vanish everywhere as we can see from Eqs. (4.16) and (4.17). Equations (4.7) and (4.9) become

$$\begin{aligned} &\partial f - i(2a_1 + a_2)f = 0 , \\ &\partial g + i(a_1 + a_2)g = 0 , \\ &\bar{\partial} a_1 - \partial \bar{a}_1 + 4im^2(2F - G - 1)F = 0 , \\ &\bar{\partial} a_2 - \partial \bar{a}_2 - 4im^2(2G - F - 1)G = 0 . \end{aligned} \tag{4.18}$$

Equations (4.18) are invariant under two U(1) gauge symmetries. We choose the gauge so that

$$\begin{aligned} \ln f &= \tfrac{1}{2} \ln F + i \sum_\alpha \mathrm{Arg}(\mathbf{r} - \mathbf{q}_{f\alpha}) , \\ \ln g &= \tfrac{1}{2} \ln G + i \sum_\beta \mathrm{Arg}(\mathbf{r} - \mathbf{q}_{g\beta}) , \end{aligned} \tag{4.19}$$

where $\mathbf{q}_{f\alpha}, \mathbf{q}_{g\beta}$ are positions of f, g vortices. Then, Eq. (4.18) can be written as

$$\begin{aligned} &\nabla^2 \ln F - 4m^2(4F^2 - 2G^2 - FG - 2F + G) = 4\pi \sum_\alpha \delta(\mathbf{r} - \mathbf{q}_{f\alpha}) , \\ &\nabla^2 \ln G - 4m^2(-2F^2 + 4G^2 - FG + F - 2G) = 4\pi \sum_\beta \delta(\mathbf{r} - \mathbf{q}_{g\beta}) . \end{aligned} \tag{4.20}$$

We expect type B solutions in all three phases. In phase II, one of f or g would take the vacuum expectation value $1/\sqrt{2}$ at spatial infinity. By similar argument for vortices of type A in phase II, elementary vortices of type B in phase II would have a topology Z_2. In phase III, vortices of type B would have the Z_3 topology. To see this, we assume that at spatial infinity $f \approx e^{ik\varphi}$ and $g \approx e^{il\varphi}$ with integers k, l. This is equivalent to a gauge transformation $\exp[i\varphi\, \mathrm{diag}(2k + l, -k + l, -k - 2l)/3]$ of $\langle \phi \rangle$, which is a Z_3 element of SU(3).

From Eq. (4.18), the global charge density (4.3) for type B becomes

$$\rho_Q = 2\kappa \nabla \times (\mathbf{a}_1 - \mathbf{a}_2) . \tag{4.21}$$

Since the gauge fields should be smooth functions, the total charge will get a contribution only from spatial infinity. To understand the angular momentum better, let us define

$$\begin{aligned} \tilde{a}_1 &= a_1 - [2\partial \mathrm{Arg}(f) + \partial \mathrm{Arg}(g)]/3 , \\ \tilde{a}_2 &= a_2 + [\partial \mathrm{Arg}(f) + 2\partial \mathrm{Arg}(g)]/3 . \end{aligned} \tag{4.22}$$

From Eq. (4.18) one can see they are transverse vector fields. Similar to type A, we subtract the vortex positions from the integration domain, $R_*^2 = R^2 - \{\mathbf{q}_{f\alpha}, \mathbf{q}_{g\beta}\}$ without any change of the angular momentum. With Eqs. (2.12) and (4.16) the angular momentum becomes

$$\begin{aligned} J &= 2\kappa \int_{R_*^2} d^2r \{ \nabla \times \tilde{\mathbf{a}}_1 (2\mathbf{r} \times \tilde{\mathbf{a}}_1 + \mathbf{r} \times \tilde{\mathbf{a}}_2) + \nabla \times \tilde{\mathbf{a}}_2 (2\mathbf{r} \times \tilde{\mathbf{a}}_2 + \mathbf{r} \times \tilde{\mathbf{a}}_1) \} \\ &= 2\kappa \int_{R_*^2} d^2r\, \nabla \cdot \{ \mathbf{r}\tilde{\mathbf{a}}_i^2 - 2\tilde{\mathbf{a}}_i (\mathbf{r} \cdot \tilde{\mathbf{a}}_i) + \mathbf{r}(\tilde{\mathbf{a}}_1 \cdot \tilde{\mathbf{a}}_2) - \tilde{\mathbf{a}}_1 (\mathbf{r} \cdot \tilde{\mathbf{a}}_2) - \tilde{\mathbf{a}}_2 (\mathbf{r} \cdot \tilde{\mathbf{a}}_1) \} . \end{aligned} \tag{4.23}$$

Thus, the total angular momentum would get contributions from the vortex positions and spatial infinity. For a given self-dual configuration, we can write down the total angular momentum as a function of vortex positions in principle. As discussed for type A, Eq. (4.23) would lead to a considerable understanding of the dynamics of the slowly moving vortices.

