Curved-space quantization: Toward a resolution of the Dirac versus reduced quantization question

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It is well known that Dirac quantization of gauge theories is not, in general, equivalent to reduced quantization. When both approaches are self-consistent some additional criterion must be found in order to decide which approach is more natural, or correct. Now, in many cases quantization on the physical degrees of freedom is properly curved-space quantization, with a highly nontrivial curvature: neither constant nor Ricci flat. On the other hand, the configuration space of the unreduced gauge theory is often (Ricci) flat, which makes Dirac quantization considerably simpler. We show that the natural "minimal" Dirac quantization scheme, together with certain restrictions we impose on the observables, is sufficient to make the quantum commutator of quadratic observables free of van Hove anomalies. This means the "minimal" Dirac quantization (acting in the physical Hilbert space) is actually a curved-space quantization scheme suitable for the type of curvature mentioned above, at least within a restricted (but still interesting) class of observables. In fact, we demonstrate that this curved-space quantization scheme, unlike "minimal" reduced quantization, has remarkable similarities with other curved-space quantization schemes proposed elsewhere. However, unlike these other schemes, it contains a piece which depends in an essential way on the gauge structure of the unreduced theory, and so could not have been guessed working strictly from within the classical reduced theory.

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I. INTRODUCTION

This is the second of two papers addressing the question of Dirac versus reduced quantization of gauge theories, which has received some attention in the recent literature [1–6]. Since gauge theories play a prominent role in our current understanding of nature it is important to understand them in depth, at both the classical and quantum levels (see, e.g., [7,8]). At the classical level the Lagrangian is invariant under local gauge transformations, which means the manifold M, representing the instantaneous configurations of the system, contains redundant, or gauge degrees of freedom. These extra degrees of freedom constitute so-called gauge orbits, and can be divided out, resulting in the reduced configuration space m, representing the physical degrees of freedom. In the phase space analysis the redundant degrees of freedom manifest themselves as constraints on the full phase space, $\Gamma = T^*M$, which are linear in the momenta. Solving these constraints, and again dividing out the gauge degrees of freedom, yields the reduced phase space, $\gamma = T^*m$. Classical reduction is then the map which takes an observable on Γ to the corresponding observable on γ.

For quantization one then has two choices: so-called reduced quantization of the unconstrained theory on γ, or Dirac quantization [9] on Γ. In the latter the constraints are quantized and applied as operators to select a subset of physical states from the full set of quantum states, and this subset is subsequently endowed with a Hilbert space structure, and called the physical Hilbert space. It is well known that these two approaches, Dirac versus reduced, generally lead to distinct quantum systems [1–6]. It is even known that this difference can be understood as a factor ordering ambiguity of the Hamiltonian acting in the physical Hilbert space [6]. In this series of papers we further illuminate the geometrical significance of this ambiguity, and provide clues to its resolution.

In the first paper [10] (henceforth referred to as I) we analyzed the rich geometry of the classical theory, for instance, the classical reduction, emphasizing the role of the horizontal basis in establishing a Yang-Mills connection on M. Furthermore, we calculated the Ricci tensors associated with three Levi-Cività connections on M, and discussed their interrelationships. We make use of these results in this second paper, as we proceed with the quantum analysis.

We first observe that, although the kinetic energy term in the Hamiltonian often induces a flat metric on M (or at least, as we suppose here, Ricci flat), the reduced configuration space m is usually neither Ricci flat nor of constant curvature (examples include scalar electrodynamics [11], and the "helix model" [3] used in I). This means reduced quantization is really curved-space quantization, which is highly nontrivial and ambiguous for observables quadratic (or higher order) in the momenta [12–16]. Such a nontrivial curvature also aggravates van Hove anomalies [17–19] in quantum commutators of quadratic observables, which may jeopardize, for example, the quantum realization of classical symmetries with generators quadratic in the momenta (for example, the boost generators, as well as the Hamiltonian, in the Poincaré symmetry [20]).

Dirac quantization, on the other hand, is much less troublesome, at least in this respect; instead we have subtleties associated with the gauge degrees of freedom. In particular, we find that minimal[1] Dirac quantization cannot be applied consistently to all classical observables quadratic in the momenta. In this paper we suggest restrictions on the quadratic observables that are sufficient to guarantee this consistency, as well as eliminate van Hove anomalies in quadratic-quadratic commutators. This means the Dirac factor ordering (acting in the physical Hilbert space) is actually a specific curved-space quantization scheme which is free of van Hove anomalies, at least within a restricted (but still interesting) class of observables and curvature. In fact, by using the geometrical results in I, together with the restrictions on observables mentioned above, we demonstrate a remarkable similarity between this Dirac quantization and other curved-space quantization schemes which have been proposed from time to time in the literature.

The paper is organized as follows. We begin with a brief introduction to the Dirac versus reduced factor ordering ambiguity, including a discussion of the consistency condition mentioned above. This is followed by a discussion of van Hove anomalies, along with a survey of various curved-space quantization schemes found in the literature. We then suggest sufficient restrictions for Dirac quantization to be free of van Hove anomalies for quadratic-quadratic commutators, and describe how, within these restrictions, Dirac quantization resembles the other curved-space quantization schemes. These results are illustrated using Kuchař's helix model [3] again, as we did in I (see Sec. II of I for our notation).[2] Finally we present conclusions, and suggest directions for future research.

II. THE DIRAC VERSUS REDUCED FACTOR ORDERING AMBIGUITY

The classical reduction of a generic observable $\mathcal{C}(K)$ on the full phase space Γ, which is homogeneous quadratic in the momenta, is summarized by

$$\mathcal{C}(K) := K^{AB}P_A P_B \mapsto c(k) := k^{ab}p_a p_b\,, \tag{1}$$

where $c(k)$ is the corresponding physical observable on the reduced phase space γ (refer to I for notation, etc.). Here k, a tensor on the reduced configuration space m, is the physical projection of its counterpart K on the full configuration space, M. We are interested primarily in observables quadratic in the momenta (such as the kinetic energy term in the Hamiltonian) since it is in the quantization of these that the Dirac versus reduced factor ordering ambiguity first emerges.[3]

[1]This term is defined later.

[2]As pointed out by Kuchař, this toy model is actually a finite dimensional analogue of scalar electrodynamics—the latter is discussed, in the above context, elsewhere [20].

[3]A more complete discussion, including observables lower order in momenta, is given in [21].

We begin with reduced quantization. Since $\gamma = T^*m$ it is natural to choose Schrödinger picture quantization. Furthermore, m is equipped with a (nontrivial) metric g (inherited from the metric G on M) so we choose the natural Hilbert space $\mathcal{H}_{\rm phys}{:=}L^2(m, \mathbf{e}^{(g)})$ of smooth square integrable complex-valued functions on m. Here $\mathbf{e}^{(g)}$, used in the inner product, is the volume form associated with g. The reduced quantization map is

$$\begin{aligned} c(k) = k^{ab}p_a p_b \mapsto q^{\rm red}(k) &:= -\hbar^2\tilde\nabla_a k^{ab}\tilde\nabla_b \\ &= -\hbar^2\{k^{ab}\tilde\nabla_a\tilde\nabla_b + (\tilde\nabla_a k^{ab})\tilde\nabla_b\}\,, \end{aligned}\tag{2}$$

where $\tilde\nabla$ is the Levi-Cività connection on m. This will be called "minimal" quantization in that, given the leading term (quadratic in derivatives) the additional complementary term, linear in derivatives, is the minimum one necessary to make the operator self-adjoint. Of course one may add more terms, for instance real potential terms associated with the curvature of m, without affecting self-adjointness, and indeed this freedom will become relevant shortly. For the kinetic energy ($k = \frac{1}{2}g^{-1}$), (2) yields the usual curved-space Laplacian term.

We now turn to Dirac quantization. The phase space $\Gamma = T^*M$, and M is equipped with a metric G, so we can proceed exactly as for reduced quantization (ignoring the constraints for now). Define the state space $\mathcal{F} := C^\infty(M, \mathbf{C})$ of all smooth complex-valued functions on M, then the Hilbert space $\mathcal{H} := L^2(M, \mathbf{E}^{(g)})$ consists of all those elements of $\mathcal{F}$ which are square integrable with respect to the volume form $\mathbf{E}^{(G)}$ associated with G. Again, the choice of quantization map is not unique, but in order to at least fix ideas, and facilitate a meaningful comparison with reduced quantization, let us begin with both quantization maps on an equal footing. Thus the Dirac quantization map is also chosen to be minimal:

$$\mathcal{C}(K) = K^{AB}P_A P_B \mapsto \mathcal{Q}(K) := -\hbar^2\nabla_A K^{AB}\nabla_B\,, \tag{3}$$

where ∇ is the Levi-Cività connection on M.

