Scalar Casimir effect for a D-dimensional sphere

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The Casimir stress on a D-dimensional sphere (the stress on a sphere is equal to the Casimir force per unit area multiplied by the area of the sphere) due to the confinement of a massless scalar field is computed as a function of D, where D is a continuous variable that ranges from $-\infty$ to ∞. The dependence of the stress on the dimension is obtained using a simple and straightforward Green's function technique. We find that the Casimir stress vanishes as $D \to +\infty$ (D is a noneven integer) and also vanishes when D is a negative even integer. The stress has simple poles at positive even integer values of D.

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I. INTRODUCTION

In recent papers [1–4] it was proposed that the dimension of space-time could be used as a perturbation parameter in quantum field theory calculations. The advantage of such an approach is that analytical (nonnumerical) results can be obtained which are nonperturbative in the coupling constant. This procedure was used to obtain the Green's functions of self-interacting scalar quantum field theory in the Ising limit [5,6]. One can also perform dimensional expansions in inverse powers of the dimension. Such expansions have proved useful in atomic physics calculations [7].

These perturbative investigations have led to and motivated analyses of the dimensional dependence of various physical systems. Such investigations are useful because by identifying the singularities in the complex-dimension plane one can predict the radius of convergence of a dimensional expansion. The dimensional dependence of some elementary quantum-mechanical and field-theoretic models is described in [1–3]. The dimensional dependence of classical physical models has also been investigated; for example, in [8,9] the dimensional dependence of probabilities in models of random walks was elucidated.

In this paper we investigate the dimensional dependence of the Casimir force per unit area, F/A, on a spherical shell of radius a in D space dimensions. Specifically, we study the Casimir stress (the stress on the sphere is equal to the Casimir force per unit area multiplied by the area of the sphere) that is due to quantum fluctuations of a free massless scalar field satisfying Dirichlet boundary conditions on the shell.

An interesting investigation of the dependence of the Casimir force per unit area upon the spatial dimension is already in the literature [10]. Ambjørn and Wolfram examined the case of infinite parallel plates embedded in a D-dimensional space and separated by a distance $2a$; that is, there is one longitudinal dimension and $D-1$ transverse dimensions. Their result is

$$F/A = -a^{-D-1}2^{-2D-2}\pi^{-(D+1)/2}D\,\Gamma\left(\frac{D+1}{2}\right)\times\zeta(D+1), \quad (1.1)$$

which we have plotted in Fig. 1. Note that F/A has a simple pole (due to the Γ function) at $D=-1$. However, F/A is not infinite at the other poles of the Γ function, which are located at all the negative odd integral values of D because the Riemann ζ function vanishes at all negative even values of its argument. One interesting and

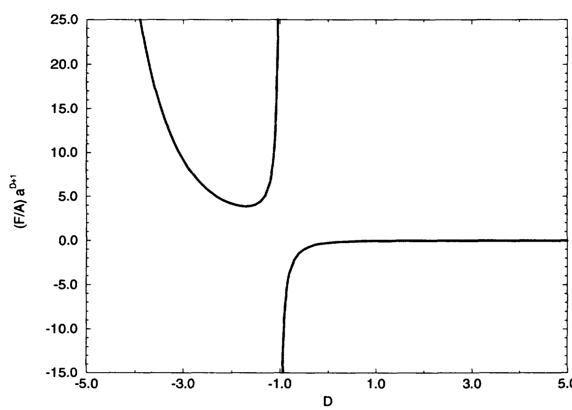

FIG. 1. A plot of the Casimir force per unit area F/A in (1.1) for $-5 < D < 5$ for the case of a slab geometry (two parallel plates).

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well-known special case of (1.1) is $D = 1$:

$$F/A|_{D=1} = -\frac{\pi}{96a^2}, \qquad (1.2)$$

where the negative sign indicates that the force is attractive. We mention this case here because the spherical geometry considered in the present paper coincides with the slab geometry of [10] when $D = 1$; we recover the result (1.2) as a special case in Sec. III.

This paper is organized very simply. In Sec. II we review the Green's function formalism required to obtain the Casimir stress. Then we apply this formalism in D-dimensional space to obtain an expression for the Casimir force per unit area on a D-dimensional spherical shell. This expression takes the form of an infinite sum of integrals over modified Bessel functions; the dimension D appears in the orders of the Bessel functions. In Sec. III we examine this expression for the Casimir force per unit area in detail. We show that each term in the series exists (each of the integrals converges) and we show how to evaluate the sum of the series numerically for all real D. When $D > 0$ the Casimir force per unit area is real; the stress is finite except when D is an even integer. When $D \leq 0$ the Casimir stress is complex; there are logarithmic singularities in the complex-D plane at $D = 0, -2, -4, -6, \ldots$.

II. MATHEMATICAL FORMALISM

The calculation in this paper of the Casimir stress on a spherical shell relies on the use of Green's functions to represent vacuum expectation values of time-ordered products of fields. The Green's function is used to obtain the vacuum expectation value of the stress-energy tensor, from which we will derive the Casimir stress. The formalism used here was developed in [11–13]. We summarize the formalism below.

