Collective coordinates and BRST symmetry

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The quantization of a particle which moves in the neighborhood of a Newtonian path is investigated as a model with typical characteristics of a field theory with classical finite energy configurations. The transformation to collective and fluctuation coordinates results in a singular Lagrangian. It is shown that the associated first class constraints generate a gauge group under which the first-order Lagrangian is invariant. It is then shown that in the BRST extension also the Hamiltonian is invariant and allows the complete quantization of the theory. Finally various gauge-fixing conditions are discussed as well as the integration of the path integral and the derivation of Schwinger-Dyson equations.

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I. INTRODUCTION

In very general terms a gauge field theory can be looked at as a theory with constraints. But the converse seems also to be true: A theory with constraints is a gauge theory in the sense that it possesses a gauge symmetry which is generated by the constraints. Constraints arise naturally with a large class of field transformations. In particular, the quantization of a field theory in the neighborhood of some classical configuration with finite action or energy (depending on whether time is Euclidean or Minkowskian, respectively) requires a transformation to collective and fluctuation variables and thus to a larger number of degrees of freedom, which has the consequence of the immediate appearance of constraints. Since instantons, solitons, bounces, and other such topological or nontopological classical configurations play an important role in many field theoretical considerations, the quantization of such theories is of considerable interest. Quantization in the background of a classical configuration implies also a perturbation expansion in its neighborhood and specifically the loop expansion if the path integral method is employed. One is therefore confronted with the problem of developing such a perturbation theory for a system with constraints. It is well known that this is a complicated task which involves in particular also the problem of gauge fixing. One way to achieve the effects of gauge fixing without breaking the invariance is to reformulate the gauge transformation as a Becchi-Rouet-Stora-Tyutin (BRST) transformation [1], the symmetry of which is now generally regarded as a fundamental requirement of any theory with local gauge invariance [2].

Frequently, there is a considerable difference between what can be done in principle and what can be done in practice. In the following we therefore consider in detail as a prototype of a theory with finite action or energy classical configurations the manageable problem of the quantization of a particle which is constrained to move in the neighborhood of a classical Newtonian path (or orbit). We later indicate briefly how an analogous procedure of gauging a theory by using collective fields can be applied to solitonlike or Skyrme-like models with topological vortex solutions.

The problem of a particle which is constrained to move in the vicinity of a classical path has been considered previously with the intention to develop a Schrödinger-like theory as an alternative to the path integral method [3,4]. Its formulation as a gauge theory has not been considered previously to our knowledge. There are, of course, some similarities with the treatment of the circular path in Ref. [2].

In Sec. II we define the basics of the theory of a particle which moves in the neighborhood of a classical orbit, and we emphasize the importance of starting from the first-order Lagrangian. In Sec. III we discuss the necessity of gauge fixing and evaluate the Dirac brackets [5] for a convenient choice. We also point out the relationship to the method of Faddeev and Jackiw [6]. In Sec. IV we define the gauge transformation generated by the constraints and point out the noninvariance of the Hamiltonian. In Secs. V–VII we define the BRST transformation and show that even with gauge fixing both the Lagrangian and Hamiltonian are invariant. We also show that with BRST and anti-BRST invariance of the Hamiltonian the physical states are precisely those which are projected out by the constraints. In Sec. VIII we discuss various choices of gauge-fixing conditions. In Sec. IX we go to the path integral and derive the Schwinger-Dyson equations. In Sec. X we make some concluding remarks and point out how the method described here may be applied to the quantization of field theories with, e.g., classical soliton or vortex configurations.