In the spirit of Dirac [9] we now account for the constraints [see (I.3)] by quantizing them, on the same footing as any other observables linear in the momenta. For such observables the minimal quantization map is

$$\mathcal{C}(V) := V^A P_A \mapsto \mathcal{Q}(V) := -i\hbar\{V^A\nabla_A + \tfrac{1}{2}(\nabla_A V^A)\}\,. \tag{4}$$

The physical state space, $\mathcal{F}_{\rm phys} \subset \mathcal{F}$, is then defined as the collection of states $\Psi_{\rm phys}$ annihilated by the constraint operators:

$$\mathcal{Q}(\phi_\alpha)\Psi_{\rm phys} = 0 \ \ \forall\alpha \Leftrightarrow \Psi_{\rm phys} \in \mathcal{F}_{\rm phys}\,. \tag{5}$$

As emphasized by Kuchař [2] the choice of basis for the gauge vectors ϕ_α is arbitrary at the classical level, but this breaks down at the quantum level, at least if one demands that the constraint operators be self-adjoint. The trouble stems from the complementary divergence term in (4), and can be eliminated naturally by restricting to "preferred" bases which are "compatible" with the Hilbert space structure:

$$\mathcal{L}_{\phi_\alpha}\mathbf{E}^{(G)} = 0\ \forall\alpha\,, \tag{6}$$

i.e., where the ϕ_α are divergence-free [cf. (I.63) with $w_\alpha := \phi_\alpha$]. This condition is natural in that (5) then implies $\Psi_{\rm phys}(Q) = \psi(q(Q))$; i.e., $\mathcal{F}_{\rm phys}$ consists of gauge invariant complex-valued functions on M. Furthermore, (5) is consistent: the quantum constraints are first class. In our case the gauge theory arises by the action of a Lie group on M, and we assume that the basis of generators of the group action satisfies (6), a condition which is obviously invariant under change of Lie algebra basis. For example, in the helix model (see I) the gauge vector ϕ_α is Killing and so (6) is satisfied.

Finally, for completeness we note that if the orbits of the gauge group are not compact then the $\Psi_{\rm phys}$ are not square integrable, in which case $\mathcal{F}_{\rm phys} \not\subset \mathcal{H}$. This is a topic which we leave outside the scope of this work (see, e.g., [22], and references therein): the purpose of this section is simply to introduce the Dirac versus reduced factor ordering ambiguity in the "standard" way [2–4,6], and then devote the remainder of our discussion to its geometrical interpretation. We only want to include a remark by DeWitt [23] to the effect that in order to factor out the gauge group in the Feynman path integral one must treat the gauge group formally as if it were compact, which is related to the traceless condition $f^{\alpha}{}_{\alpha\gamma} = 0$. So we see that the weak conditions (I.63) are sufficient for the Dirac quantization to go through (but see also (10), ahead).

Now before we can compare reduced quantization (2) with Dirac quantization (3) we must (quantum) reduce the latter: A quantum observable $\mathcal{Q}$ contains pieces which vanish when acting on $\mathcal{F}_{\rm phys}$, leaving a "physical residue" operator which we denote as $\mathcal{Q}^{\rm res}$. Using the horizontal and vertical basis introduced in I we have

$$\begin{aligned} K^{AB}\nabla_B\Psi_{\rm phys} &= \left\{K^{Ab}w_b + K^{A\beta}w_\beta\right\}\Psi_{\rm phys} \\ &= K^{Ab}\partial_b\Psi_{\rm phys}\,, \end{aligned} \tag{7}$$

where $K^{Ab} := K^{AB}e^b_B$, etc. The right-hand side of (7) is a vector field on M, denote it by V^A, say, whose Levi-Cività divergence can be calculated using the Ricci rotation coefficients derived in Sec. V of I:[4]

$$\begin{aligned} \nabla\cdot V &= w_{\hat A}V^{\hat A} + \Gamma^{\hat A}{}_{\hat A\hat B}V^{\hat B} \\ &= \partial_a v^a + v^a\partial_a \ln\sqrt{g\gamma} + \phi_\alpha V^\alpha\,, \end{aligned} \tag{8}$$

where $v^a = k^{ab}\partial_b\Psi_{\rm phys}$ is gauge invariant since $\mathcal{C}(K)$ is a classical observable [see (I.13)]. Hence

$$\begin{aligned} \mathcal{Q}(K)\Psi_{\rm phys} = (-i\hbar)^2\{&\partial_a k^{ab}\partial_b + k^{ab}(\partial_a \ln\omega)\partial_b \\ &+(\phi_\alpha K^{\alpha b})\partial_b\}\Psi_{\rm phys}\,, \end{aligned} \tag{9}$$

where $\omega := \sqrt{g\gamma}$ is gauge invariant.

[4]This result is valid even under the assumptions (I.63), which are weaker than (I.65).

Clearly, for $\mathcal{Q}(K)$ not to knock $\Psi_{\rm phys}$ out of $\mathcal{F}_{\rm phys}$, i.e., to be a quantum observable, we require $\phi_\alpha K^{\alpha b}$ to be gauge invariant. In fact, as we shall now argue, it is reasonable to demand the stronger condition

$$\phi_\alpha K^{\alpha b} = 0\ \ \forall b\,. \tag{10}$$

If this is satisfied the quantum reduction induces the quantization map

$$c(k) = k^{ab}p_a p_b \mapsto q^{\rm res}(k) = (-i\hbar)^2\omega^{-1}\partial_a\omega k^{ab}\partial_b \tag{11}$$

on the reduced space. $q^{\rm res}$ is self-adjoint with respect to the volume form

$$\mathbf{e}^{\rm res} := \omega\, dq^1\wedge\cdots\wedge dq^n = \sqrt{\gamma}\mathbf{e}^{(g)}\,, \tag{12}$$

which induces a Hilbert space structure on $\mathcal{F}_{\rm phys}$:[5] $\mathcal{H}_{\rm res} := L^2(m, \mathbf{e}^{\rm res})$. If $\phi_\alpha K^{\alpha b} \neq 0$ the quantum reduction cannot be consistent, since such a term cannot be incorporated into the $\ln\omega$ term in (9), nor can it be interpreted as the leading term of an operator for a vector on m since the factor of i would be wrong for self-adjointness. Note that (10) is automatically true for the kinetic energy[6] ($K = \frac{1}{2}G^{-1}$) since the metric on M is diagonal in the horizontal and vertical basis; for generic K this extra condition will also play an important role in our later results.

In order to facilitate a comparison between the two quantization maps given in (2) and (11) it is convenient to perform a unitary transformation on the latter so that both act in the same Hilbert space $\mathcal{H}_{\rm phys}$:

$$\mathbf{e}^{\rm res} \mapsto \mathbf{e}^{\rm dir} = \gamma^{-1/2}\mathbf{e}^{\rm res} = \mathbf{e}^{(g)}\,, \tag{13}$$

$$\psi^{\rm res} \mapsto \psi^{\rm dir} = \gamma^{1/4}\psi^{\rm res}\,, \tag{14}$$

$$q^{\rm res} \mapsto q^{\rm dir} = \gamma^{1/4}q^{\rm res}\gamma^{-1/4}\,. \tag{15}$$

A straightforward calculation yields what we henceforth refer to as Dirac quantization,

$$q^{\rm dir}(k) = q^{\rm red}(k) + \hbar^2\alpha(k)\,, \tag{16}$$

which differs from the minimal reduced quantization scheme by an additional potential term

$$\alpha(k) := \gamma^{-1/4}\tilde\nabla_a(k^{ab}\tilde\nabla_b\gamma^{1/4}) \tag{17}$$

[5]The *same* Hilbert space structure is induced by the quantum reduction of observables linear in momenta [see (4)], but only if a condition analogous to (10) is imposed on the corresponding vectors, namely, $\phi_\alpha V^\alpha = 0$.

[6]This apparently contradicts a statement by Kuchař (Eq. (3.11), p. 3048, in [2]): by substituting (I.34) into his equation (3.11) it is easy to see that in our case the kinetic term in the Hamiltonian *is* a quantum observable precisely because we are taking seriously the role of the horizontal basis as establishing a connection on M [cf. (I.44)], as well as demanding (6).

that depends on $\sqrt{\gamma}$, the volume element on the gauge orbits. Thus Dirac and reduced quantization are physically distinct [6,4,3]—they lead to different spectra for the same observable (see also cautionary remarks in [5] regarding domains of operators).

Kuchař [3] has shown that, for even the simple helix model,

$$\alpha(\tfrac{1}{2}g^{-1}) = \frac{4-\rho^2}{8(1+\rho^2)^2}\,, \tag{18}$$

is nontrivial.[7] He favors reduced quantization over Dirac, mainly because of objections to a Hilbert space structure on the state space $\mathcal{F}$ [2], which have been addressed above (for a more complete discussion, refer to [21]). Romano and Tate [4] have considered the rigid rotator and the hydrogen atom, and favor Dirac quantization over reduced because of agreement with experiment. Kunstatter [6] has recently shown that the difference is equivalent to a factor ordering ambiguity involving $\sqrt{\gamma}$, the volume element on the gauge orbits. Clearly further criteria are needed to decide which factor ordering (if either) is correct. It is the purpose of this paper to suggest a new criterion based on the theory of curved-space quantization.