A free massless scalar field $\varphi(\mathbf{x}, t)$ satisfies the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \varphi(\mathbf{x}, t) = 0, \qquad (2.1)$$

where $\mathbf{x}$ is a D-dimensional position vector. (Initially, we will think of D as a positive integer; however, once we have derived the radial equation for the Green's function we will be able to regard the parameter D as a continuous variable.) The quantum nature of the Casimir stress arises from the constraint that $\varphi(\mathbf{x}, t)$ satisfies equal-time commutation relations:

$$[\varphi(\mathbf{x}, t), \dot{\varphi}(\mathbf{x}', t)] = i\delta^{(D)}(\mathbf{x} - \mathbf{x}'). \qquad (2.2)$$

The two-point Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ is defined as the vacuum expectation value of the time-ordered product of two fields:

$$G(\mathbf{x}, t; \mathbf{x}', t') \equiv -i\langle 0|T\varphi(\mathbf{x}, t)\varphi(\mathbf{x}', t')|0\rangle. \qquad (2.3)$$

By virtue of (2.1) and (2.2), the Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ satisfies the inhomogeneous Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x}', t') = -\delta^{(D)}(\mathbf{x} - \mathbf{x}')\delta(t - t'). \qquad (2.4)$$

We will solve the above Green's function equation by dividing space into two regions: I the interior of a sphere of radius a and II the exterior of the sphere. On the sphere we will impose Dirichlet boundary conditions

$$G(\mathbf{x}, t; \mathbf{x}', t') \Big|_{|\mathbf{x}|=a} = 0. \qquad (2.5)$$

In addition, in region I we will require that G be finite at the origin $\mathbf{x} = 0$ and in region II we will require that G satisfy outgoing-wave boundary conditions at $|\mathbf{x}| = \infty$.

The stress-energy tensor $T^{\mu\nu}(\mathbf{x}, t)$ is defined as [14]

$$T^{\mu\nu}(\mathbf{x}, t) \equiv \partial^\mu\varphi(\mathbf{x}, t)\partial^\nu\varphi(\mathbf{x}, t) - \frac{1}{2}g^{\mu\nu}\partial_\lambda\varphi(\mathbf{x}, t)\partial^\lambda\varphi(\mathbf{x}, t). \qquad (2.6)$$

The radial Casimir force per unit area F/A on the sphere is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor [11]:

$$F/A = \langle 0|T^{rr}_{\rm in} - T^{rr}_{\rm out}|0\rangle \Big|_{r=a}. \qquad (2.7)$$

To calculate F/A we exploit the connection between the vacuum expectation value of the stress-energy tensor $T^{\mu\nu}(\mathbf{x}, t)$ and the Green's function at equal times $G(\mathbf{x}, t; \mathbf{x}', t)$:

$$F/A = \frac{i}{2}\left[\frac{\partial}{\partial r}\frac{\partial}{\partial r'}G(\mathbf{x}, t; \mathbf{x}', t)_{\rm in} - \frac{\partial}{\partial r}\frac{\partial}{\partial r'}G(\mathbf{x}, t; \mathbf{x}', t)_{\rm out}\right]\Bigg|_{\mathbf{x}=\mathbf{x}',\ |\mathbf{x}|=a}. \qquad (2.8)$$

To evaluate the expression in (2.8) it is necessary to solve the Green's function equation (2.4). We begin by taking the time Fourier transform of G:

$$\mathcal{G}_\omega(\mathbf{x}; \mathbf{x}') = \int_{-\infty}^{\infty} dt\, e^{-i\omega(t-t')}G(\mathbf{x}, t; \mathbf{x}', t'). \qquad (2.9)$$

The differential equation satisfied by $\mathcal{G}_\omega$ is

$$\left(\omega^2 + \nabla^2\right) \mathcal{G}_\omega(\mathbf{x}; \mathbf{x}') = \delta^{(D)}(\mathbf{x} - \mathbf{x}'). \qquad (2.10)$$

To solve this equation we introduce polar coordinates and seek a solution that has cylindrical symmetry; i.e., we seek a solution that is a function only of the two variables $r = |\mathbf{x}|$ and θ, the angle between $\mathbf{x}$ and $\mathbf{x}'$ so that $\mathbf{x} \cdot \mathbf{x}' = rr' \cos\theta$. In terms of these polar variables (2.10) becomes

$$\left(\omega^2+\frac{\partial^2}{\partial r^2}+\frac{D-1}{r}\frac{\partial}{\partial r}+\frac{\sin^{2-D}\theta}{r^2}\frac{\partial}{\partial\theta}\sin^{D-2}\theta\frac{\partial}{\partial\theta}\right)\mathcal{G}_\omega(r,r',\theta)=\frac{\delta(r-r')\delta(\theta)\Gamma\left(\frac{D-1}{2}\right)}{2\pi^{(D-1)/2}r^{D-1}\sin^{D-2}\theta}. \quad (2.11)$$

Note that the D-dimensional δ function on the right side of (2.10) has been replaced by a cylindrically symmetric δ function having the property that its volume integral in D dimensional space is unity. The D-dimensional volume integral of a cylindrically symmetric function $f(r,\theta)$ is

$$\frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty dr\, r^{D-1}\int_0^\pi d\theta\,\sin^{D-2}\theta f(r,\theta). \quad (2.12)$$

