# Kaluza-Klein-like analysis of the configuration space of gauge theories

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It is well known that Dirac quantization of gauge theories is not, in general equivalent to reduced quantization. When both approaches are self-consistent some additional criterion must be found in order to decide which approach is more natural, or correct. Now, in most cases quantization on the physical degrees of freedom is properly curved-space quantization, with a curvature that is neither constant nor Ricci flat. In a series of two papers, this being the first, we show that, unlike reduced quantization, Dirac quantization (acting in the physical Hilbert space) corresponds in a natural way to such a curved-space quantization scheme, and has remarkable similarities with other curved-space quantization schemes proposed elsewhere. We begin here with an in-depth analysis of the geometry of the classical configuration space of gauge theories. In particular, the existence of a metric on the full configuration space establishes a Yang-Mills connection with respect to the orbits—we emphasize the importance of this connection, as well as the condition the gauge theory must satisfy in order to define it. We also discuss, in Kaluza-Klein-like fashion, three Levi-Cività connections, their associated Ricci tensors, and interrelationships among them.

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## I. INTRODUCTION

This is the first in a series of two papers addressing the question of Dirac versus reduced quantization of gauge theories, which has been a point of contention in the recent literature [1–6]. It is well known that gauge theories play a prominent role in our current understanding of nature, so it is important to understand both their classical and quantum structures in detail (see, e.g., [7,8]). At the classical level the Lagrangian is invariant under local gauge transformations, which means that, on the full space of instantaneous configurations, $M$, not all points are physically distinguishable: there are redundant, or gauge degrees of freedom. These manifest themselves as the orbits of a symmetry group (the gauge group) acting on $M$. The space of such gauge orbits represents the physical degrees of freedom, and is called the reduced configuration space $m$. In this latter case there is no explicit remnant of the original gauge structure, and the theory can be quantized using "standard" canonical techniques on the phase space associated with $m$. This is called reduced quantization.

Dirac quantization [9], on the other hand, seeks to quantize on the phase space associated with $M$, initially ignoring the gauge structure. The latter manifests itself as constraints (linear in the momenta), which are then realized as operators and used to select a subset of physical states from the full set of quantum states. The other operators (such as the Hamiltonian), restricted to act on the physical state space, endow it with a Hilbert space structure. It is well known that these two approaches, Dirac versus reduced, generally lead to distinct quantum systems [1–6]. It is even known that this difference can be understood as a factor ordering ambiguity of the Hamiltonian acting on the physical Hilbert space [6]. In this series of papers we will further illuminate the geometrical significance of this ambiguity, and provide clues to its resolution.

The first paper addresses classical aspects of the problem, in particular, the rich geometry of the configuration space of gauge theories, which lays the foundation for the quantum analysis given in [10] (hereafter referred to as II). We begin with a brief introduction to gauge theories, as defined through the action of a gauge group on the full configuration space. Classical reduction is then discussed using a basis of vector fields and one-forms on $M$ adapted to the gauge orbits, i.e., the basis is split into "horizontal" and "vertical" parts (the orbits being vertical). Naturally there is some overlap with [2], a series of papers by Kuchař which develops this in detail. However, here we emphasize the role of the horizontal basis in establishing a nontrivial Yang-Mills connection on $M$, as well as the consistency condition the gauge theory must satisfy in order to admit this type of connection.

Now, if the full configuration space $M$ comes equipped with a Riemannian metric $G$, as we assume here, then there exists a *natural* horizontal basis: the orthogonal complement of the vertical basis. Natural as it may be, there is no *a priori* reason for this horizontal basis to satisfy the consistency condition mentioned above; we emphasize that an additional relation must exist between the metric and the gauge structures of the theory.[1]

The metric $G$ on $M$ induces a metric on the reduced configuration space $m$, as well as one along the orbits themselves. These three metrics give rise to three Levi-Cività connections, which we calculate and com-

[1]In the Dirac quantization discussed in II we will see that this relation plays a role in the kinetic energy operator being a quantum observable.

 

pare. Furthermore, we determine the three associated Ricci tensors, and in Kaluza-Klein-like fashion illuminate the interrelationships amongst them, as well as the Yang-Mills curvature. These results are illustrated using Kuchař's "helix model" [3] as an example. While also of intrinsic interest, the results of this paper are used in the quantum analysis in II to establish a link between the Dirac versus reduced factor ordering ambiguity and issues of curved-space quantization.

## II. GAUGE THEORIES

We begin with a brief description of the type of gauge theory to be considered here, chiefly to introduce notation. We consider a classical system whose set of configurations at any instant of time is naturally represented by a manifold $M$ with local coordinates $Q^A$, $A = 1, \ldots, N$, called the full configuration space. In a gauge theory this representation is "reducible:" not all points on $M$ correspond to physically distinct configurations. This manifests itself as the action of a Lie group (the gauge group) on $M$ via diffeomorphisms generated by a basis set $\{\phi_\alpha = \phi_\alpha^A \partial/\partial Q^A\}$ of gauge vector fields ($\alpha$ ranging from 1 to $C$, the dimension of the group). These vector fields are surface forming:

$$[\phi_\alpha, \phi_\beta] = f^\gamma{}_{\alpha\beta}\phi_\gamma \quad , \tag{1}$$

where the $f^\gamma{}_{\alpha\beta}$ are the structure constants of the group.[2] Thus, assuming the $\phi_\alpha$ to be linearly independent, they span a $C$-dimensional subspace of the tangent space $T_Q M$ at each point $Q \in M$, and these subspaces are integrable, yielding a foliation of $M$ by so-called gauge orbits. This establishes the projection $\pi : M \to m$. All points in a given gauge orbit represent physically indistinguishable configurations, and the space of orbits, $m$, is called the reduced configuration space.[3] Local coordinates $q^a$, $a = 1, \ldots, n = N - C$, on $m$ label the physically distinct configurations of the theory.

In the Hamiltonian analysis the geometrical setting is either the full phase space $\Gamma := T^*M$, with local canonical coordinates $Q^A, P_A$, or the reduced phase space $\gamma := T^*m$, with local canonical coordinates $q^a, p_a$. In the former, the notion of first class constraints linear in the momenta arises as follows (see, e.g., [12]): Let $q$ label a point in $m$, i.e., an orbit in $M$, and $Q \in M$ any point on that orbit (representing it). Then $T_q^* m$ would be the space of linear functions of variations, $\delta Q$ of $Q$, which depend only on which orbit the point $Q + \delta Q$ lies in. In other words, the allowed momenta $P \in T_Q^* M$ must satisfy

$$\langle P, \delta Q + \epsilon^\alpha \phi_\alpha \rangle = \langle P, \delta Q \rangle \ \forall \delta Q \quad , \tag{2}$$

where $\epsilon^\alpha$ are arbitrary infinitesimals. This linear condition on the momenta is equivalent to the constraints

$$\mathcal{C}(\phi_\alpha) := \phi_\alpha^A P_A \approx 0 \quad \forall \alpha \quad , \tag{3}$$

where $\approx$ denotes equality on the constraint surface $\Gamma_C \subset \Gamma$ thus defined. The constraints also canonically generate gauge transformations of states in the full phase space, which are in fact integrable on $\Gamma_C$. The set of corresponding orbits on $\Gamma_C$ comprises the reduced phase space, $\gamma$. An explicit local representation of $\gamma$ as a surface in $\Gamma$ (i.e., gauge fixing) would depend on which points $Q$ in $M$ were used to represent their respective orbits $q$. The gauge transformations should preserve the constraints, so the Poisson algebra of constraints should close, i.e., the constraints should be first class, a condition guaranteed by (1).