III. CURVED-SPACE QUANTIZATION

Reduced quantization has an advantage over Dirac quantization in the sense that the gauge degrees of freedom have been eliminated already at the classical level, and so do not contribute additional subtleties to the quantization procedure. A significant disadvantage, however, is that even if the metric G on M is flat, often the corresponding metric g induced on m has nontrivial curvature. (Examples where this is the case include Yang-Mills theory and scalar electrodynamics [11], and the helix model [3]; see (I.83).) Hence reduced quantization is essentially Schrödinger picture quantization on a curved configuration space, and as such inherits all of the difficulties and ambiguities inherent in curved-space quantization.

Now, since the work of Groenewold [17] and van Hove [18] it has been known that no quantization scheme exists which homomorphically maps the entire classical Poisson algebra into quantum commutators.[8] For example, the Poisson bracket of two physical observables homogeneous quadratic in the momenta is a third one, homogeneous cubic in the momenta: $\{c(k), c(l)\} = c(s)$. Using the minimal quantization scheme (2), the corresponding quantum commutator is

$$\frac{1}{i\hbar}[q^{\rm red}(k), q^{\rm red}(l)] = q^{\rm red}(s) + \hbar^2 q^{\rm red}(w)\,, \tag{19}$$

where $Q^{\rm red}(s)$ is the cubic counterpart to (2) (which is linear and homogeneous in s), and

$$\begin{aligned} w^d &= -\tfrac{1}{2}\tilde{\nabla}_b\tilde{\nabla}_c s^{bcd} - \tilde{\nabla}_b A^{bd}\,, \\ A^{bd} &= k^{ab}l^{cd}\tilde{\mathcal{R}}_{ac} + (\tilde{\nabla}_c k^{ab})(\tilde{\nabla}_a l^{cd}) \\ &\quad -\tfrac{1}{3}\tilde{\nabla}_c(k^{ab}\tilde{\nabla}_a l^{cd} - k^{ad}\tilde{\nabla}_a l^{cb}) - (k \leftrightarrow l)\,. \end{aligned} \tag{20}$$

The vector field w cannot be written solely in terms of s (and in general does not vanish even when $s = 0$), and represents a failure of the minimal quantization scheme to preserve the classical Poisson algebra. While such van Hove anomalies are not exclusive to curved-space quantization (of course), the presence of curvature [the Ricci tensor in (20)] amplifies the problem.

Curvature of the configuration space also opens up additional ambiguities in quantization. For example, from the point of view of either dimensional analysis or the correspondence limit, there is no reason not to add a multiple of $\hbar^2\tilde{\mathcal{R}}$ (the Ricci scalar) to the usual Laplacian term of the kinetic energy operator. There is great precedence for this in the literature [24–31]. For our purposes we need only observe that, although there is no general agreement, $(\hbar^2/8)\tilde{\mathcal{R}}$ is a commonly quoted result, but even more common is $(\hbar^2/12)\tilde{\mathcal{R}}$, obtained most notably by geometric quantization (see [32,33]). Indeed Emch [31] states that $(\hbar^2/12)\tilde{\mathcal{R}}$ "produces the best possible fit of the quantum partition function to its classical limit."

More generally, to the minimal quantization map $q^{\rm red}(k)$ in (2) one may add a potential term $\hbar^2 A(k)$, where $A(k)$ might consist of a curvature term such as $k^{ab}\tilde{\mathcal{R}}_{ab}$, or a contracted derivative term such as $\tilde{\nabla}_a\tilde{\nabla}_b k^{ab}$. Often the motivation for adding such terms is to try to cancel anomalous van Hove terms in at least the lowest order quantum commutators. For example, Vaisman [14] argues that in order to "preserve as many brackets as possible" the additional potential term should be of the form

$$A(k) = \tfrac{1}{4}[k^{ab}\tilde{\mathcal{R}}_{ab} - (\tilde{\nabla}_a\tilde{\nabla}_b k^{ab})] + B(k)\,, \tag{21}$$

where

$$B(k) = \sum_{m=1}^{r} \epsilon_m \left[F(v_m) + \tfrac{1}{4}(\tilde{\nabla}_a v_m^b)(\tilde{\nabla}_b v_m^a)\right]. \tag{22}$$

The notation in $B(k)$ refers to the local decomposition

$$k^{ab} = \sum_{m=1}^{r} \epsilon_m v_m^a v_m^b\,, \tag{23}$$

where r is the rank of k, $\epsilon_m = \pm 1$, and v_m are some vector fields on m. If we let ξ denote the physical observable $v^a p_a$, then the function $\hbar^2 F(v)$ represents the difference between the operators $\widehat{\xi^2}$ and $\hat{\xi}^2$, which is arbitrary in Vaisman's work. But, unless well chosen, $B(k)$ will depend on the choice of local bases v_m, as happens with the choice $F(v) = 0$ [whereas a choice such as $B(k) \equiv 0$ *is* of course consistent] [14].

But with such arbitrariness in F one might argue that

[7] There is a sign discrepancy between (18) and Kuchař's result that appears to stem from the statement $|\gamma|^{1/2} = (1+r^2)^{-1/2}$ above Eq. (47) in [3], which should instead read $|\gamma|^{1/2} = (1+r^2)^{+1/2}$.

[8] See, e.g., [19] for a more precise statement.

the particular form of $A(k)$ given in (21) is devoid of any special significance. That this is *not* the case is supported by recent work[9] [13], from a completely independent point of view, which shows that a generalization of the Weyl correspondence principle [34] from flat to curved configuration spaces leads to the same $A(k)$ in (21), but with $B(k) \equiv 0$.[10]

Now the question is whether any of these proposed curved-space quantization schemes, employing additional potential terms such as $A(k)$, eliminate van Hove anomalies in, say, quadratic-quadratic commutators. The answer is no, at least not in the generic case. In fact, Bloore, Assimakopoulos, and Ghobrial [15] have shown that *no* "Schrödinger-type" quantization scheme exists in which the commutator of the kinetic energy operator with a generic quadratic operator is free of van Hove anomalies, unless the configuration space is of constant curvature or Ricci flat. Thus, in order to consider more interesting curvatures, it is necessary at least to restrict the type of quadratic observables in some way.

So we conclude that the ambiguities associated with the curved-space quantization of observables quadratic in the momenta are not well understood, but that there is likely some significance to the addition of a potential term $A(k)$ of the particular form given above. Furthermore, the possibility of avoiding van Hove anomalies in quadratic-quadratic cummutators exists only within certain restrictions on the observables and the curvature.[11] It is the purpose of this paper to suggest a suitable set of such restrictions, and show that within these restrictions the potential term $\alpha(k)$, provided naturally by Dirac quantization, is remarkably similar to the $A(k)$ of curved-space quantization.

IV. RESTRICTIONS ON OBSERVABLES AND CURVATURE

We now return to Dirac quantization on the full space, and henceforth assume that the metric G on M is flat (or at least Ricci flat). Then if we use the minimal quantization map (3), the commutator of two quadratic observables $\mathcal{Q}(K)$ and $\mathcal{Q}(L)$ will have a van Hove term analogous to (20), except without the Ricci tensor term, a considerable simplification over the reduced quantization case. The remaining terms all involve covariant derivatives of K and L, which suggests the expediency of restricting all tensors involved in quadratic observables to being covariantly constant.

Within the two restrictions that we have suggested (M Ricci flat and $\nabla K = 0$), which will play a crucial role in our later results, the minimal quantization map on the full space is free of van Hove anomalies for quadratic-quadratic commutators. Now the important point is that the same is therefore true of the residual quantization map on the reduced space, $q^{\rm res}(k)$ in (11), or, equivalently, of the Dirac quantization map $q^{\rm dir}(k)$ given in (16). Furthermore, even though M is Ricci flat, the reduced space m usually has nontrivial curvature, as in the examples quoted previously, Yang-Mills theory and scalar electrodynamics [11], and the helix model [3]. In particular, the curvature may be neither constant nor Ricci flat, which takes us outside of the restrictions necessary in [15] to a more interesting class of curvature: that induced on the quotient space of a gauge group acting on a Ricci flat M. Necessarily, then, the observables have been restricted, in this case to any k which is the physical projection of a covariantly constant K. We emphasize that $\nabla K = 0$ does *not* imply $\tilde{\nabla} k = 0$, and so, from a curved-space quantization point of view, $c(k)$ represents an interesting observable [for example, the derivative term in the $A(k)$ in (21) need not be trivial]. So within suitable restrictions the Dirac quantization map $q^{\rm dir}(k)$, involving the additional potential $\alpha(k)$, is really a curved-space quantization scheme free of van Hove anomalies in a restricted, but nevertheless still interesting, setting.