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II. PARTICLE NEAR A CLASSICAL ORBIT AND ITS FIRST-ORDER LAGRANGIAN

We consider the Lagrangian [3,4]

$$L = \tfrac{1}{2}\dot{\mathbf{R}}^2 - V(R) \ , \tag{2.1}$$

where $\mathbf{R}$ is an N-dimensional Euclidean vector with canonical momentum $\mathbf{P} = \dot{\mathbf{R}}$. We assume that the path $\mathbf{R}$ of the particle of mass $m = 1$ can be approximated by some path $\mathbf{r}(f(q))$, e.g., a classical trajectory, where f (classically time) is a given arbitrary function which fixes the parametrization of the curve and q is an appropriate parameter which plays the role of a collective coordinate. $\mathbf{R}$ itself is then written

$$\mathbf{R} = \mathbf{r}(f(q)) + \sum_{a=2}^{N} \mathbf{n}_a(f(q))\eta_a \ , \tag{2.2}$$

where $\{\mathbf{n}_a(f)\}$ together with $\mathbf{r}_f(f) \equiv d\mathbf{r}/df$ form a moving local reference frame at the point $\mathbf{r}(f)$, i.e.,

$$\mathbf{n}_a \cdot \mathbf{n}_b = \delta_{ab}, \quad \mathbf{n}_a \cdot \mathbf{r}_f = 0 \ . \tag{2.3}$$

In perturbation theory it is particularly convenient to choose $\eta_1 = 0$ so that the independent variables are q and η_a and their respective conjugate momenta. In the following, perturbation theory is not our immediate aim, so that we shall make a different choice below.

It is essential to use a convenient notation. We write therefore

$$\mathbf{R} = R_i(f(q))\mathbf{e}_i = Q_\alpha \mathbf{n}_\alpha(f(q)) \ , \tag{2.4}$$

where $\{\mathbf{e}_i\}$ are the unit vectors of the fixed frame and $\{\mathbf{n}_\alpha\}$ those of the moving reference frame, with $i, \alpha = 1, \ldots, N$, and $\mathbf{n}_1 = \mathbf{r}_f / r_f$. Then

$$\begin{aligned} R_i &= \langle \mathbf{e}_i | \mathbf{R} \rangle = M_{i\alpha} Q_\alpha \ , \\ Q_\alpha &= \langle \mathbf{n}_\alpha | \mathbf{R} \rangle = W_{\alpha i} R_i \ , \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} M_{i\alpha} &= \langle \mathbf{e}_i | \mathbf{n}_\alpha \rangle = \frac{\partial R_i}{\partial Q_\alpha} \ , \\ W_{\alpha i} &= \langle \mathbf{n}_\alpha | \mathbf{e}_i \rangle = \frac{\partial Q_\alpha}{\partial R_i} \ , \end{aligned} \tag{2.6}$$

and

$$W_{\alpha i} M_{i\beta} = \delta_{\alpha\beta}, \quad M_{i\alpha} W_{\alpha j} = \delta_{ij} \ , \tag{2.7}$$

the latter being the normalization and orthogonality relations of the unit vectors in the moving and fixed reference frames, respectively. Now

$$p_\alpha := \frac{\partial L}{\partial \dot{Q}_\alpha} = \frac{\partial L}{\partial \dot{R}_i} \frac{\partial R_i}{\partial Q_\alpha} = P_i M_{i\alpha} = \langle \mathbf{P} | \mathbf{n}_\alpha \rangle = P^*_\alpha = P_\alpha \tag{2.8}$$

and

$$p := \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{R}_i} \frac{\partial R_i}{\partial q} = P_i T_i \ , \tag{2.9}$$

where

$$T_i = \frac{\partial R_i}{\partial q} = \frac{\partial}{\partial q} \langle \mathbf{e}_i | \mathbf{n}_\alpha Q_\alpha \rangle = f' \langle \mathbf{e}_i | \mathbf{n}_{\alpha,f} \rangle Q_\alpha \ , \tag{2.10}$$

where $\mathbf{n}_{\alpha,f} = \partial \mathbf{n}_\alpha / \partial f$. From (2.8) we obtain $P_i = p_\alpha W_{\alpha i}$. Inserting this into (2.9), we obtain the constraint

$$\varphi := p - p_\alpha W_{\alpha i} T_i = 0 \ . \tag{2.11}$$

Since

$$\begin{aligned} \frac{\partial Q_\alpha}{\partial q} &= W_{\alpha i} T_i = \langle \mathbf{n}_\alpha | \mathbf{e}_i \rangle f' \langle \mathbf{e}_i | \mathbf{n}_{\beta,f} \rangle Q_\beta \\ &= f' \Gamma_{\alpha\beta} Q_\beta \ , \end{aligned} \tag{2.12}$$