To make this discussion more concrete we now introduce an example, namely, the "helix model" developed in detail by Kuchař [3], which will also serve to illustrate results in later sections. The full configuration space $M = \mathbf{R}^3$ is flat, with Cartesian coordinates $Q^A = (X, Y, Z)$; we may imagine this corresponding to a nonrelativistic particle of unit mass in ordinary three-space. Let the one parameter translation group act on $M$ by helical gauge transformations:

$$\begin{aligned} X(\lambda) &= X(0)\cos(\lambda) - Y(0)\sin(\lambda) \\ Y(\lambda) &= X(0)\sin(\lambda) + Y(0)\cos(\lambda) \\ Z(\lambda) &= Z(0) + \lambda \quad . \end{aligned} \tag{4}$$

These are generated by the vector field $\phi_\alpha = \phi_\alpha^A \partial/\partial Q^A$ ($\alpha$ just takes the single value, 1), where

$$\phi_\alpha^A(Q) = \frac{\partial Q^A(\lambda)}{\partial \lambda}\Big|_{\lambda=0} = (-Y, X, 1) \quad . \tag{5}$$

In the extended phase space $\Gamma = T^*M$ the "Gauss law" constraint is

$$\mathcal{C}(\phi_\alpha) = P_Z - YP_X + XP_Y = 0 \quad . \tag{6}$$

With the flat metric $G_{AB} = \delta_{AB}$ on $M$, the vector $\phi_\alpha$, being a combination of translation and rotation, is a Killing vector.

## III. CLASSICAL REDUCTION

Briefly, classical reduction consists of obtaining an explicit representation of the reduced phase space, $\gamma$, i.e.,

[2]The choice of basis $\{\phi_\alpha\}$ is, of course, arbitrary up to linear transformations with constant coefficients, corresponding to the same arbitrariness at the Lie algebra level. However, Kuchař [2] advocates that the theory should be covariant under arbitrary change of basis at the vector field level, which would necessitate structure *functions*—i.e., it is not a particular basis, but rather the space of vector fields spanned by it, that is relevant in a gauge theory. We shall entertain this possibility on occasion, and point out subtleties, but unless otherwise noted we assume the "natural" family of bases, modulo linear transformations with constant coefficients.

[3]We shall ignore subtleties arising from boundary points of $m$ (associated with points in $M$ that remain invariant under the action of the gauge group [11]), or topological considerations.

a complete set of gauge invariant functions on the constraint surface $\Gamma_C$. The Poisson algebra structure and observables on $\gamma$ are then inherited from the full phase space. A convenient way to understand this process is to introduce coordinates (see, e.g., [6]) or other objects (see, e.g., [2]) adapted to the orbit structure $\pi : M \to m$.

For gauge theories with constraints linear in the momenta the bulk of the task is really just finding an explicit representation of the reduced configuration space. Thus suppose we can find a set $\{q^a\}$ of independent gauge invariant functions on $M$:

$$\phi_\alpha q^a = 0 \;\; \forall \alpha, a \;\; . \tag{7}$$

These are of course just the pull back under $\pi$ of coordinates (with the same name) on $m$, and serve as a subset of coordinates on $M$, labeling the orbits. The coordinates can be completed by any set $\{F^\alpha\}$ of independent gauge *variant* functions, whose "Faddeev-Popov" matrix

$$F^\beta_\alpha := \phi_\alpha F^\beta \tag{8}$$

is nondegenerate on $M$. The $F^\alpha$ label points within a given orbit, and any surface $\bar{m}$ defined by $F^\alpha - C^\alpha = 0$, where $C^\alpha$ are constants, may be used as a local representative of $m$ (although such explicit gauge fixing will not be necessary here).

This establishes a coordinate basis of one-forms $(dq^a, dF^\alpha)$ and vector fields $(\partial/\partial q^a, \partial/\partial F^\alpha)$, from which we can construct any other (generally nonholonomic) basis of one-forms $e^{\hat{A}} := (e^a, e^\alpha)$, and dual vector fields $w_{\hat{A}} := (w_a, w_\alpha)$. There already exist some natural choices to begin such a construction, namely, $e^a = dq^a$, the pull back (under $\pi$) of the coordinate one-form on $m$ with the same name, as well as $w_\alpha = \phi_\alpha$, which span the orbits and define the action of the gauge group on $M$. With these choices the condition $\langle e^{\hat{A}}, w_{\hat{B}}\rangle = \delta^{\hat{A}}_{\hat{B}}$ defines the remainder of the basis elements, up to a set of one-forms $A^\alpha = A^\alpha_a dq^a$ :

$$e^\alpha = \phi^\alpha + A^\alpha, \quad w_a = \frac{\partial}{\partial q^a} - A^\alpha_a \phi_\alpha \;\; . \tag{9}$$

Here

$$\phi^\alpha = (F^{-1})^\alpha_\beta dF^\beta \;\; \text{or} \;\; (F^{-1})^\beta_\alpha \phi_\beta = \frac{\partial}{\partial F^\alpha} \;\; ; \tag{10}$$

i.e., the inverse Faddeev-Popov matrix is an integrating factor which Abelianizes the constraints. (This can always be done [13].) The choice of the functions $A^\alpha_a$ on $M$ controls the "vertical" piece of the "horizontal" vectors $w_a$, and we leave this arbitrary for now.

A classical observable is a function on the full phase space which is well defined on the reduced phase space, i.e., whose restriction to the constraint surface $\Gamma_C$ is gauge invariant. We will be interested in observables which are polynomial in the momenta: define

$$\mathcal{C}(S) := S^{A_1 \cdots A_s}(Q) P_{A_1} \cdots P_{A_s} \tag{11}$$

as the classical observable corresponding to the symmetric valence $s$ contravariant tensor $S$ on $M$. Using the completeness relation $\delta^A_B = w^A_c e^c_B + w^A_\gamma e^\gamma_B$ together with (3) yields

$$\mathcal{C}(S) \approx s^{a_1 \cdots a_s} p_{a_1} \cdots p_{a_s} =: c(s) \tag{12}$$

on the constraint surface, which should be gauge invariant. Here $p_a := \mathcal{C}(w_a)$, and

$$s^{a_1 \cdots a_s} := e^{a_1}_{A_1} \cdots e^{a_s}_{A_s} S^{A_1 \cdots A_s} \tag{13}$$

is the push forward (or physical projection) of $S$ to the tensor $s$ on $m$ induced by $\pi : M \to m$, which is well defined if and only if (iff) the right hand side of (13) is gauge invariant. Since $\mathcal{L}_{w_\alpha} e^a = 0$, this requirement is equivalent to the projected Lie derivative condition

$$e^{a_1}_{A_1} \cdots e^{a_s}_{A_s} (\mathcal{L}_{w_\alpha} S)^{A_1 \cdots A_s} = 0 \;\; \forall \alpha \;\; . \tag{14}$$