However, the price for quantizing on the unreduced space first is that we have to deal with quantum reduction, which is not always consistent. Thus we must impose a third restriction, namely the quantum reduction consistency condition (10) (a condition which does *not* follow from $\nabla K = 0$).

This set of three restrictions seems quite narrow, and one might ask if there exist any physically interesting models, containing a special subset of observables, in which they are satisfied. The answer is yes. In the case of scalar electrodynamics on flat spacetime, M is flat, and, furthermore, the generators of the Poincaré group acting on Γ form an important subset of observables, whose quadratic pieces satisfy $\nabla K = 0$ and $\phi_\alpha K^{\alpha b} = 0$ [20]. (Note that both the Hamiltonian and the boost generators have pieces quadratic in the momenta, so one encounters quadratic-quadratic commutators in the Poincaré algebra.) These properties likely also apply in a large class of Poincaré invariant gauge theories on flat spacetime.

The $\nabla K = 0$ restriction is probably stronger than necessary for the elimination of van Hove terms, and the fact that in the case of scalar electrodynamics it is realized by a symmetry subalgebra suggests a way in which it might be relaxed. For instance, in any symmetry subalgebra, with generators containing pieces at most quadratic in the momenta, the commutators of such pieces must vanish for the algebra to close. If the Hamiltonian is to be included in the subalgebra (as emphasized in dynamical quantization [36]) then all of the other associated tensors must be Killing tensors: $\nabla^{(A}K^{BC)} = 0$. For configuration spaces of *constant* curvature (including flat), Underhill and Taraviras [37] have presented a quantization map in which the commutator of the kinetic energy

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[9] See also earlier work by Underhill [16].

[10] Zhang-Ju and Min [13] actually consider a one parameter family of quantization maps for observables quadratic in the momenta, but the one referred to here is the only one of these which is self-adjoint (at least for generic k).

[11] The notion of a preferred subset of observables also occurs in the group-theoretic approach to canonical quantization, see, e.g., [35].

operator with any other quadratic observable, with K a Killing tensor, is free of van Hove anomalies. Through a different quantization map Bloore, Assimakopoulos, and Ghobrial [15] achieve the same end, but for generic K; unfortunately, flat spaces are excluded from consideration. Perhaps these results, or suitable generalizations stressing the role of symmetry subalgebras, may be used in quantization on the full space, leading to an even more useful and interesting class of observables for which Dirac quantization succeeds as a curved-space quantization scheme on the reduced space. In any case, the restriction $\nabla K = 0$ is sufficient for our present purposes.

Now let us turn to the question of how the three restrictions discussed above can be applied to illuminate the similarity between $\alpha(k)$ and $A(k)$. In order to simplify the following analysis we shall henceforth assume the strong conditions (I.65), instead of the weak conditions (I.63) in effect heretofore, which means, in particular, that the metric components $\gamma_{\alpha\beta}$ are gauge invariant, and that the gauge vector fields are Killing vector fields. We begin with the covariant constancy restriction, but, instead of $\nabla K = 0$, it is instructive to first examine the analogous condition $\nabla V = 0$ for a vector field V on M. Assume that $\mathcal{C}(V)$ is an observable on Γ:

$$(\mathcal{L}_{w_\gamma} V)^A = \xi^\alpha_\gamma w^A_\alpha \,, \tag{24}$$

for arbitrary scalar fields ξ^α_γ on M, which is equivalent to the physical projection $V^a(Q) =: v^a(q(Q))$ being gauge invariant, where $V^a := e^a_A V^A$ [cf. (I.14)]. This means

$$w_\gamma V^\alpha = \mathcal{L}_{w_\gamma}(e^\alpha_A V^A) = \xi^\alpha_\gamma - f^\alpha{}_{\gamma\beta} V^\beta \,, \tag{25}$$

where we used (I.27).

Using the Ricci rotation coefficients in (I.66)–(I.71), the components of ∇V in the horizontal and vertical basis are

$$\nabla_a V^b = \tilde{\nabla}_a v^b + \tfrac{1}{2}\mathcal{F}_{\gamma a}{}^b V^\gamma \,, \tag{26}$$

$$\nabla_a V^\beta = w_a V^\beta - \tfrac{1}{2}\mathcal{F}^\beta{}_{ac} v^c + \tfrac{1}{2}\gamma^{\beta\alpha}(\tilde{\nabla}_a \gamma_{\alpha\gamma}) V^\gamma \,, \tag{27}$$

$$\nabla_\alpha V^b = \tfrac{1}{2}\mathcal{F}_{\alpha c}{}^b v^c - \tfrac{1}{2}(\tilde{\nabla}^b \gamma_{\alpha\gamma}) V^\gamma \,, \tag{28}$$

$$\nabla_\alpha V^\beta = w_\alpha V^\beta + \tfrac{1}{2}\gamma^{\beta\gamma}(\tilde{\nabla}_c \gamma_{\gamma\alpha}) v^c + \tfrac{1}{2} f^\beta{}_{\alpha\gamma} V^\gamma \,. \tag{29}$$

The consequences of demanding $\nabla V = 0$ can now be read off from these equations. For instance, the scalars ξ^α_γ can be expressed algebraically in terms of the components of V by using the forth of these equations in (25):

$$\xi^\alpha_\gamma = -\tfrac{1}{2}\gamma^{\alpha\beta}(\tilde{\nabla}_b \gamma_{\beta\gamma}) v^b + \tfrac{1}{2} f^\alpha{}_{\gamma\beta} V^\beta \,. \tag{30}$$

Now, if we quantized $\mathcal{C}(V)$ using minimal quantization, the quantum reduction consistency condition analogous to (10) would be

$$0 = w_\gamma V^\gamma = \xi^\gamma_\gamma - f^\gamma{}_{\gamma\beta} V^\beta = \xi^\gamma_\gamma \,, \tag{31}$$

which, by (30), is equivalent to

$$v^a \tilde{\nabla}_a \ln\sqrt{\gamma} = 0 \,; \tag{32}$$

i.e., the volume element on the gauge orbit should be constant along the integral curves of v. Furthermore, from (26) we learn that, although v is *not* covariantly constant on m in the generic case,

$$\tilde{\nabla}_a v^b = -\tfrac{1}{2}\mathcal{F}_{\gamma a}{}^b V^\gamma \,, \tag{33}$$

it must be a Killing vector, $\tilde{\nabla}^{(a} v^{b)} = 0$ (and so also divergence free).

Let us show that nontrivial observables $\mathcal{C}(V)$ exist in which $\nabla V = 0$, and the quantum reduction is consistent, by constructing one in the helix model. $\nabla V = 0$ means that the components V^A in the Cartesian coordinates X, Y, Z are constants. Using (I.5) we find that the only solution to (24) is

$$V = (0, 0, \zeta) \ \text{ and } \xi^\alpha_\gamma = 0 \,, \tag{34}$$

where ζ is an arbitrary constant. The components of V in the horizontal and vertical basis are

$$v^a = e^a_A V^A = \begin{pmatrix} \zeta \\ 0 \end{pmatrix} , \tag{35}$$

$$V^\alpha = e^\alpha_A V^A = \gamma^{\alpha\beta} G_{AB} w^B_\beta V^A = \zeta(1+\rho^2)^{-1} \,, \tag{36}$$

where we used (I.39) and (I.49). Thus, since $v = \zeta\partial/\partial B$, whereas $\sqrt{\gamma}$ depends only on ρ, we see that v also satisfies the quantum reduction consistency condition (32). Finally, using (33) and the fact that the Yang-Mills curvature $\mathcal{F}^\gamma{}_{ab}$ is nonzero, we observe that v is not covariantly constant. We will make use of this v shortly.