We solve (2.11) using the method of separation of variables. Let

$$\mathcal{G}_\omega(r,r',\theta)=A(r)B(z), \quad (2.13)$$

where $z=\cos\theta$. The equation satisfied by $B(z)$ is then

$$\left[(1-z^2)\frac{d^2}{dz^2}-z(D-1)\frac{d}{dz}+n(n+D-2)\right]B(z)=0, \quad (2.14)$$

where we have anticipated a convenient form for the separation constant. The equation satisfied by $A(r)$ is

$$\left[\frac{d^2}{dr^2}+\frac{D-1}{r}\frac{d}{dr}-\frac{n(n+D-2)}{r^2}+\omega^2\right]A(r)=0 \quad (r\neq r'). \quad (2.15)$$

The solution to (2.14) that is regular at $|z|=1$ is the ultraspherical (Gegenbauer) polynomial [15]

$$B(z)=C_n^{(-1+D/2)}(z) \quad (n=0,\,1,\,2,\,3,\,\ldots). \quad (2.16)$$

The solution in region I to (2.15) that is regular at $r=0$ involves the Bessel function [16]

$$A(r)=r^{1-D/2}J_{n-1+\frac{D}{2}}(|\omega|r). \quad (2.17)$$

In (2.17) we assume that $D\geq 2$ in order to eliminate the linearly independent solution $A(r)=r^{1-D/2}Y_{n-1+\frac{D}{2}}(|\omega|r)$, which is singular at $r=0$ for all n. The solution in region II to (2.15) that corresponds to an outgoing wave at $r=\infty$ involves a Hankel function of the first kind [16]:

$$A(r)=r^{1-D/2}H^{(1)}_{n-1+\frac{D}{2}}(|\omega|r). \quad (2.18)$$

The general solution to (2.11) is an arbitrary linear combination of separated-variable solutions; in region I the Green's function has the form

$$\mathcal{G}_\omega(r,r',\theta)=\sum_{n=0}^\infty a_n r^{1-D/2}J_{n-1+\frac{D}{2}}(|\omega|r)C_n^{(-1+D/2)}(z) \quad (r<r'<a) \quad (2.19a)$$

and

$$\mathcal{G}_\omega(r,r',\theta)=\sum_{n=0}^\infty r^{1-D/2}\left[b_nJ_{n-1+\frac{D}{2}}(|\omega|r)+c_nJ_{-n+1-\frac{D}{2}}(|\omega|r)\right]C_n^{(-1+D/2)}(z) \quad (r'<r<a). \quad (2.19b)$$

[Note that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent so long as ν is not an integer. Thus, (2.19b) assumes explicitly that D is not an even integer.] The general solution to (2.11) in region II has the form

$$\mathcal{G}_\omega(r,r',\theta)=\sum_{n=0}^\infty d_n r^{1-D/2}H^{(1)}_{n-1+\frac{D}{2}}(|\omega|r)C_n^{(-1+D/2)}(z) \quad (r>r'>a) \quad (2.20a)$$

and

$$\mathcal{G}_\omega(r,r',\theta)=\sum_{n=0}^\infty r^{1-D/2}\left[e_nH^{(1)}_{n-1+\frac{D}{2}}(|\omega|r)+f_nH^{(2)}_{n-1+\frac{D}{2}}(|\omega|r)\right]C_n^{(-1+D/2)}(z) \quad (r'>r>a). \quad (2.20b)$$

The arbitrary coefficients a_n, b_n, c_n, d_n, e_n, and f_n are uniquely determined by six conditions: namely, the Dirichlet boundary condition (2.5) at $r=a$,

$$b_nJ_{n-1+\frac{D}{2}}(|\omega|a)+c_nJ_{-n+1-\frac{D}{2}}(|\omega|a)=0 \quad (2.21a)$$

and

$$e_nH^{(1)}_{n-1+\frac{D}{2}}(|\omega|a)+f_nH^{(2)}_{n-1+\frac{D}{2}}(|\omega|a)=0, \quad (2.21b)$$

the condition of continuity at $r=r'$,

$$a_n J_{n-1+\frac{D}{2}}(|\omega|r') = b_n J_{n-1+\frac{D}{2}}(|\omega|r') + c_n J_{-n+1-\frac{D}{2}}(|\omega|r') \qquad (2.21c)$$

and

$$d_n H^{(1)}_{n-1+\frac{D}{2}}(|\omega|r') = e_n H^{(1)}_{n-1+\frac{D}{2}}(|\omega|r') + f_n H^{(2)}_{n-1+\frac{D}{2}}(|\omega|r'), \qquad (2.21d)$$

and the jump condition in the first derivative of the Green's function at $r = r'$,

$$b_n J'_{n-1+\frac{D}{2}}(|\omega|r') + c_n J'_{-n+1-\frac{D}{2}}(|\omega|r') - a_n J'_{n-1+\frac{D}{2}}(|\omega|r') = \frac{(2n+D-2)\Gamma\left(\frac{D-2}{2}\right)}{4(\pi r')^{\frac{D}{2}}|\omega|} \qquad (2.21e)$$

and

$$e_n H^{(1)\prime}_{n-1+\frac{D}{2}}(|\omega|r') + f_n H^{(2)\prime}_{n-1+\frac{D}{2}}(|\omega|r') - d_n H^{(1)\prime}_{n-1+\frac{D}{2}}(|\omega|r') = \frac{(2n+D-2)\Gamma\left(\frac{D-2}{2}\right)}{4(\pi r')^{\frac{D}{2}}|\omega|}. \qquad (2.21f)$$