This is satisfied for $S = w_a$ itself:

$$\mathcal{L}_{w_\alpha} w_b = -\left[ w_\alpha A^\gamma_b + f^\gamma{}_{\alpha\beta} A^\beta_b + (F^{-1})^\gamma_\beta \frac{\partial}{\partial q^b} F^\beta_\alpha \right] w_\gamma \tag{15}$$

has only a vertical piece, regardless of our choice of $A^\alpha$, and so we see that the $q^a, p_a$ form a complete set of gauge invariant functions on the constraint surface. In fact, they are canonical:

$$\{q^a, q^b\} = 0 \;\; , \tag{16}$$

$$\{q^a, p_b\} = \delta^a_b \;\; , \tag{17}$$

$$\{p_a, p_b\} = \mathcal{C}(-\mathcal{L}_{w_a} w_b) \approx 0 \;\; , \tag{18}$$

where in the last line we used

$$\begin{aligned} \mathcal{L}_{w_a} w_b = &-\left( \frac{\partial}{\partial q^a} A^\gamma_b - \frac{\partial}{\partial q^b} A^\gamma_a + f^\gamma{}_{\alpha\beta} A^\alpha_a A^\beta_b \right) w_\gamma \\ &- A^\alpha_a \mathcal{L}_{w_\alpha} w_b + A^\beta_b \mathcal{L}_{w_\beta} w_a \;\; , \end{aligned} \tag{19}$$

which follows from the definition of the horizontal vectors. Note that it has only a vertical component, a fact which is again independent of our choice of $A^\alpha$.

Finally, for any two observables $\mathcal{C}(S), \mathcal{C}(T)$ it is straightforward to show that

$$\{\mathcal{C}(S), \mathcal{C}(T)\} \approx \{c(s), c(t)\}, \tag{20}$$

where the Poisson brackets on the right hand side (on the reduced phase space) are inherited via (16)–(17). In other words the classical reduction map (12) preserves the Poisson algebra.

## IV. YANG-MILLS CONNECTION

The astute reader will have recognized in (15) and (19) the fact that $A^\alpha$ appears in formulas which suggest its natural role as a Yang-Mills connection, but that this role is "dormant" as far as the classical reduction is concerned. Actually, as we shall see shortly, it is $e^\alpha = A^\alpha + \phi^\alpha$ we are interested in, and we will vitalize its role as a connection one-form, as well as discuss the condition on the gauge theory under which this interpretation is possible.

The action of the gauge group on $M$ establishes the fiber bundle projection $\pi : M \to m$. The $w_\alpha$ allow us to move vertically within the fibers, and the $w_a$ define a horizontal motion connecting neighboring fibers. In other words, at each point $Q \in M$ the tangent space $T_Q M$ is split into a vertical subspace $V_Q M$, spanned by the $w_\alpha$, and a horizontal subspace $H_Q M$, spanned by the $w_a$:

$$T_Q M \simeq V_Q M \oplus H_Q M \quad . \tag{21}$$

However, establishing this splitting is not sufficient to endow $M$ with a connection (see, e.g., [15]); we also need the splitting to be *compatible* with the action of the gauge group on $M$. This means

$$\mathcal{L}_{w_\alpha} w_\beta \in VM \quad , \tag{22}$$

$$\mathcal{L}_{w_\alpha} w_b \in HM \quad ; \tag{23}$$

i.e., the vertical and horizontal subspaces are invariant under gauge transformations generated by the $w_\alpha$. The first condition is obviously satisfied, being just the integrability condition for the gauge orbits [cf. (1)]. Returning to (15) we learn that the second condition is satisfied iff

$$\mathcal{L}_{w_\alpha} w_b = [w_\alpha, w_b] = 0, \tag{24}$$

or equivalently

$$w_\alpha A^\gamma_a = -f^\gamma{}_{\alpha\beta} A^\beta_a + F^\beta_\alpha \frac{\partial}{\partial q^a} (F^{-1})^\gamma_\beta \tag{25}$$

(assuming linear independence of the $w_\alpha$). So in order to raise the splitting to the status of a connection the $A^\alpha_a$ are no longer arbitrary, but must (of course) transform in a certain way under gauge transformations.

We remark that condition (23) is invariant under an arbitrary linear transformation of the *horizontal* basis vectors. It has been suggested [2,14], however, that the formalism of a gauge theory (both at the classical and quantum level) be covariant under an arbitrary linear transformation of the *vertical* basis vectors. In this regard (22) presents no obstacle, but (23) does: for

$$\mathcal{L}_{\mu^\alpha w_\alpha} w_b = \mu^\alpha \mathcal{L}_{w_\alpha} w_b - (w_b \mu^\alpha) w_\alpha \tag{26}$$

to be in $HM$ requires the coefficients $\mu^\alpha$ to be constant in the horizontal direction (i.e., $w_b \mu^\alpha = 0$). This tells us that the notion of full symmetry under change of gauge vector basis is not compatible with the notion of the horizontal basis giving rise to a Yang-Mills connection on $M$. (But, of course, a transformation of basis with constant coefficients, i.e., at the Lie algebra level, is allowed.)

An equivalent way to define this connection on $M$ is to recognize that the $e^\alpha$ are components of a standard connection one-form (see, e.g., [15]) in the sense that $\langle e^\alpha, w_\beta \rangle = \delta^\alpha_\beta$ and

$$\begin{aligned} \mathcal{L}_{w_\alpha} e^\gamma &= \langle \mathcal{L}_{w_\alpha} e^\gamma, w_b \rangle e^b + \langle \mathcal{L}_{w_\alpha} e^\gamma, w_\beta \rangle e^\beta \\ &= -\langle e^\gamma, \mathcal{L}_{w_\alpha} w_b \rangle e^b - \langle e^\gamma, \mathcal{L}_{w_\alpha} w_\beta \rangle e^\beta \\ &= -f^\gamma{}_{\alpha\beta} e^\beta \quad , \end{aligned} \tag{27}$$

a result which relies upon (23). In other words, $e^\gamma$ transforms according to the adjoint representation of the gauge group. Using the $e^\alpha$, horizontal vectors $V \in HM$ are defined by

$$\langle e^\alpha, V \rangle = 0 \quad \forall \alpha \quad . \tag{28}$$

In fact, in the decomposition $e^\alpha = A^\alpha + \phi^\alpha$ the $\phi^\alpha$ essentially correspond to the Maurer-Cartan form, and the $A^\alpha$ to a Yang-Mills field.[4] Then the condition (25) corresponds to the requirement that this field have a standard gauge transformation law.