The analysis of the $\nabla K = 0$ restriction is analogous to the one just given for V, and the details may be found in the Appendix. To account for the Ricci flat restriction we write

$$0 = k^{ab}\mathcal{R}_{ab} + \mu 2K^{a\beta}\mathcal{R}_{a\beta} + \nu K^{\alpha\beta}\mathcal{R}_{\alpha\beta} \,, \tag{37}$$

for arbitrary μ, ν, since each term on the right hand side vanishes independently. Here $\mathcal{R}_{\hat{A}\hat{B}}$ are the components of the Ricci tensor on M in the horizontal and vertical basis. The nominal choice $\mu = \nu = 1$ corresponds to the natural contraction $K^{\hat{A}\hat{B}}\mathcal{R}_{\hat{A}\hat{B}}$, but any choice is equally valid. Then, using the analysis of the $\nabla K = 0$ restriction, together with the results of the Kaluza-Klein-like study of curvature in gauge theories given in I, we can extract from (37) an expression for $\alpha(k)$:

$$\begin{aligned} 2(1+\nu)\alpha(k) = {} & [k^{ab}\tilde{\mathcal{R}}_{ab} - (2\mu - \nu)(\tilde{\nabla}_a \tilde{\nabla}_b k^{ab})] \\ & + \tfrac{1}{4}k^{ab}(\tilde{\nabla}_a \gamma^{\alpha\beta})(\tilde{\nabla}_b \gamma_{\alpha\beta}) + \tfrac{1}{4}(2\mu - \nu)K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^b \mathcal{F}_{\beta b}{}^a - \frac{\nu}{4}K^{\alpha\beta} f^\delta{}_{\gamma\alpha} f^\gamma{}_{\delta\beta} \\ & + \tfrac{1}{2}(1-\mu)k^{ab}\mathcal{F}_{\gamma c a}\mathcal{F}^\gamma{}_b{}^c + \tfrac{1}{2}(1-\nu)k^{ab}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}_b \ln\sqrt{\gamma}) \\ & + [1 - (2\mu - \nu)](\tilde{\nabla}_a k^{ab})(\tilde{\nabla}_b \ln\sqrt{\gamma}) + \tfrac{1}{2}(\mu - \nu)(\tilde{\nabla}_a \gamma_{\alpha\beta})\mathcal{F}^\alpha{}_b{}^a K^{b\beta} \,. \end{aligned} \tag{38}$$

The details of this calculation are outlined in the Appendix. This is the central result of our work.

V. DISCUSSION

Two features of this result are immediately striking. First, the simple potential term $\alpha(k)$ in (17) actually contains terms such as $k^{ab}\tilde{\mathcal{R}}_{ab}$ and $\tilde{\nabla}_a\tilde{\nabla}_b k^{ab}$, in accordance with virtually all other quantization schemes on a curved configuration space. Second, it contains terms in addition to these, which appear to require knowledge of the gauge structure not available in the classically reduced theory.

The nominal choice $\mu = \nu = 1$ eliminates the last four terms in (38), leaving

$$\alpha(k) = \tfrac{1}{4}[k^{ab}\tilde{\mathcal{R}}_{ab} - (\tilde{\nabla}_a\tilde{\nabla}_b k^{ab})] + \beta(k)\,, \tag{39}$$

where

$$\beta(k) := \tfrac{1}{16}[k^{ab}(\tilde{\nabla}_a\gamma^{\alpha\beta})(\tilde{\nabla}_b\gamma_{\alpha\beta}) + K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} - K^{\alpha\beta}f^{\delta}{}_{\gamma\alpha}f^{\gamma}{}_{\delta\beta}]\,. \tag{40}$$

Comparing $\alpha(k)$ with the $A(k)$ in Vaisman's quantization scheme (21), we see a remarkable agreement of the leading terms.

At this point one might argue that, at worst, all we have done is artificially extract certain desirable terms from a relatively simple object, $\alpha(k)$, leaving behind a more complicated object, $\beta(k)$. To see that this is not the case we deepen the correspondence with Vaisman's quantization scheme by establishing similarities between his $B(k)$, given in (22), and our $\beta(k)$. In analogy with (23) we construct the rank r tensor

$$K^{AB} := \sum_{m=1}^{r} \epsilon_m V_m^A V_m^B\,, \tag{41}$$

$\epsilon_m = \pm 1$, out of some set of covariantly constant vector fields V_m on M, which also satisfy (24) and (31) as per our previous discussion. This K is covariantly constant, produces a gauge invariant physical projection

$$k^{ab} = e^a_A K^{AB} e^b_B = \sum_{m=1}^{r} \epsilon_m v_m^a v_m^b\,, \tag{42}$$

and satisfies the quantum reduction consistency condition (10):

$$w_\beta K^{\alpha\beta} = w_\beta(e^\alpha_A K^{AB} e^\beta_B) = \sum_{m=1}^{r} \epsilon_m v_m^a w_\beta V_m^\beta = 0\,. \tag{43}$$

But the vector fields V_m satisfy (33), so the middle term in $\beta(k)$ is

$$\tfrac{1}{16}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} = \tfrac{1}{16}\sum_{m=1}^{r} \epsilon_m (V_m^\alpha\mathcal{F}_{\alpha a}{}^{b})(V_m^\beta\mathcal{F}_{\beta b}{}^{a}) = \tfrac{1}{4}\sum_{m=1}^{r} \epsilon_m (\tilde{\nabla}_a v_m^b)(\tilde{\nabla}_b v_m^a)\,, \tag{44}$$

exactly the term in Vaisman's $B(k)$.

Furthermore, Dirac quantization now fixes the function $F(v)$ in $B(k)$, which remained arbitrary in Vaisman's work [14]. Comparing (44) with (40) and (22) yields

$$F(v) = \tfrac{1}{16}[v^a v^b(\tilde{\nabla}_a\gamma^{\alpha\beta})(\tilde{\nabla}_b\gamma_{\alpha\beta}) - V^\alpha V^\beta f^{\delta}{}_{\gamma\alpha}f^{\gamma}{}_{\delta\beta}] \tag{45}$$

(at least for this special class of covariantly constant V). We notice that $F(v)$ depends on the gauge structure—in general, it cannot be calculated using information available only in the classical reduced theory. It *can*, however, be written exclusively in terms of gauge orbit quantities simply by squaring (29) (with zero on the left hand size) and observing that the cross terms vanish, so in fact

$$F(v) = -\tfrac{1}{4}(w_\alpha V^\beta)(w_\beta V^\alpha)\,. \tag{46}$$

[Notice that in the case of a one dimensional gauge group $F(v)$ is identically zero on account of (31).] It is interesting that $F(v)$ has the same *form* as the other term in $B(k)$ [see (22)], but is its gauge orbit counterpart.

We can pursue this "duality" further by writing $F(v)$ in terms of $\tilde{\nabla}_\alpha$, the Levi-Cività connection associated with $\gamma_{\alpha\beta}$, instead of w_α. With regard to our discussion in Sec. V of I, we write

$$\tilde{\nabla}_\alpha V^\beta = w_\alpha V^\beta + \tfrac{1}{2}f^{\beta}{}_{\alpha\gamma}V^\gamma\,. \tag{47}$$

On squaring this we find

$$F(v) = -\tfrac{1}{4}(\tilde{\nabla}_\alpha V^\beta)(\tilde{\nabla}_\beta V^\alpha) + \tfrac{1}{4}V^\alpha V^\beta \tilde{\mathcal{R}}_{\alpha\beta}\,, \tag{48}$$

where we used (29) again, as well as the expression (I.88) for the Ricci tensor $\tilde{\mathcal{R}}_{\alpha\beta}$ *within* a gauge orbit. Applying this result to the tensor case, (40) becomes

$$\beta(k) = \tfrac{1}{4}K^{\alpha\beta}\tilde{\mathcal{R}}_{\alpha\beta} + \sum_{m=1}^{r}\epsilon_m[-\tfrac{1}{4}(\tilde{\nabla}_\alpha V_m^\beta)(\tilde{\nabla}_\beta V_m^\alpha) + \tfrac{1}{4}(\tilde{\nabla}_a v_m^b)(\tilde{\nabla}_b v_m^a)]\,. \tag{49}$$

It is remarkable that for every term in $\alpha(k)$, associated with m, there appears to be a "gauge orbit complement" term; for example, $\frac{1}{4}K^{\alpha\beta}\tilde{\mathcal{R}}_{\alpha\beta}$ is the gauge orbit complement of $\frac{1}{4}k^{ab}\tilde{\mathcal{R}}_{ab}$. (It can be shown that $\tilde{\nabla}_\alpha\tilde{\nabla}_\beta K^{\alpha\beta}$, analogous to the term $\tilde{\nabla}_a\tilde{\nabla}_b k^{ab}$, identically vanishes with our assumptions, which may be the only reason it does not appear like the others.)

