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Differential equations for definition and evaluation of Feynman integrals

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It is shown that every Feynman integral can be interpreted as a Green function of some linear differential operator with constant coefficients. This definition is equivalent to the usual one but needs no regularization and application of the R operation. It is argued that the presented formalism is convenient for practical calculations of Feynman integrals.

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Though fundamental results in renormalization theory were obtained many years ago in classical works of Feynman, Tomonaga, Schwinger, Dyson, Salam, Bogolubov, Parasiuk, Hepp, and Zimmermann,[1] renormalization problems continue to attract the attention of theorists. In particular, during the last twenty years very many papers were devoted to investigations of various regularization schemes.

Of course, all known regularization schemes are equivalent, in principle, at the perturbative level. However, their practical value is different. For instance, only the discovery of dimensional regularization [4] allowed the possibility to carry out systematical calculations in gauge theories. Moreover, different regularization schemes that are equivalent at the perturbative level can be inequivalent beyond perturbation theory. For instance, a partial summing of a perturbation series by means of renormalization group methods can give scheme-dependent results (see, for instance, [6]). This fact stimulates a further search of "the most natural" and convenient regularization scheme.

In this paper we will show that Feynman integrals can be defined and evaluated without any regularization at all. Of course, in itself it is not a surprise. In particular, the recently proposed differential regularization [5] also needs no regularization in the usual sense. But the simplicity of our results is the real surprise. *We will show that any Feynman diagram without internal vertices can be treated as a Green function of some linear differential operator with constant coefficients.* This result allows us also to define and evaluate Feynman diagrams with internal vertices because such diagrams can be considered as certain diagrams without internal vertices at a zero value of some external momenta. For instance, the value of the diagram with internal vertices in Fig. 1 coincides with the value at $k = q = 0$ of diagram in Fig. 2.

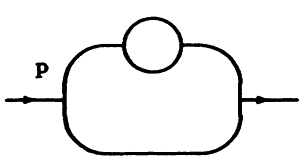

FIG. 1. Example of a diagram with internal vertices.

The renormalization scheme, given in this paper, is equivalent to the usual R operation scheme. But "equivalent" does not mean "the same." Indeed, in the standard R operation renormalization scheme one must, first, regularize the initial divergent (in general) Feynman integral. Then it is necessary to use a rather complicated subtraction prescription (forest formula) to obtain a finite result. Nothing similar is needed in my renormalization scheme. To obtain a finite expression for a given Feynman integral, one must only solve some well-defined differential equations. Neither any regularization, nor any manipulations with counterterm diagrams are needed to obtain a finite result.

For simplicity, in this paper we will consider only scalar Feynman integrals. The general case will be investigated in the forthcoming longer paper.

Let us consider an arbitrary (Euclidean) Feynman diagram without internal vertices in coordinate space. This is the well-defined function

$$\tilde{\Gamma}_n(x_1, ..., x_n; \{m_{ij}^2\}) = \prod_{\text{all lines of } \Gamma} D(x_i - x_j; m_{ij}^2), \quad (1)$$

where

$$D(x, m^2) = \int d^4p \frac{e^{ipx}}{p^2 + m^2}.$$

But their Fourier image

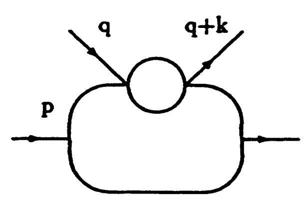

FIG. 2. The diagram without internal vertices that corresponds to one in Fig. 1.

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[1]A beautiful account of foundations and modern achievements of renormalization theory can be found, for instance, in monographs [1–3].

$$\Gamma_n(p_1,...,p_{n-1};\{m_{ij}^2\}) = \frac{1}{(2\pi^4)^{n-1}}\int d^4x_1\cdots d^4x_{n-1}\exp(ip_1x_1+\cdots+ip_{n-1}x_{n-1})\tilde{\Gamma}(x_1,...,x_{n-1},0;\{m_{ij}^2\}) \quad (2)$$

is not, in general, well defined. The problem of renormalization theory is to define the function $\Gamma(p_1,...,p_{n-1};\{m_{ij}^2\})$.