Given this connection on $M$ we can calculate its curvature:

$$[w_a, w_b] =: -\mathcal{F}^\gamma{}_{ab} w_\gamma \quad . \tag{30}$$

Returning to (19) we see that the last two terms vanish on account of (24), and so the curvature takes the standard form

$$\mathcal{F}^\gamma{}_{ab} = \frac{\partial}{\partial q^a} A^\gamma_b - \frac{\partial}{\partial q^b} A^\gamma_a + f^\gamma{}_{\alpha\beta} A^\alpha_a A^\beta_b \quad . \tag{31}$$

A result we will need later concerns the gauge dependence of $\mathcal{F}^\gamma{}_{ab}$. Taking the Lie derivative of both sides of (30) with respect to $w_\alpha$, and using (24) together with the Jacobi identity, shows that the curvature transforms according to the adjoint representation, as expected:

$$w_\alpha \mathcal{F}^\gamma{}_{ab} = -f^\gamma{}_{\alpha\beta} \mathcal{F}^\beta{}_{ab} \quad . \tag{32}$$

Finally, the curvature can also be calculated using the Cartan structural equation

$$\mathcal{F}^\gamma = \tfrac{1}{2} \mathcal{F}^\gamma{}_{ab} e^a \wedge e^b = de^\gamma + \tfrac{1}{2} f^\gamma{}_{\alpha\beta} e^\alpha \wedge e^\beta \quad , \tag{33}$$

a formula which is convenient for applications (such as the helix model to be considered shortly).

Of course we cannot apply any of the standard results discussed above until we find a suitable $A^\alpha$ such that (24) is satisfied. So far there is no natural choice, unless we introduce some additional geometrical object into the theory. To this end we suppose that the Hamiltonian of the gauge theory has a standard kinetic energy term $\mathcal{C}(\frac{1}{2} G^{-1})$, which is nondegenerate[5] and positive definite.

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[4]Strictly speaking (see, e.g., [15]), a Yang-Mills field is a local representative of $e^\alpha$ obtained by defining a local section $\bar{m} : m \to M$ of $M$, which in our case would correspond to an embedded surface representing the reduced configuration space $m$, and then defining the pullback $\bar{e}^\alpha := \bar{m}_*(e^\alpha)$. If, for example, we take the gauge-fixing condition $F^\alpha = 0$ to define $\bar{m}$, we get

$$\bar{e}^\alpha(q) = A^\alpha(q, F = 0). \tag{29}$$

[5]In scalar electrodynamics, as well as certain other gauge theories, one must first eliminate the Lagrange multiplier pair $A_0, \Pi^0$ from the original phase space to achieve a nondegenerate kinetic energy (see, e.g., [16], and a comment to the same effect in [17]).

This induces a Riemannian metric $G$ on $M$ which, beyond certain restrictions we will discuss, need not be specified. To begin with, we assume that this kinetic term is an observable so that (14) is satisfied, with $S = G^{-1}$; this can be written as the projected Killing equation

$$(\mathcal{L}_{w_\alpha} G^{-1})^{AB} = \zeta_\alpha^{\beta(A} w_\beta^{B)} \quad , \tag{34}$$

where $\zeta_\alpha^\beta = \zeta_\alpha^{\beta A}\partial/\partial Q^A$ may be any vector fields on $M$. They encode important information about the relationship between the gauge and metric structure on $M$. We then have the gauge invariant physical projection

$$g^{ab} = G^{-1}(e^a, e^b) \quad . \tag{35}$$

Now, using the completeness relation for the nonholonomic basis we can write

$$G(w_\alpha, w_\beta)e^\beta = G(w_\alpha, \ ) - G(w_\alpha, w_b)e^b \quad . \tag{36}$$

The object

$$\gamma_{\alpha\beta} := G(w_\alpha, w_\beta) \tag{37}$$

on the left hand side is positive definite (a fact which follows from the positive definiteness of $G$ and linear independence of the $w_\alpha$), and serves as a metric within a given orbit. Using its inverse, $\gamma^{\alpha\beta}$, we can then uniquely fix $A^\alpha$ (or equivalently $e^\alpha$) by demanding that $H_Q M$ be orthogonal to $V_Q M$:

$$G(w_\alpha, w_b) = 0 \quad , \tag{38}$$

yielding

$$e_A^\alpha = \gamma^{\alpha\beta} G_{AB} w_\beta^B \quad . \tag{39}$$

With this choice of basis the metric takes a "Kaluza-Klein-like" form

$$G = g_{ab} e^a \otimes e^b + \gamma_{\alpha\beta} e^\alpha \otimes e^\beta \quad , \tag{40}$$

where $g_{ab} = G(w_a, w_b)$, the inverse of $g^{ab}$ above, is positive definite, and serves as a metric on $m$. Finally, this fixes[6]

$$w_a^A = g_{ab} G^{AB} e_B^b \quad . \tag{41}$$

The corresponding Yang-Mills field is easily determined to be

$$A_a^\alpha = \gamma^{\alpha\beta} F_\beta^\gamma G\left(\frac{\partial}{\partial F^\gamma}, \frac{\partial}{\partial q^a}\right) \quad . \tag{42}$$

So by introducing a metric on $M$ one can fix the horizontal subspaces in a natural way, but it is not *a priori* obvious that this choice will be gauge invariant, i.e., satisfy (23). This is a question about the relationship between the metric and gauge structure, and must be checked in any given theory. Using (41) and (34) we find

$$(\mathcal{L}_{w_\alpha} w_a)^A = g_{ab}(\mathcal{L}_{w_\alpha} G^{-1})^{AB} e_B^b = \tfrac{1}{2} g_{ab} \zeta_\alpha^{\beta b} w_\beta^A \quad , \tag{43}$$

where $\zeta_\alpha^{\beta b} := \langle e^b, \zeta_\alpha^\beta \rangle$ is the horizontal, or physical, projection of the vector $\zeta_\alpha^\beta$. Now, since (43) has only a vertical piece, (23) is equivalent to

$$\zeta_\alpha^{\beta a} = 0 \quad . \tag{44}$$

This is an extra condition relating the metric and gauge structures, beyond the usual preservation of the classical constraints under time evolution. In the language of (34) it appears as

$$(\mathcal{L}_{w_\alpha} G^{-1})^{AB} = \zeta_\alpha^{\beta\gamma} w_\beta^{(A} w_\gamma^{B)} \quad , \tag{45}$$

where $\zeta_\alpha^{\beta\gamma} := \langle e^\gamma, \zeta_\alpha^\beta \rangle$, the vertical projection of the vector $\zeta_\alpha^\beta$, is still unrestricted. Equation (44) plays an important role in the quantum Hamiltonian being an observable, as we shall see in II.