Here we have used the orthogonality property of the ultraspherical polynomials [15],

$$\int_{-1}^{1} dz\,(1-z^2)^{\alpha-1/2} C_n^{(\alpha)}(z) C_m^{(\alpha)}(z) = \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!\,(n+\alpha)\Gamma^2(\alpha)}\delta_{nm} \quad (\alpha \neq 0), \qquad (2.22)$$

the value of the ultraspherical polynomials at $z = 1$,

$$C_n^{(\alpha)}(1) = \frac{\Gamma(n+2\alpha)}{n!\,\Gamma(2\alpha)} \quad (\alpha \neq 0), \qquad (2.23)$$

and the duplication formula $\Gamma(2x) = 2^{2x-1}\Gamma(x) \times\Gamma(x+1/2)/\sqrt{\pi}$.

Having determined the coefficients in the expressions for the Green's function, we can immediately evaluate the right side of (2.8). The contribution to F/A from the interior region (region I) is

$$(F/A)_{\rm in} = i\sum_{n=0}^{\infty} \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^D \pi^{\frac{D+1}{2}} a^D n!\,\Gamma\left(\frac{D-1}{2}\right)} \times \int_{-\infty}^{\infty} d\omega \left[\frac{|\omega|aJ'_{n-1+\frac{D}{2}}(|\omega|a)}{J_{n-1+\frac{D}{2}}(|\omega|a)} + 1 - \frac{D}{2}\right]. \qquad (2.24a)$$

The contribution to F/A from the exterior region (region II) is

$$(F/A)_{\rm out} = i\sum_{n=0}^{\infty} \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^D \pi^{\frac{D+1}{2}} a^D n!\,\Gamma\left(\frac{D-1}{2}\right)} \times \int_{-\infty}^{\infty} d\omega \left[\frac{|\omega|aH^{(1)\prime}_{n-1+\frac{D}{2}}(|\omega|a)}{H^{(1)}_{n-1+\frac{D}{2}}(|\omega|a)} + 1 - \frac{D}{2}\right]. \qquad (2.24b)$$

The integrals in (2.24) are oscillatory and therefore very difficult to evaluate numerically. Thus, it is advantageous to perform a rotation of 90° in the complex-ω plane. The resulting final expression for F/A is

$$F/A = -\sum_{n=0}^{\infty} \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^{D-1}\pi^{\frac{D+1}{2}} a^{D+1} n!\,\Gamma\left(\frac{D-1}{2}\right)} \int_0^{\infty} dx \left[\frac{xI'_{n-1+\frac{D}{2}}(x)}{I_{n-1+\frac{D}{2}}(x)} + \frac{xK'_{n-1+\frac{D}{2}}(x)}{K_{n-1+\frac{D}{2}}(x)} + 2 - D\right]. \qquad (2.25)$$

The $D = 2$ result, where the $n = 0$ term appears with weight $\frac{1}{2}$, was derived in [13]. This result can be recovered by setting $D = 2 + \epsilon$ and letting ϵ tend to 0.

III. NUMERICAL EVALUATION OF F/A

Our objective now is to evaluate the formal expression in (2.25) for arbitrary dimension D. Recall that this expression was derived under the assumption that $D > 2$ and that D is not an even integer. However, we will now seek an interpretation of (2.25) that is generally valid; to do so we will apply a summation procedure that enables us to continue (2.25) to *all* values of D.

The expression in (2.25) does not exist *a priori* for all D. Furthermore, as we will see, the individual terms in the series, which are integrals in x, do not exist. Fortunately, it is possible to modify the terms in the series so that the integrals do exist; this modification requires a delicate and detailed argument. However, there is one simple case, namely that for which $D = 1$, where the series (2.25) is well defined and easy to evaluate. We examine this case in the next section.