Let us now realize this discussion in a concrete example. Recall that for the helix model we had constructed a covariantly constant vector V [see (35) and (36)], which is suitable for use in (41): $K^{AB} = \epsilon V^A V^B$. The components of this K in the horizontal and vertical basis are

$$k^{ab} = \epsilon v^a v^b = \epsilon\zeta^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \tag{50}$$

$$K^{a\beta} = \epsilon v^a V^\beta = \epsilon\zeta^2\gamma^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \tag{51}$$

$$K^{\alpha\beta} = \epsilon V^\alpha V^\beta = \epsilon\zeta^2\gamma^{-2}\,, \tag{52}$$

where $\gamma = \det\gamma_{\alpha\beta} = 1 + \rho^2$. Although k is a Killing tensor, it is not covariantly constant; instead we find

$$\tilde{\nabla}_a \tilde{\nabla}_b k^{ab} = \epsilon\zeta^2 (3\rho^2 - 2)\gamma^{-3} . \tag{53}$$

Furthermore, using (I.83) we have

$$k^{ab}\tilde{\mathcal{R}}_{ab} = \epsilon\zeta^2 3\rho^2 \gamma^{-3} , \tag{54}$$

so the leading terms in $\alpha(k)$ are

$$\tfrac{1}{4}\left[k^{ab}\tilde{\mathcal{R}}_{ab} - (\tilde{\nabla}_a \tilde{\nabla}_b k^{ab})\right] = \tfrac{1}{2}\epsilon\zeta^2\gamma^{-3} . \tag{55}$$

As mentioned earlier, $F(v)$ vanishes identically for a one dimensional gauge group, so the only term that contributes to $\beta(k)$ is

$$\tfrac{1}{16} K^{\alpha\beta} \mathcal{F}_{\alpha a}{}^{b} \mathcal{F}_{\beta b}{}^{a} = -\tfrac{1}{2}\epsilon\zeta^2\gamma^{-3} , \tag{56}$$

where we made use of (I.51).

Observe that although $\beta(k)$ is nontrivial for this particular k, in this model, it cancels with the leading terms, so that $\alpha(k)$ vanishes. In other words, the Dirac quantization of this k we constructed coincides with minimal quantization, and so it is not as interesting as one might have hoped for. However, this is not true in general, as demonstrated in the important case of the kinetic energy; see (18), and also [4].

Specializing now to $k^{ab} = \frac{1}{2} g^{ab}$, the Dirac quantization (16) tells us that the kinetic energy operator

$$\hat{T} := q^{\rm dir}(\tfrac{1}{2}g^{-1}) = -\hbar^2\{\tfrac{1}{2}\tilde{\Delta} - \tfrac{1}{8}\tilde{\mathcal{R}} - \beta(\tfrac{1}{2}g^{-1})\} , \tag{57}$$

where [see (39) and (40)]

$$\beta(\tfrac{1}{2}g^{-1}) = \tfrac{1}{32}[(\tilde{\nabla}_c\gamma_{\alpha\beta})(\tilde{\nabla}^c\gamma^{\alpha\beta}) - \mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} + f_{\gamma\alpha\beta}f^{\gamma\alpha\beta}] . \tag{58}$$

This result exhibits the $(\hbar^2/8)\tilde{\mathcal{R}}$ curvature scalar term mentioned earlier, but the β term is new. The natural question is whether or not $\beta(\frac{1}{2}g^{-1})$ is zero, and, if not, is it proportional to $\tilde{\mathcal{R}}$? The answer to the first question is no: even in an example as simple as the helix model we find

$$(\tilde{\nabla}_c\gamma_{\alpha\beta})(\tilde{\nabla}^c\gamma^{\alpha\beta}) = -4\rho^2(1+\rho^2)^{-2} , \tag{59}$$

$$\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} = 8(1+\rho^2)^{-2} , \tag{60}$$

whose difference does not vanish. The answer to the second question is also no—but comparison of (60) with (I.84) suggests the identification $\mathcal{F}^2 \leftrightarrow \frac{4}{3}\tilde{\mathcal{R}}$, true at least for this example.

To generalize this result, recall that in the Appendix we derived expressions for $k^{ab}\mathcal{R}_{ab}$, $K^{\alpha\beta}\mathcal{R}_{a\beta}$, and $K^{\alpha\beta}\mathcal{R}_{\alpha\beta}$ [see (A22), (A27), and (A25)], each of which vanishes for Ricci flat M. In the special case of the kinetic energy, an interesting mechanism arises: $k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^{\gamma}{}_{b}{}^{c}$ is the same as $K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a}$, so the cross term equation becomes trivial, and the remaining two reduce to

$$\tilde{\mathcal{R}} = \tfrac{1}{2}\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} + \tilde{\Delta}\ln\sqrt{\gamma} - \tfrac{1}{4}(\tilde{\nabla}_c\gamma_{\alpha\beta})(\tilde{\nabla}^c\gamma^{\alpha\beta}) , \tag{61}$$

$$0 = \tfrac{1}{4}\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} - \tilde{\Delta}\ln\sqrt{\gamma} - (\tilde{\nabla}_c\ln\sqrt{\gamma})(\tilde{\nabla}^c\ln\sqrt{\gamma}) + \tfrac{1}{4}f_{\gamma\alpha\beta}f^{\gamma\alpha\beta} . \tag{62}$$

It is then natural to add these two equations, and thereby cancel the terms highest order in derivatives of γ, yielding

$$\mathcal{F}_{\gamma ab}\mathcal{F}^{\gamma ab} = \tfrac{4}{3}\tilde{\mathcal{R}} - \tfrac{1}{3}f_{\gamma\alpha\beta}f^{\gamma\alpha\beta} + \tfrac{1}{3}(\tilde{\nabla}_c\gamma_{\alpha\beta})(\tilde{\nabla}^c\gamma^{\alpha\beta}) + \tfrac{4}{3}(\tilde{\nabla}_c\ln\sqrt{\gamma})(\tilde{\nabla}^c\ln\sqrt{\gamma}) , \tag{63}$$

which has the desired $\frac{4}{3}\tilde{\mathcal{R}}$ leading term. Using this in (58) we can then eliminate the Yang-Mills curvature term, and the kinetic energy operator becomes

$$\hat{T} = -\hbar^2\{\tfrac{1}{2}\tilde{\Delta} - \tfrac{1}{12}\tilde{\mathcal{R}} - \beta'(\tfrac{1}{2}g^{-1})\} , \tag{64}$$

where

$$\beta'(\tfrac{1}{2}g^{-1}) := \tfrac{1}{24}[\tfrac{1}{2}(\tilde{\nabla}_c\gamma_{\alpha\beta})(\tilde{\nabla}^c\gamma^{\alpha\beta}) - (\tilde{\nabla}_c\ln\sqrt{\gamma})(\tilde{\nabla}^c\ln\sqrt{\gamma}) + f_{\gamma\alpha\beta}f^{\gamma\alpha\beta}] . \tag{65}$$

Remarkably, this reproduces the $(\hbar^2/12)\tilde{\mathcal{R}}$ result of geometric quantization.[12] This mechanism of eliminating the Yang-Mills curvature term may provide a hint why $(\hbar^2/4)k^{ab}\tilde{\mathcal{R}}_{ab}$ usually appears in curved-space quantization schemes with generic k, whereas $(\hbar^2/12)\tilde{\mathcal{R}}$ is more common in treatments that deal exclusively with the kinetic energy case, $k^{ab} = \frac{1}{2}g^{ab}$.

Nevertheless, $\hat{T}$ still contains a "nonremovable" β (or β') term which cannot be completely absorbed into the $\hbar^2\tilde{\mathcal{R}}$ term, as proven by the helix model example, and this term depends on the gauge structure of the unreduced theory. But the *only* natural object associated with m, and having the correct dimensions, that can be added to the Laplace-Beltrami operator is a multiple of $\hbar^2\tilde{\mathcal{R}}$. So, despite its remarkable similarity with other curved-space quantization schemes, the Dirac scheme could not have been guessed working strictly within the framework of the classical reduced theory.

Let us summarize our results. For many gauge theories, quantization on the physical degrees of freedom is essentially curved-space quantization, with a configuration space whose curvature is neither constant nor Ricci flat (examples include scalar electrodynamics [11] and the helix model [3]). For such cases Bloore, Assimakopoulos, and Ghobrial [15] have shown that no Schrödinger-type quantization scheme exists in which the quantum commutator of the kinetic energy with another arbitrary observable, quadratic in the momenta, is free of van Hove anomalies. In particular, minimal quantization does not work. Hence, in order to avoid such anomalies, our considerations must be restricted to some subset of quadratic observables. But the nature of this restriction and, moreover, the correct curved-space quantization scheme, are not *a priori* obvious.

[12] We remark that (64) can be obtained with the choice $\mu = 0$, $\nu = 2$ in (38).

On the other hand, we observe that the gauge theories in question usually have a (Ricci) flat full configuration space, which makes Dirac quantization simpler than reduced: in this case it is *natural* to use the minimal quantization scheme. Although van Hove anomalies in quadratic-quadratic commutators are still present, they are now easily eliminated by restricting our considerations to quadratic observables $\mathcal{C}(K)$ in which the tensors K are covariantly constant. Modulo an additional consistency condition on K [see (10)], quantum reduction then automatically results in a curved-space quantization scheme (16) on the physical degrees of freedom, in which quadratic-quadratic commutators are free of van Hove anomalies, at least within a restricted, but nevertheless still interesting, class of observables $c(k)$. This restricted class includes, for example, the generators of the Poincaré symmetry in scalar electrodynamics [20]. Although the additional potential term $\alpha(k)$, naturally present in Dirac quantization, has been studied before [4,6,5] (at least for the kinetic energy,[13] $k = \frac{1}{2}g^{-1}$), its connection with curved-space quantization, as well as the elimination of van Hove anomalies, within a suitably restricted class of observables, is new.