where

$$\Gamma_{\alpha\beta} := \langle \mathbf{n}_\alpha | \mathbf{n}_{\beta,f} \rangle = -\Gamma_{\beta\alpha} \ , \tag{2.13}$$

we can write the constraint

$$\varphi = p - f' p_\alpha \Gamma_{\alpha\beta} Q_\beta \ . \tag{2.14}$$

This is a primary constraint [5] which is indicative of the singularity of the Lagrangian [7] expressed in terms of collective and fluctuation coordinates. Since the constraint involves momenta, we have to start from the first-order Lagrangian expressed in terms of coordinates and momenta and not from the momentum-integrated second-order form in the path integral formulation. In the original variables the first-order form is given by

$$L = P_i \dot{R}_i - H \ , \tag{2.15}$$

where $H = \frac{1}{2}P_i^2 + V(R) = \frac{1}{2}p_\alpha p_\alpha + V(R)$, with $\dot{R}_i$ in (2.15) given by $\dot{R}_i = \partial H / \partial P_i$ in the first-order Lagrangian. Now

$$\dot{R}_i = \frac{\partial R_i}{\partial Q_\alpha} \dot{Q}_\alpha + \frac{\partial R_i}{\partial q} \dot{q} \ ,$$

so that

$$\begin{aligned} P_i \dot{R}_i &= P_i (M_{i\alpha} \dot{Q}_\alpha + T_i \dot{q}) \\ &= p_\alpha \dot{Q}_\alpha + p\dot{q} \end{aligned} \tag{2.16}$$

and so

$$L = p\dot{q} + p_\alpha \dot{Q}_\alpha - H, \quad H = \tfrac{1}{2} p_\alpha p_\alpha + V(Q) \ . \tag{2.17}$$

This is the new first-order form of the Lagrangian on which all subsequent considerations will be based. It may be observed here that $H = H(p_\alpha, Q_\alpha)$ is independent of q since $V = V(R) = V(\sqrt{Q_\alpha^2})$ and

$$\begin{aligned} \frac{dQ_\alpha^2}{dq} &= 2Q_\alpha \frac{dQ_\alpha}{dq} = 2Q_\alpha \frac{dQ_\alpha}{dR_i} \frac{dR_i}{dq} \\ &= 2Q_\alpha W_{\alpha i} T_i = 2f' Q_\alpha \Gamma_{\alpha\beta} Q_\beta = 0 \ . \end{aligned} \tag{2.18}$$

III. GAUGE FIXING AND DIRAC QUANTIZATION

The canonical quantization of the theory is achieved with Dirac's method [5]. In Dirac's terminology, φ of (2.14) is a primary constraint. Before one can proceed one has to check the presence of further constraints. We define therefore the total Hamiltonian

$$H_\lambda = H + \lambda\varphi \tag{3.1}$$

and evaluate the Poisson bracket $\{\varphi, H_\lambda\}$. A somewhat lengthy calculation yields

$$\{\varphi, H_\lambda\} = 0 \ , \tag{3.2}$$

so that $\dot{\varphi} = 0$; i.e., the constraint is stable, and there is no new, i.e., secondary, constraint. The first-class nature of φ (and, in fact, $p_\lambda = \partial L/\partial\dot{\lambda} = 0$) is indicative of the gauge symmetry of the theory and implies, in the procedure of Dirac, that the symmetry has to be broken by imposition of a further constraint or gauge-fixing condition which makes the set of constraints and gauge-fixing conditions second class. In the present context it is convenient to choose, as a gauge-fixing condition,