A. Special case $D = 1$

When $D = 1$ the series in (2.25) truncates after two terms. This happens because of the identity

$$\lim_{D\to 1} \frac{\Gamma(n+D-2)}{\Gamma\left(\frac{D-1}{2}\right)} = -\frac{1}{2}\delta_{n0} + \frac{1}{2}\delta_{n1}. \tag{3.1}$$

When this identity is inserted into the sum in (2.25) we obtain

$$F/A|_{D=1} = -\frac{1}{4\pi a^2}\int_0^\infty dx \left[\frac{xI'_{\frac{1}{2}}(x)}{I_{\frac{1}{2}}(x)} + \frac{xI'_{-\frac{1}{2}}(x)}{I_{-\frac{1}{2}}(x)} + \frac{xK'_{\frac{1}{2}}(x)}{K_{\frac{1}{2}}(x)} + \frac{xK'_{-\frac{1}{2}}(x)}{K_{-\frac{1}{2}}(x)} + 2\right]. \tag{3.2}$$

Next we use the identities

$$\sqrt{\frac{\pi}{2}}I_{\frac{1}{2}}(x) = \frac{\sinh x}{\sqrt{x}}, \quad \sqrt{\frac{\pi}{2}}I_{-\frac{1}{2}}(x) = \frac{\cosh x}{\sqrt{x}},$$

$$\sqrt{\frac{2}{\pi}}K_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}K_{-\frac{1}{2}}(x) = \frac{e^{-x}}{\sqrt{x}}, \tag{3.3}$$

to reduce (3.2) to the (convergent) integral

$$\begin{aligned} F/A|_{D=1} &= -\frac{1}{4\pi a^2}\int_0^\infty dx \left[x\frac{d}{dx}\ln\left(\frac{\cosh x \sinh x}{x^2 e^{2x}}\right) + 2\right] \\ &= -\frac{1}{4\pi a^2}\int_0^\infty dx\, x\frac{d}{dx}\ln\left(1-e^{-4x}\right) \\ &= -\frac{1}{\pi a^2}\int_0^\infty dx \frac{x}{e^{4x}-1} \\ &= -\frac{\pi}{96a^2}. \end{aligned} \tag{3.4}$$

This result agrees with the well-known result given in (1.2).

B. Convergent reformulation of (2.25)

In this section we modify the form of the series (2.25) so that each term in the series exists and we apply a summation procedure to evaluate the resulting series numerically. We begin by rewriting (2.25) in the form

$$F/A = -\sum_{n=0}^\infty \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^{D-1}\pi^{\frac{D+1}{2}}a^{D+1}n!\,\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty dx\left[x\frac{d}{dx}\ln[I_{n-1+\frac{D}{2}}(x)K_{n-1+\frac{D}{2}}(x)] + 2 - D\right]. \tag{3.5}$$

In this form it is easy to investigate the convergence of the individual integrals in the series. To do so we recall the asymptotic behavior as $x \to \infty$:

$$I_\nu(x)K_\nu(x) \sim \frac{1}{2x} \quad (x \to +\infty). \tag{3.6}$$

From (3.6) it is clear that the integrals in (3.5) do not converge except for the special case $D = 1$. However, as we will now argue, one can replace the quantity $2 - D$ in (3.5) by 1 without changing the value of F/A, provided that $D < 1$. This replacement will render the integrals convergent.

Consider the series

$$\sum_{n=0}^\infty \frac{\Gamma(n+\alpha)}{n!}. \tag{3.7}$$

This series converges so long as $\alpha < 0$ and $\alpha \neq -N$, where $N = 1, 2, 3, \ldots$ (so that individual terms in the series exist). This series can be summed in closed form because it is a special case of the binomial expansion

$$\sum_{n=0}^\infty x^n \frac{\Gamma(n+\alpha)}{n!} = \Gamma(\alpha)(1-x)^{-\alpha}. \tag{3.8}$$

Note that if we let $x \to 1$ we obtain the identity

$$\sum_{n=0}^\infty \frac{\Gamma(n+\alpha)}{n!} \equiv 0. \tag{3.9}$$

One can also show that the identity (3.9) holds in the limit as α approaches a negative integer $-N$. To do so we let $\alpha = -N + \epsilon$, where $N = 1, 2, 3, \ldots$. We then decompose the series into two parts, the first whose terms are finite and the second whose terms diverge as $\epsilon \to 0$:

$$\sum_{n=0}^\infty \frac{n!}{(n+N+1)!} + \frac{1}{\epsilon}\sum_{n=0}^N \frac{(-1)^n}{(N-n)!\,\Gamma(1+n-\epsilon)}. \tag{3.10}$$

The first sum in (3.10) can be easily evaluated as

$$\frac{1}{N\,N!}. \tag{3.11}$$

If the second sum in (3.10) is expanded in a Laurent series in ϵ, the coefficient of $1/\epsilon$ vanishes. However, the coefficient of ϵ^0 is given by

$$\frac{1}{N!}\sum_{n=0}^N (-1)^n \binom{N}{n} \psi(n+1). \tag{3.12}$$

Finally, using the integral representation

$$\psi(n+1) = -\gamma + \int_0^1 dt \frac{1-t^n}{1-t}, \quad (3.13)$$

we show that (3.12) cancels (3.11).

This argument shows that if the integral in (3.5) is replaced by 1 then the sum vanishes:

$$\sum_{n=0}^{\infty} \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{n!} \equiv 0 \quad (D<1). \quad (3.14)$$

It follows that in the region $D < 1$ (if we sum first over n) we can add any constant to the integrand in each term in the series (3.5) without changing the value of the sum. We conclude that we may replace $2 - D$ by 1 in (3.5). Our new improved expression for the Casimir force per unit area is thus

$$F/A = \sum_{n=0}^{\infty} \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^{D-1}\pi^{\frac{D+1}{2}} a^{D+1} n!\,\Gamma\left(\frac{D-1}{2}\right)} \times \int_0^{\infty} dx\, \ln[2x I_{n-1+\frac{D}{2}}(x) K_{n-1+\frac{D}{2}}(x)]. \quad (3.15)$$

For $D \geq 0$ each term in this series exists for all n. (For $D < 0$ there is yet another subtlety that we will address shortly.) Before we proceed, we must emphasize that while (3.15) has a different and more compact form than that in (2.25) we have not changed the value of F/A; we have in effect added zero to the series representing F/A.