We now apply the above analysis to the helix model. The classical reduction is carried out in [3]; we simply quote the results we will need. The adapted coordinates can be chosen as

$$q^a = (B, \rho) \quad , \tag{46}$$
$$F^\alpha = \theta, \ 0 \le \theta < 2\pi \quad , \tag{47}$$

where $B := (Z - \theta) \bmod 2\pi$ and $X + iY =: \rho \exp(i\theta)$. $B$ and $\rho$ are obviously gauge invariant, and the Faddeev-Popov matrix (just the identity in this case) is nondegenerate on $M$. The inverse metric on the reduced space $m$ is

$$g^{ab} = G^{-1}(e^a, e^b) = \begin{pmatrix} 1 + \frac{1}{\rho^2} & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{48}$$

whereas the metric within a given orbit turns out to be

$$\gamma_{\alpha\beta} = G(w_\alpha, w_\beta) = 1 + \rho^2 \quad . \tag{49}$$

As mentioned before, the gauge vector (5) is Killing, so the condition (44) is automatically satisfied: $e^\alpha$ given by (39) can be interpreted as a connection one-form. Using (42) we find

$$A^\alpha = \frac{1}{1+\rho^2} dB \quad , \tag{50}$$

which is not closed. Furthermore, since the Faddeev-Popov matrix is just the identity we see from (10) that the $\phi^\alpha$ are exact. Finally, with just one gauge vector there are no structure constants, so, according to (33),

$$\mathcal{F}^\alpha = de^\alpha = dA^\alpha = \frac{2\rho}{(1+\rho^2)^2} dB \wedge d\rho \quad . \tag{51}$$

Note that a nonvanishing $\mathcal{F}^\alpha$ means the distribution of horizontal subspaces $H_Q M$ is not integrable, which in our case means it is not possible, even locally, to find

[6] Kuchař [2] writes down the same expressions, (39) and (41), but here we point out the decomposition $e^\alpha = A^\alpha + \phi^\alpha$, which emphasizes the natural role of $e^\alpha$ as a connection one-form (see also [18]), as well as discuss its relation to other (Levi-Cività) connections.

a surface $\bar{m}$ embedded in $M$, representing the reduced configuration space $m$, which is orthogonal to the gauge orbits.

## V. THREE LEVI-CIVITÀ CONNECTIONS

So far we have discussed the Yang-Mills connection (and its curvature) associated with the induced fiber bundle structure of $M$. There are three other connections to discuss: the Levi-Civitià connections associated by the metrics $G_{AB}$, $g_{ab}$, and $\gamma_{\alpha\beta}$, on the extended configuration space $M$, the reduced configuration space $m$, and the orbits, respectively.

We begin with the Levi-Civitià connection $\nabla$ associated with the metric $G$ on $M$. It is convenient to use the basis $e^{\hat{A}}$ and $w_{\hat{A}}$ of one forms and vectors. The action of $\nabla$ is defined by

$$\nabla e^{\hat{C}} := -\Gamma^{\hat{C}}{}_{\hat{A}\hat{B}} e^{\hat{A}} \otimes e^{\hat{B}} \quad , \tag{52}$$

$$\nabla w_{\hat{B}} := \Gamma^{\hat{C}}{}_{\hat{A}\hat{B}} e^{\hat{A}} \otimes w_{\hat{C}} \quad , \tag{53}$$

where the $\Gamma^{\hat{C}}{}_{\hat{A}\hat{B}}$ are Ricci rotation coefficients.

As usual, the Levi-Civitià connection is uniquely determined by the conditions of no torsion and metricity. For arbitrary vector fields $U$ and $V$ on $M$, the torsion

$$\begin{aligned} T(U,V) &:= \nabla_U V - \nabla_V U - [U,V] \\ &= U^{\hat{A}} V^{\hat{B}} \{ (\Gamma^{\hat{C}}{}_{\hat{A}\hat{B}} - \Gamma^{\hat{C}}{}_{\hat{B}\hat{A}}) - C^{\hat{C}}{}_{\hat{A}\hat{B}} \} w_{\hat{C}} \quad , \end{aligned} \tag{54}$$

where the $C$'s are defined by the commutators

$$[w_{\hat{A}}, w_{\hat{B}}] =: C^{\hat{C}}{}_{\hat{A}\hat{B}} w_{\hat{C}} \quad . \tag{55}$$

No torsion then means

$$\Gamma^{\hat{C}}{}_{\hat{A}\hat{B}} - \Gamma^{\hat{C}}{}_{\hat{B}\hat{A}} = C^{\hat{C}}{}_{\hat{A}\hat{B}} \quad . \tag{56}$$

Collecting the results of (30), (24), and (1) we learn that the nonvanishing $C$'s are

$$C^{\gamma}{}_{ab} = -\mathcal{F}^{\gamma}{}_{ab} \quad , \tag{57}$$

$$C^{\gamma}{}_{\alpha\beta} = f^{\gamma}{}_{\alpha\beta} \quad . \tag{58}$$

Using (56) with the metricity condition $\nabla G = 0$ then determines the Ricci rotation coefficients:

$$\begin{aligned} \Gamma^{\hat{C}}{}_{\hat{A}\hat{B}} = \tfrac{1}{2} G^{\hat{C}\hat{D}} \{ & w_{\hat{A}} G_{\hat{B}\hat{D}} + w_{\hat{B}} G_{\hat{A}\hat{D}} - w_{\hat{D}} G_{\hat{A}\hat{B}} - C_{\hat{A}\hat{B}\hat{D}} \\ & - C_{\hat{B}\hat{A}\hat{D}} + C_{\hat{D}\hat{A}\hat{B}} \} \quad , \end{aligned} \tag{59}$$

where $C_{\hat{C}\hat{A}\hat{B}} := G_{\hat{C}\hat{D}} C^{\hat{D}}{}_{\hat{A}\hat{B}}$.

The calculation of the $\Gamma$'s involves computing derivatives of components of the metric in (40). We recall that the components $g_{ab}$ are gauge invariant—a necessary condition for the Hamiltonian to be a classical observable. The same is not necessarily true for the $\gamma_{\alpha\beta}$ components. Using (34) and (27) we learn that

$$w_\gamma \gamma^{\alpha\beta} = \mathcal{L}_{w_\gamma} G^{-1}(e^\alpha, e^\beta) = \zeta_\gamma^{\alpha\beta} - f^{\alpha}{}_{\gamma\delta}\gamma^{\delta\beta} - f^{\beta}{}_{\gamma\delta}\gamma^{\alpha\delta} \quad . \tag{60}$$

Tracing with $\gamma_{\alpha\beta}$ we find

$$w_\gamma \ln \sqrt{\gamma} = \nabla \cdot w_\gamma - f^{\alpha}{}_{\alpha\gamma} \tag{61}$$

(as in [18]), where we used the fact that the divergence of a gauge vector is given by

$$\nabla \cdot w_\gamma = -\tfrac{1}{2} G_{AB} \left( \mathcal{L}_{w_\gamma} G^{-1} \right)^{AB} = -\tfrac{1}{2} \zeta_\gamma^{\alpha\beta} \gamma_{\alpha\beta} \quad . \tag{62}$$

Here $\sqrt{\gamma} := \sqrt{\det \gamma_{\alpha\beta}}$ is the volume element on the gauge orbits.

In II we shall argue that a natural requirement in Dirac quantization is the existence of a basis of gauge vectors which are divergence-free. Furthermore, for a gauge theory based on a compact semisimple Lie group the $f^{\gamma}{}_{\alpha\beta}$ are trace-free [18]. Hence we shall assume

$$\nabla \cdot w_\gamma = 0 \quad \text{and} \quad f^{\alpha}{}_{\alpha\gamma} = 0 \quad . \tag{63}$$

These conditions can always be realized in Yang-Mills theories [11], and they mean that $\sqrt{\gamma}$ is gauge invariant, and, as we shall see, that the Dirac quantization goes through.