$$\chi := Q_1 = 0 \ . \tag{3.3}$$

Demanding that this condition be maintained in the course of time, i.e.,

$$\dot{Q}_1 = \{Q_1, H_\lambda\} = 0 \ , \tag{3.4}$$

we obtain, for the Lagrange multiplier,

$$\lambda = \frac{p_1}{f'\Gamma_{1a}Q_a} \ . \tag{3.5}$$

Since $\{\varphi, \chi\} = f'\Gamma_{1a}Q_a$, the constraints $\{\psi_i\}$, $\psi_1 \equiv \varphi$, $\psi_2 \equiv \chi$, are second class. The Dirac bracket of any two quantities F and G is defined by

$$\{F,G\}_D := \{F,G\} + \sum_{i,j}\{F,\psi_i\}\{\psi_i,\psi_j\}^{-1}\{\psi_j,G\} \ . \tag{3.6}$$

Evaluation of this expression for the canonical variables q, p and Q_α, p_α, $\alpha = 1, \ldots, N$ yields the normal relations

$$\{q,p\}_D = 1, \quad \{Q_a, p_b\}_D = \delta_{ab} \tag{3.7}$$

and the abnormal ones

$$\begin{aligned} &\{q,p_1\}_D = \frac{1}{f'\Gamma_{1a}Q_a}, \quad \{p,p_1\}_D = \frac{f'p_\alpha\Gamma_{\alpha\beta,f}Q_\beta}{\Gamma_{1a}Q_a} \ , \\ &\{Q_1,p_1\}_D = 0, \quad \{Q_a,p_1\}_D = \frac{\Gamma_{a\gamma}Q_\gamma}{\Gamma_{1c}Q_c} \ , \\ &\{p_a,p_1\}_D = \frac{p_\beta\Gamma_{\beta\alpha}}{\Gamma_{1c}Q_c} \ . \end{aligned} \tag{3.8}$$

We observe that the abnormal relations are related to p_1, the dependent variable. Especially, the bracket $\{Q_1, p_1\}_D = 0$ shows that the canonical variables Q_1, p_1 can be simultaneously determined in quantum theory. The Dirac brackets of Q_1 with any variable are zero, so that Q_1 is a classical quantity. Q_1 and p_1 can be fixed by the conditions $\varphi = 0$, $\chi = 0$:

$$p_1 = \frac{p - p_a W_{ai}T_i}{W_{1i}T_i} = \frac{p - f'p_a\Gamma_{ab}Q_b}{f'\Gamma_{1c}Q_c}, \quad Q_1 = 0 \ . \tag{3.9}$$

In the reduced physical subspace of the independent variables p, q, p_a, Q_a, $a = 2, \ldots, N$, there are only the normal type of Dirac brackets. As usual, quantization is effected by the replacement

$$\{A,B\}_D \to [\hat{A},\hat{B}]/i\hbar \ ,$$

i.e., (with $\hbar = 1$),

$$[\hat{p},\hat{q}] = -i, \quad [\hat{p}_a,\hat{Q}_b] = -i\delta_{ab} \ . \tag{3.10}$$

The method of quantization advocated by Faddeev and Jackiw [6] is based on the idea of using the constraints in order to go to a reduced phase space. Thus, if $\delta L(q, Q_\alpha; \dot{q}, \dot{Q}_\alpha)$ is expressed in terms of $\delta q, \delta p, \delta Q_\alpha, \delta p_\alpha$ (where p and p_α are the momenta associated with q and Q_α), one would use the constraint (2.14) in order to eliminate, e.g., δp_1. δL will then be expressed in terms of $2N + 1$ δ quantities, i.e., an odd number of phase space variable increments. If one now proceeds as in the usual derivation of Hamilton's equations, one encounters a new problem: The inverse of an odd-dimensional antisymmetric matrix does not exist. It is therefore necessary to demand a further condition (e.g., $Q_1 = 0$), the gauge-fixing condition, in order to cure this problem. In the present problem with one constraint, this condition suffices, and the canonical commutation relations of the (then) independent variables can be read off the inverted matrix. The same results are obtained with Faddeev's argument [8], which is based on the initial conditions of the equations of motion. This method of going to a reduced phase space differs considerably from the method to be discussed in the following sections, which is based on the consideration of an enlarged phase space.