Unfortunately, the formula in (3.15) is still not satisfactory because the series does not converge. To examine the convergence of this series we need to know the asymptotic behavior of the integrals for large n. We make use of the uniform asymptotic approximation to the product $I_\nu(\nu x) K_\nu(\nu x)$:

$$I_\nu(\nu x) K_\nu(\nu x) \sim \frac{t}{2\nu}\left(1 + \frac{t^2 - 6t^4 + 5t^6}{8\nu^2} + \ldots\right) \quad (\nu \to \infty), \quad (3.16)$$

where $t = (1+x^2)^{-1/2}$. This asymptotic behavior implies that the integral, which we will abbreviate by Q_n, in the nth term in (3.15) grows linearly with increasing n:

$$Q_n \equiv -\int_0^{\infty} dx\, \ln[2x I_\nu(x) K_\nu(x)] \sim \pi\left(\frac{\nu}{2} + \frac{1}{128\nu} - \frac{35}{32768\nu^3} + \cdots\right) \quad (n \to \infty), \quad (3.17)$$

where $\nu = n - 1 + \frac{D}{2}$. Because of this linear growth in n it is apparent that the series in (3.15) *does not converge* if $D > 0$ except for the special case $D = 1$, where the series truncates.

To solve this problem we introduce an analytic summation procedure based on the properties of the Riemann ζ function. Specifically, we consider the leading large-n behavior of the summand in (3.15):

$$-\frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^{D-1}\pi^{\frac{D+1}{2}} a^{D+1} n!\,\Gamma\left(\frac{D-1}{2}\right)} Q_n \sim -\frac{1}{2^D \pi^{\frac{D-1}{2}} a^{D+1} \Gamma\left(\frac{D-1}{2}\right)} (n^{D-1} + \cdots) \quad (n \to \infty). \quad (3.18)$$

Then, regarding the parameter D as being less than 0, we sum the expression on the right side of (3.18) over n from 1 to ∞. This gives

$$-\frac{1}{2^D \pi^{\frac{D-1}{2}} a^{D+1} \Gamma\left(\frac{D-1}{2}\right)} \zeta(1-D), \quad (3.19)$$

which is a well-defined function of D. We now add (3.19) to (3.15) and correspondingly subtract the right side of (3.18) from each term (except the $n = 0$ term) in the expression for F/A in (3.15). This produces a new series for F/A that is convergent for $D < 1$:

$$F/A = \frac{1}{a^{D+1}\pi^{\frac{D-1}{2}}} \left\{ \frac{\Gamma(\frac{D}{2})}{4\pi^{\frac{3}{2}}} \int_0^{\infty} dx\, \ln[2x I_{-1+\frac{D}{2}}(x) K_{-1+\frac{D}{2}}(x)] + \frac{1}{2^{D-1}\pi\Gamma\left(\frac{D-1}{2}\right)} \sum_{n=1}^{\infty} \left[\frac{\pi}{2} n^{D-1} + \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{n!} \times \int_0^{\infty} dx\, \ln[2x I_{n-1+\frac{D}{2}}(x) K_{n-1+\frac{D}{2}}(x)] \right] - \frac{1}{2^D \Gamma\left(\frac{D-1}{2}\right)} \zeta(1-D) \right\}. \quad (3.20)$$

We have finally achieved our objective; we have obtained a convergent series representation for F/A for a finite range of positive D, namely, $0 < D < 1$. Each term in this series exists. In the next section we will use this series to calculate analytically the Casimir stress on a sphere in zero dimensions.

C. Casimir stress on a zero-dimensional sphere

If we substitute $D = 0$ in (3.20) we find that $F/A = \infty$; this is because the Riemann ζ function $\zeta(1-D)$ is singular when $D = 0$. The divergence in F/A is a consequence of that fact that the surface area of a zero dimensional sphere is zero. Hence, we will compute the Casimir stress, rather than the Casimir force per unit area. The equation for F is obtained by multiplying (3.20) by the surface area of a D-dimensional sphere of radius a, $A = 2a^{D-1}\pi^{D/2}/\Gamma(D/2)$:

$$F = \frac{1}{2\pi a^2}\left\{ \int_0^\infty dx\, \ln[2xI_{-1+\frac{D}{2}}(x)K_{-1+\frac{D}{2}}(x)] + \frac{2}{\Gamma(D-1)}\sum_{n=1}^{\infty}\left[\frac{\pi}{2}n^{D-1} + \frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{n!} \times \int_0^\infty dx\, \ln[2xI_{n-1+\frac{D}{2}}(x)K_{n-1+\frac{D}{2}}(x)]\right] - \frac{\pi}{\Gamma(D-1)}\zeta(1-D)\right\}. \tag{3.21}$$

We can now let D tend to 0 in (3.21). We obtain the result

$$F|_{D=0} = -\frac{1}{2a^2}, \tag{3.22}$$

where we have used $\zeta(z) \sim \frac{1}{z-1}$ as $z \to 1$.