In particular, $\alpha(k)$ contains the curvature and derivative terms present in virtually all proposed curved-space quantization schemes, even with the "correct"[14] numerical factors. And since m is curved we should *expect* such terms. It is for this reason we believe that Dirac quantization is more natural than minimal reduced quantization which, by decree, contains no such terms.

Also, recalling that $\alpha(k)$ represents a factor ordering ambiguity [6], these results suggest a connection between the curvature and derivative terms usually added in curved-space quantization, and issues of factor ordering. Furthermore, with M Ricci flat and K covariantly constant, our choice of minimal quantization for the unreduced theory is consistent in the sense that any analogous curvature or derivative terms on the full configuration space would vanish anyway.

However, besides these terms, $\alpha(k)$ contains some unexpected terms: it appears that to each "expected" term in $\alpha(k)$ there corresponds an additional, enigmatic "complementary orbit term," having a similar form, but requiring knowledge of the gauge structure not available in the classical reduced theory. Whatever role these complementary terms play in eliminating van Hove anomalies, the point is $\alpha(k)$ cannot be constructed using objects found only in the classical reduced theory: adding $\alpha(k)$ by hand to minimal reduced quantization in order to achieve the Dirac curved-space quantization scheme would be unnatural. Perhaps not all information needed for a correct curved-space quantization (which avoids certain van Hove anomalies, for instance) is available in the classical reduced theory. Perhaps information about other structures on M is required, in which case we may have to take seriously the *natural* metric on M, as well as the *natural* basis of gauge vectors (modulo linear transformations with constant coefficients): we may not be able to assume, *a priori*, that the gauge vector basis is arbitrary [2], or that the kinetic term in the Hamiltonian is arbitrary up to terms that vanish on the constraint surface [37].

A possible future research direction is to understand the nature of the "complementary orbit terms." In this regard, we observe that the terms in β', for example, [see (65)] are essentially horizontal derivatives of the orbit metric, and as such are reminiscent of an "extrinsic curvature" of the orbits embedded in M. Another direction might be to relax the $\nabla K = 0$ restriction, as discussed in Sec. IV.

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APPENDIX: ANALYSIS OF RESTRICTIONS

In this Appendix we analyze the three restrictions, $\nabla K = 0$, $w_\beta K^{\alpha\beta} = 0$, and M Ricci flat, used to derive (38).

1. Consequences of restrictions on K

First, suppose that $\mathcal{C}(K)$ is an observable on Γ,

$$(\mathcal{L}_{w_\gamma} K)^{AB} = \xi_\gamma^{\beta(A} w_\beta^{B)} , \tag{A1}$$

for arbitrary vector fields $\xi_\gamma^{\beta A}$ on M, which is equivalent to the physical projection $K^{ab}(Q) =: k^{ab}(q(Q))$ being gauge invariant [cf. (I.14)]. Using (I.27) we then find

$$w_\gamma K^{a\beta} = \mathcal{L}_{w_\gamma}(e_A^a K^{AB} e_B^\beta) = \tfrac{1}{2}\xi_\gamma^{\beta a} - f^\beta{}_{\gamma\delta} K^{a\delta} , \tag{A2}$$

$$w_\gamma K^{\alpha\beta} = \mathcal{L}_{w_\gamma}(e_A^\alpha K^{AB} e_B^\beta) = \xi_\gamma^{\beta\alpha} - 2f^{(\alpha}{}_{\gamma\delta} K^{\beta)\delta} , \tag{A3}$$

where $\xi_\gamma^{\beta a}$ and $\xi_\gamma^{\beta\alpha}$ are the horizontal and vertical components of $\xi_\gamma^{\beta A}$, and $\xi_\gamma^{\beta\alpha} = \xi_\gamma^{\alpha\beta}$ by definition.

Using the Ricci rotation coefficients in (I.66)–(I.71), the components of ∇K in the horizontal and vertical basis are

$$\nabla_c K^{ab} = \tilde{\nabla}_c k^{ab} + \mathcal{F}_{\delta c}{}^{(a} K^{b)\delta} , \tag{A4}$$

$$\nabla_\gamma K^{ab} = \mathcal{F}_{\gamma d}{}^{(a} k^{b)d} - (\tilde{\nabla}^{(a}\gamma_{\gamma\delta}) K^{b)\delta} , \tag{A5}$$

---

[13]For more generic quadratic observables we find that we must introduce the additional consistency condition (10) to ensure that they are quantum observables. This condition plays a nontrivial role in our calculations leading to the curved-space quantization interpretation of $\alpha(k)$.

[14]Remember that the agreement on these factors is not unanimous, and, furthermore, none of the proposed quantization schemes claims to eliminate van Hove anomalies in quadratic-quadratic commutators, at least not for the type of curvature considered here.

$$\nabla_c K^{\alpha\beta} = w_c K^{\alpha\beta} + \tilde{\Gamma}^a{}_{cd} K^{d\beta} + \tfrac{1}{2}[\gamma^{\beta\alpha}(\tilde{\nabla}_c \gamma_{\alpha\delta})K^{a\delta} - \mathcal{F}^\beta{}_{cd} k^{ad} + \mathcal{F}_{\delta c}{}^a K^{\delta\beta}], \quad \text{(A6)}$$

$$\nabla_\gamma K^{a\beta} = w_\gamma K^{a\beta} + \tfrac{1}{2}[\mathcal{F}_{\gamma d}{}^a K^{d\beta} + f^\beta{}_{\gamma\delta} K^{a\delta} + \gamma^{\beta\alpha}(\tilde{\nabla}_d \gamma_{\alpha\gamma}) k^{ad} - (\tilde{\nabla}^a \gamma_{\gamma\delta}) K^{\delta\beta}], \quad \text{(A7)}$$

$$\nabla_c K^{\alpha\beta} = w_c K^{\alpha\beta} + \gamma^{\gamma(\alpha}(\tilde{\nabla}_c \gamma_{\gamma\delta}) K^{\beta)\delta} - \mathcal{F}^{(\alpha}{}_{cd} K^{\beta)d}, \quad \text{(A8)}$$

$$\nabla_\gamma K^{\alpha\beta} = w_\gamma K^{\alpha\beta} + f^{(\alpha}{}_{\gamma\delta} K^{\beta)\delta} + \gamma^{\delta(\alpha}(\tilde{\nabla}_d \gamma_{\delta\gamma}) K^{\beta)d}. \quad \text{(A9)}$$

In analogy with the vector case, demanding $\nabla K = 0$ (in particular the fourth and sixth relations above) then fixes the $\xi^{\beta A}_\gamma$ algebraically in terms of the components of K:

$$\xi^{\beta a}_\gamma = -\mathcal{F}_{\gamma d}{}^a K^{d\beta} + (\tilde{\nabla}^a \gamma_{\gamma\delta}) K^{\delta\beta} - \gamma^{\beta\alpha}(\tilde{\nabla}_d \gamma_{\alpha\gamma}) k^{ad} + f^\beta{}_{\gamma\delta} K^{a\delta}, \quad \text{(A10)}$$

$$\xi^{\beta\alpha}_\gamma = -\gamma^{\delta(\beta}(\tilde{\nabla}_d \gamma_{\delta\gamma}) K^{\alpha)d} + f^{(\beta}{}_{\gamma\delta} K^{\alpha)\delta}. \quad \text{(A11)}$$

Now we see that the quantum reduction consistency condition (10),

$$0 = w_\beta K^{a\beta} = \tfrac{1}{2}\xi^{\beta a}_\beta - f^\beta{}_{\beta\delta} K^{a\delta} = \tfrac{1}{2}\xi^{\beta a}_\beta, \quad \text{(A12)}$$

is equivalent to

$$k^{ab}\tilde{\nabla}_b \ln\sqrt{\gamma} + \tfrac{1}{2}\mathcal{F}_{\delta b}{}^a K^{b\delta} - \tfrac{1}{2}K^{\alpha\beta}\tilde{\nabla}^a \gamma_{\alpha\beta} = 0. \quad \text{(A13)}$$

Furthermore, from (A4) we learn that, as in the vector case, k is not, in general, covariantly constant on m,

$$\tilde{\nabla}_c k^{ab} = -\mathcal{F}_{\delta c}{}^{(a} K^{b)\delta}, \quad \text{(A14)}$$

but must, of course, be a Killing tensor: $\tilde{\nabla}^{(c} k^{ab)} = 0$. But, unlike in the vector case, the "divergence"

$$\tilde{\nabla}_b k^{ab} = -\tfrac{1}{2}\mathcal{F}_{\delta b}{}^a K^{b\delta} \quad \text{(A15)}$$

$$= k^{ab}\tilde{\nabla}_b \ln\sqrt{\gamma} - \tfrac{1}{2}K^{\alpha\beta}\tilde{\nabla}^a \gamma_{\alpha\beta} \quad \text{(A16)}$$

$$= \tfrac{1}{2}\tilde{\nabla}^a(\gamma_{\alpha\beta} K^{\alpha\beta}) \quad \text{(A17)}$$

does not necessarily vanish. The second line, which will be useful later, follows from the quantum consistency condition (A13), and the third line, an interesting alternative form for $\tilde{\nabla}_b k^{ab}$, follows from contracting (A8) with $\gamma_{\alpha\beta}$. [That $\gamma_{\alpha\beta}K^{\alpha\beta}$ is gauge invariant can be seen by contracting (A9) with $\gamma_{\alpha\beta}$, and then using (A5).]