It turns out, though, that the calculations involved in *interpreting* the Dirac factor ordering in II are considerably simplified if we make the stronger assumption that the metric $\gamma_{\alpha\beta}$, instead of just its determinant, is gauge invariant. We remark that we are already assuming $\zeta_\gamma^{ab} = 0$ in connection with the gauge invariance of the horizontal subspaces [see (44)], and that the trace $\zeta_\gamma^{\alpha\beta}\gamma_{\alpha\beta} = 0$ for $\nabla \cdot w_\gamma = 0$; the further assumption $\zeta_\gamma^{\alpha\beta} = 0$ (instead of just its trace) means that the $\phi_\gamma$ are Killing. It is not clear what is to be gained by not taking this last step. If we do, then

$$w_\gamma \gamma^{\alpha\beta} = 2 f^{(\alpha\beta)}{}_{\gamma} \quad . \tag{64}$$

Thus, to make the components $\gamma_{\alpha\beta}$ gauge invariant let us assume, unless otherwise noted, the conditions

$$\mathcal{L}_{w_\gamma} G = 0 \quad \text{and} \quad f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma} \quad . \tag{65}$$

The Killing condition restricts our freedom of choice of gauge vector basis to those which can be reached by linear transformations with constant coefficients, and we assume the $f^{\gamma}{}_{\alpha\beta}$ are structure constants, as is the case for a Lie group acting on $M$ with linearly independent $w_\gamma$. The antisymmetry condition makes $f_{\alpha\beta\gamma}$ antisymmetric in all pairs of indices. Of course these conditions imply (63), and are realized, for example, in the helix model.

The Ricci rotation coefficients are evaluated using (59):

$$\Gamma^{c}{}_{ab} = \Gamma^{c}{}_{ba} = \tfrac{1}{2} g^{cd} \{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \} \quad , \tag{66}$$

$$\Gamma^{c}{}_{a\beta} = \Gamma^{c}{}_{\beta a} = \tfrac{1}{2} \mathcal{F}_{\beta a}{}^{c} \quad , \tag{67}$$

$$\Gamma^{c}{}_{\alpha\beta} = \Gamma^{c}{}_{\beta\alpha} = -\tfrac{1}{2} g^{cd} \partial_d \gamma_{\alpha\beta} \quad , \tag{68}$$

$$\Gamma^{\gamma}{}_{ab} = -\Gamma^{\gamma}{}_{ba} = -\tfrac{1}{2} \mathcal{F}^{\gamma}{}_{ab} \quad , \tag{69}$$

$$\Gamma^{\gamma}{}_{a\beta} = \Gamma^{\gamma}{}_{\beta a} = \tfrac{1}{2} \gamma^{\gamma\alpha} \partial_a \gamma_{\alpha\beta} \quad , \tag{70}$$

$$\Gamma^{\gamma}{}_{\alpha\beta} = -\Gamma^{\gamma}{}_{\beta\alpha} = \tfrac{1}{2} f^{\gamma}{}_{\alpha\beta} \quad . \tag{71}$$

Here $\partial_a := \partial/\partial q^a$, and indices are raised and/or lowered using the appropriate metric $g_{ab}$ or $\gamma_{\alpha\beta}$. Now let us denote the Levi-Cività connection associated with $g_{ab}$ on $m$ as $\tilde{\nabla}$. We immediately recognize the corresponding Ricci rotation coefficient (with respect to $\partial_a$):

$$\tilde{\Gamma}^c{}_{ab} = \Gamma^c{}_{ab} \quad . \tag{72}$$

Finally we come to the third Levi-Cività connection, associated with the metric $\gamma_{\alpha\beta}$. The components of this metric refer to the nonholonomic basis $\phi^\alpha$, and its dual $\phi_\alpha = w_\alpha$ within a given orbit [recall (10)]. We restrict ourselves to a single orbit, labeled by $q$, and, where confusion might arise, indicate this explicitly with $|_q$. We also temporarily drop the restrictions (63) and (65).

Using (10) we find

$$\begin{aligned} d(\phi^\gamma\ |_q) &= \frac{\partial}{\partial F^\epsilon}(F^{-1})^\gamma_\delta\ |_q\ dF^\epsilon \wedge dF^\delta \\ &= F^\delta_{[\alpha} w_{\beta]}(F^{-1})^\gamma_\delta\ |_q\ \phi^\alpha \wedge \phi^\beta \quad , \end{aligned} \tag{73}$$

where $[\alpha\beta]$ denotes antisymmetrization of $\alpha\beta$. But

$$\begin{aligned} [w_\alpha, w_\beta] &= \left[ F^\delta_\alpha \frac{\partial}{\partial F^\delta}, F^\epsilon_\beta \frac{\partial}{\partial F^\epsilon} \right] \\ &= 2F^\delta_{[\alpha} w_{\beta]}(F^{-1})^\gamma_\delta w_\gamma = f^\gamma{}_{\alpha\beta} w_\gamma \quad , \end{aligned} \tag{74}$$

so

$$0 = d(\phi^\gamma\ |_q) + \tfrac{1}{2} f^\gamma{}_{\alpha\beta}\phi^\alpha \wedge \phi^\beta \quad . \tag{75}$$

This is essentially the Maurer-Cartan equation (see, e.g., [15]), and should be contrasted with (33), where $e^\gamma$ differs from $\phi^\gamma$ by the "Yang-Mills field" $A^\alpha$—see (9).

The Ricci rotation coefficients in an orbit, which we denote as $\tilde{\Gamma}^\gamma{}_{\alpha\beta}$, are calculated exactly as was done earlier for $\Gamma^{\hat{C}}{}_{\hat{A}\hat{B}}$:

$$\tilde{\Gamma}^\gamma{}_{\alpha\beta} - \tilde{\Gamma}^\gamma{}_{\beta\alpha} = \tilde{C}^\gamma{}_{\alpha\beta} \tag{76}$$

is the analogue of (56), with

$$[w_\alpha w_\beta] = \tilde{C}^\gamma{}_{\alpha\beta} w_\gamma \quad , \tilde{C}^\gamma{}_{\alpha\beta} = f^\gamma{}_{\alpha\beta} \quad , \tag{77}$$

corresponding to (55). Inspection of (59) reveals that $\tilde{\Gamma}^\gamma{}_{\alpha\beta}$ is precisely $\Gamma^\gamma{}_{\alpha\beta}$ [before any of the restrictions (63) or (65)]:

$$\begin{aligned} \tilde{\Gamma}^\gamma{}_{\alpha\beta} = \Gamma^\gamma{}_{\alpha\beta} = \tfrac{1}{2}\gamma^{\gamma\delta}\{ & w_\alpha\gamma_{\beta\delta} + w_\beta\gamma_{\alpha\delta} - w_\delta\gamma_{\alpha\beta} \\ & - f_{\alpha\beta\delta} - f_{\beta\alpha\delta} + f_{\delta\alpha\beta}\} \quad . \end{aligned} \tag{78}$$

We now discuss the Ricci tensors associated with these Levi-Cività connections.