Below we will interpret m_{ij}^2 as the square of some Euclidean two-dimensional vector. Then we can write

$$m_{ij}^2 = (m_{ij,1})^2 + (m_{ij,2})^2.$$

Further, let us define the Fourier image of $D(x,m^2)$ with respect to variables m_1 and m_2 (here $m_1^2+m_2^2=m^2$):

$$\hat{D}(x,u) = \int d^2m e^{i\mathbf{u}\cdot\mathbf{m}}D(x,m) = \int d^4pd^2m\frac{e^{ipx+i\mathbf{u}\cdot\mathbf{m}}}{p^2+m^2}$$

where $\mathbf{u}=(u_1,u_2)$, $u^2=u_1^2+u_2^2$. It follows from the definition that $\hat{D}(x,u)$ is the Green function of a six-dimensional Laplace operator:

$$\triangle_{xu} = \sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2}+\sum_{i=1}^{2}\frac{\partial^2}{\partial u_i^2}.$$

Therefore,

$$\hat{D}(x,u) = 16\pi^3\frac{1}{(x^2+u^2)^2}$$

and

$$\hat{\Gamma}_n(x_1,...,x_n;\{u_{ij}^2\}) \equiv \int\prod_{\text{all lines of }\Gamma}d^2m_{ij}\exp\left(i\sum_{\text{all lines of }\Gamma}\mathbf{m}_{ij}\mathbf{u}_{ij}\right)\tilde{\Gamma}_n(x_1,...,x_n;\{m_{ij}^2\})$$
$$= (16\pi^3)^N\frac{1}{P(x_1,...,x_n;\{u_{ij}^2\})},$$

where N is the total number of lines in diagram Γ and P is the polynomial:

$$P = \prod_{\text{all lines of }\Gamma}[(x_i-x_j)^2+u_{ij}^2]^2. \quad (3)$$

We see that $\hat{\Gamma}$ satisfies the simple algebraic equation

$$P(x_1,...,x_n;\{u_{ij}^2\})\hat{\Gamma}(x_1,...,x_n;\{u_{ij}\}) = (16\pi^3)^N. \quad (4)$$

Comparing (2), (3), and (4), we see that it is very natural *to define* $\Gamma(p_1,...,p_{n-1})$ as a solution of the differential equation

$$P\left(i\frac{\partial}{\partial p_1},...,i\frac{\partial}{\partial p_{n-1}},0;\{\triangle_{m_{ij}}\}\right)\Gamma_n(p_1,...,p_{n-1};\{m_{ij}^2\})$$
$$= (16\pi^3)^N\delta(p_1)\cdots\delta(p_{n-1})\prod_{\text{all lines of }\Gamma}\delta(\mathbf{m}_{ij}). \quad (5)$$

This means that Γ is the Green function of the linear differential operator $P\,(i\partial/\partial p_1,...)$. For instance, the diagram in Fig. 2 is defined by

$$(\triangle_{pm_1})^2(\triangle_{(p-q)m_2})^2(\triangle_{km_3})^2(\triangle_{qm_4})^2(\triangle_{qm_5})^2\Gamma$$
$$= (16\pi^3)^5\delta(p)\delta(q)\delta(k)\prod_{i=1}^{5}\delta(\mathbf{m}_i),$$

where

$$\triangle_{(p-q)m_2} = \sum_{i=1}^{4}\left(\frac{\partial}{\partial p_i}-\frac{\partial}{\partial q_i}\right)^2+\sum_{i=1}^{2}\frac{\partial^2}{\partial m_{2,i}{}^2}.$$

Equation (5) defines Γ up to a solution of the homogeneous equation:

$$P\left(i\frac{\partial}{\partial p_1},...,i\frac{\partial}{\partial p_{n-1}},0;\{\triangle_{m_{ij}}\}\right)\Gamma = 0.$$

This arbitrariness can be fixed inductively in the following way.

All diagrams with $(L-1)$ loops are already defined. For a given L-loop diagram with a divergent index $\omega(\Gamma)$ one defines the $(L-1)$-loop diagram Γ_{ij} as a diagram Γ without the line (ij) with the propagator $[(p-k)^2+m_{ij}^2]^{-1}$ where p is the external momentum. We can always define external momenta in such way that Γ_{ij} does not depend on p. (See Fig. 3, where Γ_{ij} is represented as shaded block.)