Now, the quantity we are particularly interested in is

$$\tilde{\nabla}_a \tilde{\nabla}_b k^{ab} = w_a(\tilde{\nabla}_b k^{ab}) + \tilde{\Gamma}^a{}_{ac}\tilde{\nabla}_b k^{cb}, \quad \text{(A18)}$$

where we have put w_a in place of ∂_a, with impunity. On substituting (A15) we find

$$\tilde{\nabla}_a \tilde{\nabla}_b k^{ab} = -\tfrac{1}{2}[w_a \mathcal{F}_{\delta b}{}^a + \tilde{\Gamma}^a{}_{ac}\mathcal{F}_{\delta b}{}^c - \tilde{\Gamma}^c{}_{ab}\mathcal{F}_{\delta c}{}^a] K^{b\delta} - \tfrac{1}{4}[\mathcal{F}^\beta{}_{ac} k^{cb} - \gamma^{\beta\gamma}(\tilde{\nabla}_a \gamma_{\gamma\delta}) K^{b\delta} - \mathcal{F}_{\alpha a}{}^b K^{\alpha\beta}]\mathcal{F}_{\beta b}{}^a, \quad \text{(A19)}$$

where (A6) was used to obtain an expression algebraic in the components of K.

2. Consequences of M Ricci flat

In I we used a Kaluza-Klein-like approach to calculate the horizontal and vertical components of the Ricci tensor on M; see (I.81), (I.85), and (I.86). Contracting (I.81) with k^{ab} yields

$$k^{ab}\mathcal{R}_{ab} = k^{ab}\tilde{\mathcal{R}}_{ab} + \tfrac{1}{2}k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^\gamma{}_b{}^c - k^{ab}\tilde{\nabla}_a\tilde{\nabla}_b \ln\sqrt{\gamma} + \tfrac{1}{4}k^{ab}(\tilde{\nabla}_a \gamma^{\alpha\beta})(\tilde{\nabla}_b \gamma_{\alpha\beta}). \quad \text{(A20)}$$

The third term on the right hand side, second order in derivatives of γ, closely resembles $\alpha(k)$ in (17). In fact,

$$k^{ab}\tilde{\nabla}_a\tilde{\nabla}_b \ln\sqrt{\gamma} = \tilde{\nabla}_a(k^{ab}\tilde{\nabla}_b \ln\sqrt{\gamma}) - (\tilde{\nabla}_a k^{ab})(\tilde{\nabla}_b \ln\sqrt{\gamma})$$
$$= 2\alpha(k) - \tfrac{1}{2}k^{ab}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}_b \ln\sqrt{\gamma}) - (\tilde{\nabla}_a k^{ab})(\tilde{\nabla}_b \ln\sqrt{\gamma}), \quad \text{(A21)}$$

and so

$$k^{ab}\mathcal{R}_{ab} = -2\alpha(k) + k^{ab}\tilde{\mathcal{R}}_{ab} + \tfrac{1}{2}k^{ab}\mathcal{F}_{\gamma ca}\mathcal{F}^\gamma{}_b{}^c + \tfrac{1}{4}k^{ab}(\tilde{\nabla}_a \gamma^{\alpha\beta})(\tilde{\nabla}_b \gamma_{\alpha\beta}) + \tfrac{1}{2}k^{ab}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}_b \ln\sqrt{\gamma}) + (\tilde{\nabla}_a k^{ab})(\tilde{\nabla}_b \ln\sqrt{\gamma}). \quad \text{(A22)}$$

When M is Ricci flat the left hand side vanishes, suggesting a relationship between $\alpha(k)$ and $k^{ab}\tilde{\mathcal{R}}_{ab}$.

Next we contract (I.86) with $K^{\alpha\beta}$,

$$K^{\alpha\beta}\mathcal{R}_{\alpha\beta} = -\tfrac{1}{4}K^{\alpha\beta}f^\delta{}_{\gamma\alpha}f^\gamma{}_{\delta\beta} - \tfrac{1}{4}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^b\mathcal{F}_{\beta b}{}^a - \tfrac{1}{2}K^{\alpha\beta}\tilde{\Delta}\gamma_{\alpha\beta} + \tfrac{1}{2}K^{\alpha\beta}\gamma^{\gamma\delta}(\tilde{\nabla}_a \gamma_{\gamma\alpha})(\tilde{\nabla}^a \gamma_{\delta\beta}) - \tfrac{1}{2}(\tilde{\nabla}_a \ln\sqrt{\gamma})K^{\alpha\beta}(\tilde{\nabla}^a \gamma_{\alpha\beta}). \quad \text{(A23)}$$

Again, the third term on the right hand side, second order in derivatives of γ, can be written in terms of $\alpha(k)$:

$$K^{\alpha\beta}\tilde{\Delta}\gamma_{\alpha\beta} = 4\alpha(k) - 2\tilde{\nabla}_a\tilde{\nabla}_b k^{ab} - k^{ab}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}_b \ln\sqrt{\gamma}) + K^{\alpha\beta}\gamma^{\gamma\delta}(\tilde{\nabla}_a \gamma_{\gamma\alpha})(\tilde{\nabla}^a \gamma_{\delta\beta}) + \mathcal{F}^\alpha{}_b{}^a(\tilde{\nabla}_a \gamma_{\alpha\beta})K^{b\beta}. \quad \text{(A24)}$$

The calculation is similar to (A21), but, in addition, makes use of (A16) and (A8). Using (A16) again, now in the right hand side of (A23), we finally obtain

$$K^{\alpha\beta}\mathcal{R}_{\alpha\beta} = -2\alpha(k) + \tilde{\nabla}_a\tilde{\nabla}_b k^{ab} - \tfrac{1}{4}K^{\alpha\beta}f^\delta{}_{\gamma\alpha}f^\gamma{}_{\delta\beta} - \tfrac{1}{4}K^{\alpha\beta}\mathcal{F}_{\alpha a}{}^b\mathcal{F}_{\beta b}{}^a - \tfrac{1}{2}k^{ab}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}_b \ln\sqrt{\gamma}) + (\tilde{\nabla}_a k^{ab})(\tilde{\nabla}_b \ln\sqrt{\gamma}) - \tfrac{1}{2}\mathcal{F}^\alpha{}_b{}^a(\tilde{\nabla}_a \gamma_{\alpha\beta})K^{b\beta}. \quad \text{(A25)}$$

Lastly, we contract (I.85) with $K^{\alpha\beta}$:

$$2K^{\alpha\beta}\mathcal{R}_{\alpha\beta} = [w_b\mathcal{F}_{\beta a}{}^{b} + \tilde{\Gamma}^{b}{}_{bc}\mathcal{F}_{\beta a}{}^{c} - \tilde{\Gamma}^{c}{}_{ba}\mathcal{F}_{\beta c}{}^{b} + (\tilde{\nabla}_b \ln\sqrt{\gamma})\mathcal{F}_{\beta a}{}^{b}]K^{a\beta}\,. \qquad (A26)$$

Comparing this with $\tilde{\nabla}_a\tilde{\nabla}_b k^{ab}$ in (A19), and using (A15) again, we have the third and final Ricci curvature relation,

$$2K^{\alpha\beta}\mathcal{R}_{\alpha\beta} = -2\tilde{\nabla}_a\tilde{\nabla}_b k^{ab} - 2(\tilde{\nabla}_a k^{ab})(\tilde{\nabla}_b \ln\sqrt{\gamma}) - \tfrac{1}{2}[\mathcal{F}^{\beta}{}_{ac}k^{cb} - \gamma^{\beta\gamma}(\tilde{\nabla}_a\gamma_{\gamma\delta})K^{b\delta} - \mathcal{F}_{\alpha a}{}^{b}K^{\alpha\beta}]\mathcal{F}_{\beta b}{}^{a}\,. \qquad (A27)$$

These three relations (A22), (A27), and (A25), which also embody the restrictions on K, are combined in the form (37) to yield (38).

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