IV. GAUGE TRANSFORMATION

We have seen above that the theory possesses one primary constraint and no secondary or higher-order constraints. We incorporated this primary constraint into the Hamiltonian by forming H_λ of (3.1). We now consider the corresponding first-order Lagrangian in the enlarged phase space:

$$\begin{aligned} L(q, Q_\alpha, \lambda; p, p_\alpha, p_\lambda) &= p\dot{q} + p_\alpha\dot{Q}_\alpha \\ &\quad - H_\lambda(q, Q_\alpha, \lambda; p, p_\alpha, p_\lambda), \\ H_\lambda &= \tfrac{1}{2}p_\alpha^2 + V(R(f(q))) + \lambda\varphi \ . \end{aligned} \tag{4.1}$$

This new first-order Lagrangian now has two constraints, i.e.,

$$\psi_1 := \varphi = 0, \quad \psi_2 := p_\lambda = 0 \ ,$$

and we see that they commute: i.e.,

$$\{\psi_i, \psi_j\} = 0, \quad i,j = 1,2 \ . \tag{4.2}$$

Thus ψ_1 and ψ_2 are first class, and (4.2) expresses the fact that the gauge transformation can be constructed from ψ_1 and ψ_2. We define the generator of this time-dependent gauge transformation by

$$Q_g := -ig(t)\varphi - i\dot{g}(t)p_\lambda \ , \tag{4.3}$$

where $g(t)$ is an arbitrary real function of t and the corresponding unitary operator is

$$U_g = \exp(Q_g) \approx 1 + Q_g \ . \tag{4.4}$$

The infinitesimal transformation of an operator Ω is then given by

$$\delta\Omega = U_g \Omega U_g^{-1} - \Omega = [Q_g, \Omega] \ . \tag{4.5}$$

With this we can compute the variations of all dynamical variables. One obtains

$$\begin{aligned} \delta p &= [Q_g, p] = -ig[\varphi, p] = -gM \ , \\ \delta q &= [Q_g, q] = -ig[\varphi, q] = -g \ , \\ \delta Q_\alpha &= [Q_g, Q_\alpha] = -ig[\varphi, Q_\alpha] = gR_\alpha \ , \\ \delta p_\alpha &= [Q_g, p_\alpha] = -ig[\varphi, p_\alpha] = -gN_\alpha \ , \\ \delta\lambda &= [Q_g, \lambda] = -\dot{g} \ , \\ \delta p_\lambda &= 0 \ , \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} M &= f'^2 \Gamma_{\alpha\beta,f} Q_\beta p_\alpha \ , \\ R_\alpha &= f' \Gamma_{\alpha\beta} Q_\beta \ , \\ N_\alpha &= f' \Gamma_{\beta\alpha} p_\beta \ . \end{aligned} \tag{4.7}$$

We also have

$$\begin{aligned} &[\varphi, q] = -i, \quad [\varphi, p] = -iM \ , \\ &[\varphi, Q_\alpha] = iR_\alpha, \quad [\varphi, p_\alpha] = -iN_\alpha \ , \end{aligned} \tag{4.8}$$

and, since $V(R_i(f(q)))$is a scalar depending only on $R_i^2 = Q_\alpha^2$,

$$\begin{aligned} \delta V &= \frac{\partial V}{\partial q}\delta q = \frac{\partial V}{\partial Q_\alpha^2}\frac{\partial Q_\beta^2}{\partial q}\delta q \\ &= \frac{dV}{dQ_\gamma^2} 2Q_\alpha f' \Gamma_{\alpha\beta} Q_\beta \delta q \\ &= 0 \ , \end{aligned}$$

in view of the antisymmetry of Γ. It is now straightforward to verify that

$$\delta L = [Q_g, L] = \frac{d}{dt}[g(t)\varphi] \ ; \tag{4.9}$$

i.e., the action is invariant under the gauge transformation (4.6). However,

$$\delta H_\lambda = [Q_g, H_\lambda] = -\dot{g}(t)\varphi \ . \tag{4.10}$$

Thus the Hamiltonian is only invariant for time-independent gauge transformations. Of course, since the physical states must be gauge invariant, we must also have a Hamiltonian which is gauge invariant. This is exactly what we shall achieve with the BRST transformation below. Equation (4.10) allows us only to infer that in the subspace of physical states $|\psi\rangle$ with $\varphi|\psi\rangle = 0$ we have $\delta H_\lambda|\psi\rangle = 0$, but Q_g is not a conserved charge.