## VI. RICCI CURVATURES AND THEIR INTERRELATIONSHIPS

For vector fields $U$ and $V$ on $M$ (defining an infinitesimal parallelogram), the curvature operator acting on a vector field $W$ is

$$\begin{aligned} R(U,V)W &:= \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W \\ &= U^{\hat{A}} V^{\hat{B}} W^{\hat{C}} \{ w_{\hat{A}} \Gamma^{\hat{D}}{}_{\hat{B}\hat{C}} - w_{\hat{B}} \Gamma^{\hat{D}}{}_{\hat{A}\hat{C}} + \Gamma^{\hat{D}}{}_{\hat{A}\hat{E}} \Gamma^{\hat{E}}{}_{\hat{B}\hat{C}} - \Gamma^{\hat{D}}{}_{\hat{B}\hat{E}} \Gamma^{\hat{E}}{}_{\hat{A}\hat{C}} - \Gamma^{\hat{D}}{}_{\hat{E}\hat{C}} C^{\hat{E}}{}_{\hat{A}\hat{B}} \} w_{\hat{D}} \\ &=: -U^{\hat{A}} V^{\hat{B}} W^{\hat{C}} R_{\hat{A}\hat{B}\hat{C}}{}^{\hat{D}} w_{\hat{D}} \quad . \end{aligned} \tag{79}$$

In particular, we shall be interested in the Ricci tensor

$$\begin{aligned} \mathcal{R}_{\hat{A}\hat{B}} &:= R_{\hat{A}\hat{D}\hat{B}}{}^{\hat{D}} \\ &= w_{\hat{D}} \Gamma^{\hat{D}}{}_{\hat{A}\hat{B}} - w_{\hat{A}} \Gamma^{\hat{D}}{}_{\hat{D}\hat{B}} + \Gamma^{\hat{D}}{}_{\hat{D}\hat{E}} \Gamma^{\hat{E}}{}_{\hat{A}\hat{B}} \\ &\quad - \Gamma^{\hat{D}}{}_{\hat{A}\hat{E}} \Gamma^{\hat{E}}{}_{\hat{D}\hat{B}} - \Gamma^{\hat{D}}{}_{\hat{E}\hat{B}} C^{\hat{E}}{}_{\hat{D}\hat{A}} \quad . \end{aligned} \tag{80}$$

Using (66)–(71) and (57) and (58) it can be shown that the $ab$ components are

$$\begin{aligned} \mathcal{R}_{ab} = \tilde{\mathcal{R}}_{ab} &+ \tfrac{1}{2}\mathcal{F}_{\gamma ca}\mathcal{F}^\gamma{}_b{}^c - \tilde{\nabla}_a \tilde{\nabla}_b \ln\sqrt{\gamma} \\ &+ \tfrac{1}{4}(\tilde{\nabla}_a \gamma^{\alpha\beta})(\tilde{\nabla}_b \gamma_{\alpha\beta}) \quad . \end{aligned} \tag{81}$$

Here $\tilde{\mathcal{R}}_{ab}$ is the Ricci tensor associated with the Levi-Cività connection $\tilde{\nabla}$ on $m$, which has a form analogous to (80), but with no $C$ term. We used the fact that $w_a$ reduces to $\partial_a$ when acting on a gauge invariant function [see (9)], as well as

$$w_\gamma \mathcal{F}^\gamma{}_{ab} = -f^\gamma{}_{\gamma\beta}\mathcal{F}^\beta{}_{ab} = 0 \tag{82}$$

by (32) and (63). Notice that if the full configuration space $M$ is (at least Ricci) flat then $\mathcal{R}_{ab} = 0$ and the Kaluza-Klein-like equation (81) yields an expression for $\tilde{\mathcal{R}}_{ab}$ in terms of the square of the Yang-Mills curvature $\mathcal{F}$, as well as other objects involving the gauge orbit structure on $M$.[7] For example, in the helix model (refer to the end of Sec. IV) a simple calculation reveals that all terms on the right hand side of (81) contribute to $\tilde{\mathcal{R}}_{ab}$, yielding

$$\tilde{\mathcal{R}}_{ab} = \tfrac{1}{2} g_{ab} \tilde{\mathcal{R}} \quad , \tag{83}$$

[7] I am indebted to G. Kunstatter for suggesting this method of computing the curvature on $m$ (see also [19]). The original approach was a Gauss-Codazzi-like analysis, relating the intrinsic and extrinsic curvature of a surface $\bar{m}$ (representing $m$) with the curvature of the embedding space $M$. However, the immediate obstacle is that, in order for $\bar{m}$ to inherit the correct metric from $M$, the vectors tangent to $\bar{m}$ must be orthogonal to the gauge orbits (with respect to the metric on $M$). But we know that the existence of a nontrivial Yang-Mills curvature $\mathcal{F}$ means that such a surface cannot be found. There is also the complication of multiple normal vectors. Although this approach may offer an interesting perspective, the Kaluza-Klein-like approach is more straightforward.

(which is of course true for any two dimensional space). Here the Ricci scalar

$$\tilde{\mathcal{R}} := g^{ab}\tilde{\mathcal{R}}_{ab} = 6(1+\rho^2)^{-2} \quad , \tag{84}$$

as noted in [3].

The $a\beta$ (or mixed) components of the Ricci tensor on $M$ are

$$2\mathcal{R}_{a\beta} = w_b \mathcal{F}_{\beta a}{}^{b} + \tilde{\Gamma}^{b}{}_{bc}\mathcal{F}_{\beta a}{}^{c} - \tilde{\Gamma}^{c}{}_{ba}\mathcal{F}_{\beta c}{}^{b} + (\tilde{\nabla}_b \ln\sqrt{\gamma})\mathcal{F}_{\beta a}{}^{b} \quad , \tag{85}$$

where we made use of (65). As before, we suppose the left hand side vanishes; the right hand side has the form of a divergence of the Yang-Mills curvature $\mathcal{F}$: a source equation, but whose physical interpretation is not clear.