Note that we could not have obtained this result from (3.15). Indeed, if we naively let $D \to 0+$ in the formula obtained by multiplying (3.15) by the surface area A of the sphere we appear to get the value 0. This is because only the $n = 0$ and $n = 2$ terms survive in this limit and these two terms cancel as a result of the identities $I_{-n}(z) = I_n(z)$ and $K_{-\nu}(z) = K_\nu(z)$. However, the result $F = 0$ at $D = 0$ is incorrect because the series in (3.15) does not converge.

D. Numerical results

The expression in (3.20) may in principle be used to compute F/A numerically for $D < 1$; to wit, we may evaluate the integrals for a large number N of terms in the series, compute the Nth partial sum, and extrapolate the result to its value at $N = \infty$. However, this procedure is rather inefficient because the sum in (3.20) is very slowly converging. Therefore, to prepare for evaluating F/A we subtract not just the one term in (3.18) but many terms in this asymptotic expansion. The first three terms are

$$-\frac{(n-1+\frac{D}{2})\Gamma(n+D-2)}{2^{D-1}\pi^{\frac{D+1}{2}}a^{D+1}n!\,\Gamma\left(\frac{D-1}{2}\right)}Q_n \sim -\frac{1}{2^D\pi^{\frac{D-1}{2}}a^{D+1}\Gamma\left(\frac{D-1}{2}\right)}\left[n^{D-1} + \frac{(D-1)(D-2)}{2}n^{D-2} + \frac{24D^4 - 176D^3 + 504D^2 - 688D + 387}{192}n^{D-3} + \cdots\right] \quad (n \to \infty). \tag{3.23}$$

If we use K terms in this asymptotic expansion we then have K corresponding Riemann ζ functions appearing in the final form of the series. The series converges more rapidly (the nth term in the series vanishes like n^{D-K-1}) and it also converges for a larger range of the dimension: $D < K$. We have used this method to graph F/A and F as functions of D (see Figs. 2 and 3).

From Figs. 2 and 3 it appears that F and F/A are singular at $D = 2$ and $D = 4$. If fact, as we will now explain, the Casimir stress is singular at *all* even positive integer values of D; F and F/A have simple poles at $D = 2N$, $N = 1, 2, 3, \ldots$. To verify this, we examine the generalization of (3.20) obtained by making many subtractions of the asymptotic behavior in (3.23). This formula for F/A will contain many Riemann ζ functions, one for each subtraction. The kth ζ function will have the form $\zeta(k - D)$. Furthermore, if k is even the coefficient of $\zeta(k - D)$ contains the factor $(k - 1 - D)$. (No such factor occurs if k is odd.) Thus, when $D = k - 1$ and k is an even positive integer the simple pole of the zeta function is canceled by this factor and when $D = k - 1$ and k is an odd positive integer, the simple pole persists.

As explained above, the Casimir stress is finite at all odd-integer dimensions. For example,

$$F|_{D=3} = \frac{1}{a^2}0.002\,816\,8\ldots, \tag{3.24}$$

where the positive value indicates that the stress is repulsive (tends to inflate the sphere). The numerical value in (3.24) is much smaller than that obtained

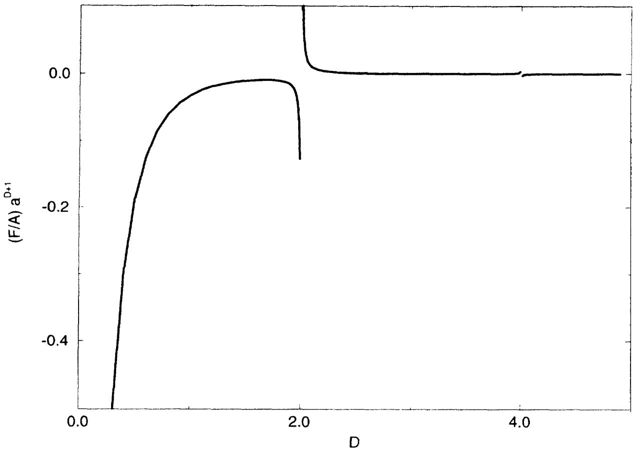

FIG. 2. A plot of the Casimir force per unit area F/A for $0 < D < 5$ on a spherical shell.

by Boyer [17] for the case of an electromagnetic field confined in a three-dimensional spherical cavity ($F = 0.046\,176\ldots\, a^{-2}$) and that obtained by Milton [18] for the case of a spinor field confined in a three-dimensional spherical cavity ($F = 0.0204\ldots\, a^{-2}$).

Although we have not proved it, it does appear from Fig. 3 that $F \to 0$ as $D \to \infty$. This is probably associated with the fact that the volume and surface area of a D-dimensional sphere of radius a tend to 0 as D tends to ∞.