It is easy to see that if Γ satisfies

$$(\triangle_{pm_{ij}})^2\Gamma = (16\pi^3)\delta(\mathbf{m}_{ij})\Gamma_{ij}, \quad (6)$$

then Γ also satisfies Eq. (5). Finally, we impose the asymptotic conditions

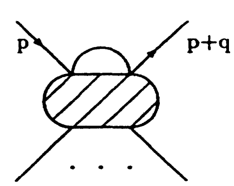

FIG. 3. Illustration to the proof of equivalence of proposed renormalization scheme and the usual one.

$$\lim_{|p|\to\infty} \frac{1}{p^{\omega(\Gamma)+\epsilon}}\Gamma = 0\,, \quad \lim_{m_{ij}\to\infty} \frac{1}{m_{ij}^{\omega(\Gamma)+\epsilon}}\Gamma = 0\,, \qquad (7)$$

for any $\epsilon > 0$.

It is easy to prove that Eq. (6) together with analogous equations for other lines with asymptotic conditions (7) define Γ up to a polynomial of degree $\omega(\Gamma)$ with respect to external momenta and masses. [For $\omega(\Gamma) < 0$ the diagram Γ is defined unequivocally if $(L-1)$-loop diagrams are already defined.] So our definition reproduces the usual renormalization arbitrariness in the definition of Feynman integrals.

Now let us prove that for renormalizable theories with a divergent index less or equal to two, our definition is equivalent to the usual one. Consider again the diagram in Fig. 3 regularized by means of the cutoff at large momentum Λ. We will denote this diagram as Γ_Λ. The renormalized diagram $\Gamma^{\text{ren}}(\Lambda)$ is the sum of Γ_Λ and counterterm diagrams. The latter ones can be divided into two sets. The first set of counterterm diagrams contains the line (ij). The sum of Γ_Λ and these diagrams can be written as

$$\int_{|k|<\Lambda} d^4k \frac{1}{(p-k)^2+m_{ij}^2}\Gamma_{ij}^{\text{ren}}(\Lambda), \qquad (8)$$

where $\Gamma_{ij}^{\text{ren}}(\Lambda)$ is renormalized diagram Γ_{ij}. This diagram does not depend on p.

The second set of counterterm diagrams does not contain the line (ij). They are produced by the change of some divergent subdiagrams of Γ that contain the line (ij), on polynomials not more than second degree with respect to external momenta and masses (see, for instance [1–3]). In particular, these polynomials are not more than second degree with respect to p and m_{ij}. The diagram $\Gamma^{\text{ren}}(\Lambda)$ is the sum of (8) and these polynomials.

Using the formulas

$$(\triangle_{pm_{ij}})^2 \frac{1}{(p-k)^2+m_{ij}^2} = 16\pi^3\delta(p-k)\delta(\mathbf{m}_{ij}) \qquad (9)$$

and (8), one can prove that

$$(\triangle_{pm_{ij}})^2\Gamma^{\text{ren}}(\Lambda) = 16\pi^3\delta(\mathbf{m}_{ij})\theta(\Lambda-|p_i|)\Gamma_{ij}^{\text{ren}}(\Lambda) \quad (10)$$

[because the above-mentioned polynomials are annihilated by $(\triangle_{pm_{ij}})^2$]. In the limit $\Lambda \to \infty$ one obtains the Eq. (6). Asymptotic conditions (6) are satisfied due to Weinberg's theorem [8] . This finishes the proof.

Now let us consider an illustrative example that shows how our definition works in practical calculations.