Finally, the $\alpha\beta$ components are

$$\mathcal{R}_{\alpha\beta} = -\tfrac{1}{4}f^{\delta}{}_{\gamma\alpha}f^{\gamma}{}_{\delta\beta} - \tfrac{1}{4}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} - \tfrac{1}{2}\tilde{\Delta}\gamma_{\alpha\beta} + \tfrac{1}{2}\gamma^{\gamma\delta}(\tilde{\nabla}_a\gamma_{\gamma\alpha})(\tilde{\nabla}^a\gamma_{\delta\beta}) - \tfrac{1}{2}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}^a\gamma_{\alpha\beta}) \quad , \tag{86}$$

where $\tilde{\Delta} := \tilde{\nabla}_a\tilde{\nabla}^a$ acting on scalars. With $\mathcal{R}_{\alpha\beta} = 0$ this is another condition on $\mathcal{F}^2$. Notice that the Ricci tensor $\tilde{\mathcal{R}}_{ab}$ of the (dimensionally) reduced space $m$ appears on the right hand side of (81), in standard Kaluza-Klein fashion. Since $M$ is locally a product of $m$ and the gauge orbit, one would expect an analogous result: the Ricci tensor of the gauge orbit should appear on the right hand side of (86). This is indeed true. It can be shown that in general the Ricci tensor

$$\tilde{\mathcal{R}}_{\alpha\beta} = \phi_\delta\tilde{\Gamma}^{\delta}{}_{\alpha\beta} - \phi_\alpha\tilde{\Gamma}^{\delta}{}_{\delta\beta} + \tilde{\Gamma}^{\delta}{}_{\delta\epsilon}\tilde{\Gamma}^{\epsilon}{}_{\alpha\beta} - \tilde{\Gamma}^{\delta}{}_{\alpha\epsilon}\tilde{\Gamma}^{\epsilon}{}_{\delta\beta} - \tilde{\Gamma}^{\delta}{}_{\epsilon\beta}\tilde{C}^{\epsilon}{}_{\delta\alpha} \tag{87}$$

[cf. (80)] corresponding to (78) appears as stated. Now when we reinstate the conditions (65), $\tilde{\Gamma}^{\gamma}{}_{\alpha\beta}$ reduces to $\Gamma^{\gamma}{}_{\alpha\beta}$ in (71), and $\tilde{\mathcal{R}}_{\alpha\beta}$ reduces to

$$\begin{aligned}\tilde{\mathcal{R}}_{\alpha\beta} &= \tfrac{1}{2}\phi_\delta f^{\delta}{}_{\alpha\beta} - \tfrac{1}{2}\phi_\alpha f^{\delta}{}_{\delta\beta} + \tfrac{1}{4}f^{\delta}{}_{\delta\epsilon}f^{\epsilon}{}_{\alpha\beta} - \tfrac{1}{4}f^{\delta}{}_{\alpha\epsilon}f^{\epsilon}{}_{\delta\beta} - \tfrac{1}{2}f^{\delta}{}_{\epsilon\beta}f^{\epsilon}{}_{\delta\alpha} \\ &= -\tfrac{1}{4}f^{\delta}{}_{\gamma\alpha}f^{\gamma}{}_{\delta\beta} \quad ,\end{aligned} \tag{88}$$

the first term on the right hand side of (86). Here we used the fact that, quite generally,

$$w_\gamma f^{\gamma}{}_{\alpha\beta} = w_\alpha f^{\gamma}{}_{\gamma\beta} - w_\beta f^{\gamma}{}_{\gamma\alpha} - f^{\epsilon}{}_{\alpha\beta}f^{\gamma}{}_{\gamma\epsilon} \quad , \tag{89}$$

which follows from taking the Lie derivative of (1) with respect to $w_\delta$, and then contracting $\delta$ and $\gamma$. The right hand side vanishes when $f^{\gamma}{}_{\gamma\beta} = 0$. The Ricci scalar within an orbit is

$$\gamma^{\alpha\beta}\tilde{\mathcal{R}}_{\alpha\beta} = \tfrac{1}{4}f_{\gamma\alpha\beta}f^{\gamma\alpha\beta} \quad . \tag{90}$$

With $f^{\gamma}{}_{\alpha\beta}$ structure *constants*, and $\gamma_{\alpha\beta}$ gauge invariant, this Ricci scalar is constant, but may change from orbit to orbit.

This allows us to rewrite (86) as

$$\mathcal{R}_{\alpha\beta} = \tilde{\mathcal{R}}_{\alpha\beta} - \tfrac{1}{4}\mathcal{F}_{\alpha a}{}^{b}\mathcal{F}_{\beta b}{}^{a} - \tfrac{1}{2}\tilde{\Delta}\gamma_{\alpha\beta} + \tfrac{1}{2}\gamma^{\gamma\delta}(\tilde{\nabla}_a\gamma_{\gamma\alpha})(\tilde{\nabla}^a\gamma_{\delta\beta}) - \tfrac{1}{2}(\tilde{\nabla}_a \ln\sqrt{\gamma})(\tilde{\nabla}^a\gamma_{\alpha\beta}) \quad . \tag{91}$$

The term by term analogy of this expression with (81) is very interesting; a similar duality will show up when we use these results to provide a geometrical interpretation of Dirac quantization in II. Furthermore, notice that all terms in (81) and (91) are naturally occurring curvatures, except those involving $\gamma_{\alpha\beta}$. Roughly speaking, the latter consist of horizontal derivatives of the metric in the vertical space, or contracted products of such terms: it seems quite likely that they could be understood in terms of the extrinsic curvature of the orbits embedded in $M$ (see also footnote 7).

In conclusion, we have provided an extensive geometrical analysis of the gauge and metric structures present in gauge theories, and how they naturally give rise to a connection one-form and other Levi-Cività connections. There are interesting Kaluza-Klein-like relationships amongst the various curvatures which, of importance in their own right, also provide the classical background necessary to understand how Dirac quantization (acting in the physical Hilbert space) can be understood as a curved-space quantization scheme, as discussed in II.

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[1] A. Ashtekar and G. T. Horowitz, Phys. Rev. D **26**, 3342 (1982).

[2] K. Kuchař, Phys. Rev. D **34**, 3031 (1986); **34**, 3044 (1986).

[3] K. Kuchař, Phys. Rev. D **35**, 596 (1987).

[4] J. Romano and R. Tate, Class. Quantum Grav. **6**, 1487 (1989).

[5] K. Schleich, Class. Quantum Grav. **7**, 1529 (1990).

[6] G. Kunstatter, Class. Quantum Grav. **9**, 1469 (1992).

[7] K. Sundermeyer, *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).

[8] A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).

[9] P. A. M. Dirac, Can. J. Math. **2**, 125 (1956); *Lectures in Quantum Mechanics* (Academic, New York, 1956).

[10] R. J. Epp, following paper, Phys. Rev. D **50**, 6578 (1994).

[11] B. S. DeWitt, in *General Relativity, an Einstein Cen-*

*tenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University, Cambridge, England, 1979).

[12] R. M. Wald, *General Relativity* (University of Chicago, Chicago, 1984).

[13] I. A. Batalin and G. A. Vilkovisky, Nucl. Phys. **B234**, 106 (1984).

[14] D. McMullan and J. Paterson, J. Math. Phys. **30**, 477 (1989).

[15] C. J. Isham, *Modern Differential Geometry for Physicists*, World Scientific Lecture Notes in Physics Vol. 32 (World Scientific, Singapore, 1989).

[16] R. J. Epp and G. Kunstatter, Report No. WIN-94-03, 1994 (unpublished).

[17] C. R. Ordóñez and J. M. Pons, Phys. Rev. D **45**, 3706 (1992).

[18] G. Kunstatter, in *Gravitation 1990*, Proceedings of the Banff Summer Institute, Banff, Alberta, 1990, edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991).

[19] J. D. Gegenberg and G. Kunstatter, Class. Quantum Grav. **3**, 379 (1986).