E. Casimir stress for negative dimension

For all odd-integer $D \leq 1$ the series in (3.15) truncates and thus it is not necessary to subtract off the large-n behavior. This truncation occurs because of the identity

$$\lim_{D\to 1-2N} \frac{\Gamma(n+D-2)}{\Gamma\left(\frac{D-1}{2}\right)} = \sum_{j=0}^{2N+1} (-1)^{N+1+j} \times \frac{N!}{2(2N+1-j)!}\delta_{nj} \quad (N = 0,\, 1,\, 2,\, 3,\, \ldots), \quad (3.25)$$

which is the generalization of (3.1). If we use this identity at $N = 1$ we obtain the following integral representation for F/A at $D = -1$:

$$F/A|_{D=-1} = -\frac{1}{2}\int_0^\infty dx\, \ln\left[\left(1+\frac{1}{x}\right)^2 \left(1 - \frac{\sinh x}{x\cosh x}\right) \times \left(1 - \frac{\cosh x}{x \sinh x}\right)\right], \quad (3.26)$$

where we have inserted the expressions for the half-odd integer modified Bessel functions. An interesting aspect of this integral representation for F/A is that the argument of the logarithm has a zero for a positive real value of x. This zero may be traced to the positive zero of the function $I_{-3/2}(x)$. Thus, $F/A|_{D=-1}$ is complex. Because the contour of integration passes under the zero, we find here $F/A|_{D=-1} = 0.653\,82 + i1.884\,45$.

In general, for all $D < 0$, $D \neq -2N$ with $N = 1, 2, 3, \ldots$, the argument of the logarithm in the integrand of (3.15) always has a zero. Hence, the analytic continuation of F/A to negative values of D is complex. The zero of the argument of the logarithm comes about because $I_\nu(x)$ has a real positive zero when $-2m < \nu < -2m+1$, where m is a positive integer. Only a finite number of integrals in the series in (3.15) are complex. However, as D becomes more negative, there are more and more complex integrals in the series. In particular, each time D decreases past a negative even integer one additional integral in the series (3.15) becomes complex. Thus, in the complex-D plane, F/A has branch cuts

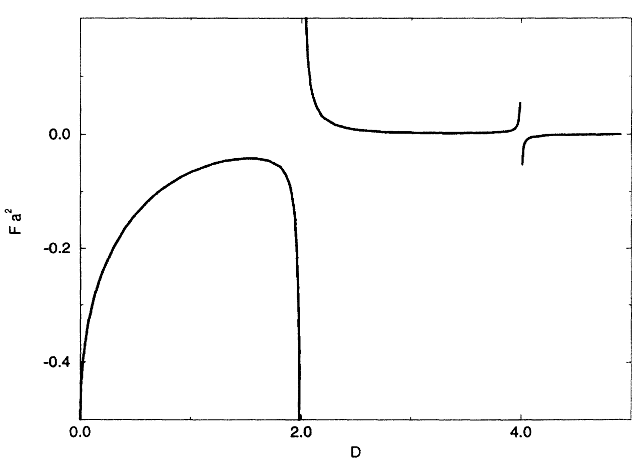

FIG. 3. A plot of the Casimir stress F for $0 < D < 5$ on a spherical shell.

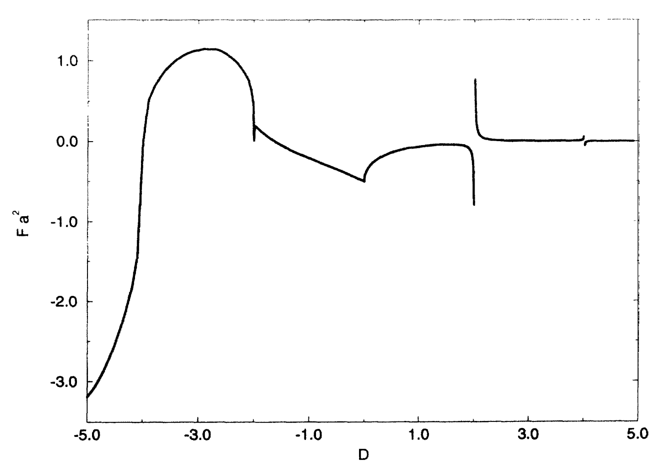

FIG. 4. A plot of the Casimir stress F for $-5 < D < 5$ on a spherical shell. For $D < 0$ the stress F is complex and we have plotted $\mathrm{Re}\, F$.

emanating from the points $D = -2k$, $k = 0, 1, 2, 3, \ldots$.

It is remarkable, however, that exactly at the negative even integers, it is possible to evaluate the Casimir stress F; we find that at these points $F = 0$. This is because the series (3.15) truncates after a finite number of terms for these values of D, and the remaining terms cancel in pairs. In Fig. 4 we plot $\mathrm{Re}\, F$ for $-5 < D < 5$. Figure 4 illustrates one interesting aspect of the Casimir stress, namely, the erratic fluctuations in the sign of F. The sign of the Casimir stress is extremely difficult to understand intuitively—we know of no simple physical argument that predicts whether the stress is attractive or repulsive.

ACKNOWLEDGMENTS

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