C. Type C solutions

From Eqs. (4.8) and (4.16) we get the equation for the u field:

$$\partial\bar{u} + i(a_1 - a_2)\bar{u} - 4m^2(2K-1)\bar{h} = 0 \,. \tag{4.24}$$

From Eqs. (4.7), (4.17), and (4.24) we get

$$\begin{aligned} &\partial(fg\bar{u}) - 4m^2(2K-1)fg\bar{h} = 0 \,, \\ &\partial(h\bar{u}) - (F+G)\mid u\mid^2 - 4m^2(2K-1)H = 0 \,, \end{aligned} \tag{4.25}$$

which implies that $\partial(h\mathbf{u})$ is a real field.

Since $M = fg\bar{h}$ is a gauge invariant quantity, the vortices of the f and g field should be closely related to that of the h field. However, it is not easy to see what kind of solutions will exist because of the self-dual equations Eqs. (4.7), (4.9), (4.17), and (4.25) are rather complicated. In principle, type C could exist in all phases of the theory. The topology of type C vortices in the broken phases would be identical to that of type A or B vortices because type C solutions become type A or type B at spatial infinity.

Note that the charge density is given as a total derivative:

$$\rho_Q = \frac{\kappa}{2i}[\bar{\partial}(a_2 - a_1) - \partial(\bar{a}_2 - \bar{a}_1)] + \kappa\partial(h\bar{u}) \,. \tag{4.26}$$

However, we have not succeeded in expressing the angular momentum as a boundary contribution as in Eq. (4.23). The self-dual equations satisfied by type C is rather complicated and needs a further consideration.

V. CONCLUSION

We have studied the self-dual Chern-Simons Higgs systems with SU(3) gauge symmetry and U(1) global symmetry. The matter field is made of a complex scalar field in the adjoint representation. Our work is a first step toward understanding the self-dual Chern-Simons Higgs systems where the non-Abelian symmetry plays a crucial role. We have analyzed the vacuum structure, particle spectrum, and unbroken symmetries. In addition, we classified the self-dual configurations into three types of increasing complexity. We have shown that vortices in phase II would have the Z_2 topology and vortices in phase III would have the Z_3 topology. We have seen the global charge of the self-dual configurations is given as a boundary contribution from spatial infinity, making topological the total energy of those configurations. In addition, the self-dual configurations are characterized by the total angular momentum, which we have shown to take a rather simple form for at least types A and B.

Ideally, we want to understand the nature of self-dual solitons completely and there are many directions to take to reach that goal. Here are some ideas to be explored: the rotationally symmetric solutions, the topological domain walls interpolate degenerate vacua, the self-dual solutions of type C, the classical dynamics of slowly moving solitons, the relation between relativistic and nonrelativistic solutions, and the possible magnetic monopole instantons in phase II. We would also like to understand the quantum aspects of these solitons. One novel possibility might be the "non-Abelian Magnus force and statistical phase" between vortices in the asymmetric phase. We finally note that some of understandings gained here could be easily generalized to the cases with more complicated gauge groups and matter fields.

ACKNOWLEDGMENTS

We thank Alexios Polychronakos for useful discussions. This work was supported in part by the NSF Presidential Young Investigator program (K.L.), the Alfred P. Sloan Foundation (K.L.), and Department of Energy (H.-C.K.).

[1] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. **64**, 2330 (1990); R. Jackiw and E. J. Weinberg, *ibid.* **64**, 2334 (1990).

[2] R. Jackiw, K. Lee, and E. Weinberg, Phys. Rev. D **42**, 3488 (1990); Y. Kim and K. Lee, *ibid.* **49**, 2041 (1994).

[3] K. Lee, Phys. Lett. B **255**, 381 (1991); K. Lee, Phys. Rev. Lett. **66**, 553 (1991); G. V. Dunne, Phys. Lett. B **324**, 359 (1994).

[4] B. Gossman, Phys. Rev. Lett. **65**, 3230 (1990); G. Dunne, R. Jackiw, S.-Y. Pi, and C. Trugenberger, Phys. Rev. D **43**, 1332 (1991); G. Dunne, Ann. Phys. (N.Y.) **223**, 180 (1993); G. V. Dunne, "Classification of Nonabelian Chern-Simons Vortices," Connecticut University Report No. UCONN-93-8, hep-th/9310182, 1993 (unpublished).

[5] E. Witten and D. Olive, Phys. Lett. **78B**, 97 (1978); P. Di Vecchia and S. Ferrara, Nucl. Phys. **B130**, 93 (1977); Z. Hlousek and D. Spector, *ibid.* **B397**, 173 (1993).

[6] C. Lee, K. Lee, and E. J. Weinberg, Phys. Lett. B **243**, 105 (1990); E. A. Ivanov, *ibid.* **268**, 203 (1991); S. J. Gates and H. Nishino, *ibid.* **281**, 72 (1992); B. H. Lee, C. Lee, and H. Min, Phys. Rev. D **45**, 4588 (1992).

[7] H.-C. Kao and K. Lee, Phys. Rev. D **46**, 4691 (1992); H.-C. Kao, Phys. Rev. D **50**, 2881 (1994).

[8] J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); S. Deser, R. Jaciw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 37 (1982)].

[9] K. Lee, Nucl. Phys. **B373**, 735 (1992); F. A. Bais, P. van Direl, and M. de Wild Propitius, *ibid.* **B393**, 547 (1993).