The diagram $\Gamma^{(1)}$ in Fig. 4 is the simplest divergent Feynman one. According to our general theory it is defined by the equations

$$(\triangle_{pm_1})^2\Gamma^{(1)} = \frac{16\pi^3\delta(\mathbf{m}_1)}{p^2+m_2^2}, \qquad (11)$$

$$(\triangle_{pm_2})^2\Gamma^{(1)} = \frac{16\pi^3\delta(\mathbf{m}_2)}{p^2+m_1^2}. \qquad (12)$$

Using the formula[2]

$$\frac{1}{p^2+m_2^2} = \frac{1}{2\pi i}\int_{\substack{c-i\infty \\ 0<\text{Re }c<1}}^{c+i\infty} ds\Gamma(s)\Gamma(1-s)\frac{(m_2^2)^{s-1}}{(p^2)^s}, \quad (13)$$

one can represent the solution of (11) in the form

$$\Gamma^{(1)} = \frac{1}{2\pi i}\int_{\substack{c-i\infty \\ 0<\text{Re }c<1}}^{c+i\infty} ds\Gamma(s)\Gamma(1-s)(m_2^2)^{s-1}\Gamma_s + f_1(m_2^2) \qquad (14)$$

where Γ_s satisfies the equation

$$(\triangle_{pm_1})^2\Gamma_s = \frac{16\pi^3\delta(\mathbf{m}_1)}{(p^2)^s} \qquad (15)$$

and $f_1(m_2^2)$ is an arbitrary function.

If in (15) $1 < \text{Re } s < 2$, then the solution of (15) is a convergent Feynman integral:

$$\Gamma_s = \int d^4k \frac{1}{[(p-k)^2+m_1^2](k^2)^s}. \qquad (16)$$

[This can be proved by using of formula (9).]

Introducing Feynman parameters, one can write Γ_s in the form

$$\Gamma_s = \frac{\pi^2}{s-1}\int_0^1 dx\left[\frac{x}{(1-x)(xp^2+m_1^2)}\right]^{s-1}. \qquad (17)$$

We see that Γ_s can be analytically continued in the strip $0 < \text{Re } s < 1$ before integration with respect to x. Substituting (17) in (14) and integrating with respect to s, one obtains

$$\Gamma^{(1)} = -\pi^2\int_0^1 dx \,\ln\left(1+\frac{(1-x)(xp^2+m_1^2)}{xm_2^2}\right) + f_1(m_2^2). \qquad (18)$$

Let μ^2 is an arbitrary constant with dimension of $[\text{mass}]^2$. Then (18) can be rewritten in the form

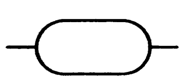

FIG. 4. The simplest divergent Feynman diagram.

[2]Representation (13) was used first for the evaluation of Feynman integrals in [9].

$$\Gamma^{(1)} = \left\{-\pi^2 \int_0^1 dx \,\ln\left(\frac{x(1-x)p^2 + xm_1^2 + (1-x)m_2^2}{\mu^2}\right)\right\} + \pi^2\left(1 + \ln\frac{\mu^2}{m_2^2}\right) + f_1(m_2^2). \tag{19}$$

Equation (12) can be solved in an analogous way. Comparing results, one can prove that the sum of terms outside the curly brackets in (19) is constant. It can be included in the definition of μ^2. So, finally, we obtain a familiar result:

$$\begin{aligned}\Gamma^{(1)}(p^2, m_1^2, m_2^2) &= -\pi^2 \int_1^0 dx \ln\left(\frac{x(1-x)p^2 + xm_1^2 + (1-x)m_2^2}{\mu^2}\right) \\ &= \pi^2\left\{\ln\frac{\mu^2}{m_1 m_2} - \frac{(m_1^2 - m_2^2)}{2p^2}\ln\frac{m_2^2}{m_1^2} + f\ln\frac{p^2 + m_1^2 + m_2^2 - 2p^2 f}{p^2 + m_1^2 + m_2^2 + 2p^2 f}\right\},\end{aligned} \tag{20}$$

where

$$f = \sqrt{\frac{(m_1^2 - m_2^2)^2}{4p^4} + \frac{m_1^2 + m_2^2}{2p^2} + \frac{1}{4}}.$$

We see that a finite result for a divergent diagram in Fig. 4 can be obtained without any regularization and application of the R operation.

Now let us consider a less trivial application of our theory. We will calculate the two-loop diagram in Fig. 5. In general, this diagram depends on three different masses m_1, m_2, and m_3. For simplicity, we will consider only the most important case $m_1 = m_2 \equiv m$, $m_3 \equiv M$. The general case can be treated analogously.

Power expansions for this diagram were investigated for the equal mass case in [10] and for the general case in a recent work [11]. See also [12] for the corresponding numerical results.

Up to a polynomial of first degree with respect to p^2, the corresponding Feynman integral can be defined by

$$(\triangle_{pM})^2\Gamma^{(2)}(p^2, M^2, m^2) = 16\pi^3\delta(\mathbf{M})\Gamma^{(1)}(p^2, m^2, m^2). \tag{21}$$

Up to an insignificant constant, $\Gamma^{(1)}$ can be represented in the form

$$\Gamma^{(1)}(p^2, m^2, m^2) = \pi^2\int_{4m^2}^{\infty} d\sigma^2\sqrt{1 - \frac{4m^2}{\sigma^2}} \times \left(\frac{1}{p^2 + \sigma^2} - \frac{1}{\sigma^2}\right). \tag{22}$$

Comparing (11), (21), and (22), we see that $\Gamma^{(2)}$ can be represented as

$$\Gamma^{(2)}(p^2, M^2, m^2) = \pi^2\int_{4m^2}^{\infty} d\sigma^2\sqrt{1 - \frac{4m^2}{\sigma^2}} \times \left(\Gamma^{(1)}(p^2, M^2, \sigma^2) - \frac{1}{\sigma^2}\varphi(p^2, M^2, \sigma^2)\right), \tag{23}$$

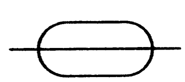

FIG. 5. Rising sun diagram.

where $\Gamma^{(1)}(p^2, M^2, \sigma^2)$ is defined by (20) and $\varphi(p^2, M^2, \sigma^2)$ satisfies the equation

$$(\triangle_{pm})^2\varphi = 16\pi^3\delta(\mathbf{M}). \tag{24}$$

Using the formula

$$\triangle_M \ln M^2 = 2\pi\delta(\mathbf{M})$$

one can represent the solution of (24) in the form

$$\varphi = \pi^2\left\{M^2\ln\frac{M^2}{\sigma^2} + M^2 f_1(\sigma^2) + p^2 f_2(\sigma^2) + f_3(\sigma^2)\right\}, \tag{25}$$

where $f_1, f_2,$ and f_3 are arbitrary functions of σ^2. These functions must be defined so that integral (23) converges. Using the explicit formula (20) for $\Gamma^{(1)}$, it is easy to prove that one of the possible choices is

$$f_1 = 0,\ f_2 = -\frac{1}{2},\ f_3 = \sigma^2\left(\ln\frac{\mu^2}{\sigma^2} - 1\right). \tag{26}$$

The replacement of functions $f_1, f_2,$ and f_3 by any other ones, for which the integral (22) converges, leads to an insignificant change of $\Gamma^{(2)}$ on the polynomial of first degree with respect to p^2.

Substituting of (25) and (26) in (23), we obtain our final result:

$$\Gamma^{(2)} = \pi^2\int_{4m^2}^{\infty} d\sigma^2\sqrt{1 - \frac{4m^2}{\sigma^2}} \times \left(\Gamma^{(1)}(p^2, M^2, \sigma^2) - \pi^2\frac{M^2}{\sigma^2}\ln\frac{M^2}{\sigma^2} - \pi^2\frac{p^2}{2\sigma^2} - \pi^2\ln\frac{\mu^2}{\sigma^2} + \pi^2\right). \tag{27}$$

The integrand in (27) is of order $O(\sigma^{-4}\ln\sigma^2)$ at $\sigma^2 \to \infty$. So integral (27) converges.

To the author's knowledge, integral (27) cannot be expressed through standard special functions. But the integrand in (23) is a rather simple elementary function and so this formula makes it possible to investigate $\Gamma^{(2)}$ in detail. This will be done in a forthcoming paper.

A onefold integral representation, that is very similar to (27), was obtained independently in works [11] and

[12] by dispersive methods. An analogous representation for five propagators self-energy diagram can be found in [13,14].

Now it is unclear whether our approach to renormalization theory has principal advantages in comparison with the standard formulation. But, at least, the calculations, represented above, show that our approach gives new effective methods of evaluating Feynman integrals. So the author believes that the proposed formalism will be useful in various investigations in quantum field theory.

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