V. BRST TRANSFORMATION

As is well known, the BRST extension of phase space allows one to maintain the gauge invariance in the sense of BRST invariance in spite of the addition of a gauge-fixing term to the first-order Lagrangian. The generator of the BRST transformation is essentially defined such that the function $g(t)$ in the generator of the gauge transformation Q_g, i.e., (4.3), is replaced by the Grassmann variable or anticommuting ghost field operator $c(t)$. This procedure then guarantees the BRST invariance of the first-order Lagrangian. Thus the BRST charge Q_B is taken to be

$$Q_B := -ic(t)\varphi - i\dot{c}(t)p_\lambda + i(\pi_{\bar{c}} - \dot{c})b \ , \tag{5.1}$$

where $c(t)$ and $\bar{c}(t)$ are ghost and antighost variables, respectively, and $b(t)$ is the bosonic Nakanishi-Lautrup variable. The momenta canonical to c and $\bar{c}$ are π_c and $\pi_{\bar{c}}$, which we define by directional derivatives: i.e.,

$$\pi_c := L\frac{\overleftarrow{\partial}}{\partial\dot{c}}, \quad \pi_{\bar{c}} = \frac{\overrightarrow{\partial}}{\partial\dot{\bar{c}}}L \ . \tag{5.2}$$

The generator Q_B is nilpotent: i.e.,

$$Q_B^2 = 0 \ ,$$

with $\{c, \dot{c}\} = 0$.

We also assume here in the BRST quantization procedure that all bosonic operators have canonical commutation relations and all fermionic operators have canonical anticommutation relations. Thus, in particular, we assume

$$[p, q] = -i, \quad [p_\alpha, Q_\beta] = -i\delta_{\alpha\beta} \tag{5.3}$$

for $\alpha, \beta = 1, \ldots, N$ and

$$\{\pi_c, c\} = i, \quad \{\pi_{\bar{c}}, \bar{c}\} = -i \ . \tag{5.4}$$

Thus $\pi_{\bar{c}}$ in (5.1) has ghost number $+1$ like c, and hence Q_B has definite ghost number $+1$. Since we do not (here) choose c to be Hermitian and $\bar{c}$ to be anti-Hermitian as is often customary in the literature, the anticommutators $\{\pi_c, c\}$ and $\{\pi_{\bar{c}}, \bar{c}\}$ here will not be independent (the convention chosen here makes the Hamiltonian formalism below more transparent), though c and $\bar{c}$ are independent canonical variables.

The infinitesimal transformation of any Heisenberg operator Ω under the BRST transformation generated by Q_B is given by

$$\delta\Omega = [Q_B, \Omega]_\mp \ , \tag{5.5}$$

depending on whether Ω is bosonic or fermionic.

We then find (with $[,]_+ \equiv \{,\}$)

$$\begin{aligned} \delta q &= [Q_B, q] = -c \ , \\ \delta p &= [Q_B, p] = -cM \ , \\ \delta Q_\alpha &= [Q_B, Q_\alpha] = cR_\alpha \ , \\ \delta p_\alpha &= [Q_B, p_\alpha] = -cN_\alpha \ , \\ \delta \lambda &= [Q_B, \lambda] = -\dot{c} \ , \\ \delta p_\lambda &= 0 \ , \\ \delta c &= \{Q_B, c\} = 0 \ , \\ \delta \pi_c &= \{Q_B, \pi_c\} = \varphi \ , \\ \delta \bar{c} &= \{Q_B, \bar{c}\} = b - i\{\dot{c}, \bar{c}\}(p_\lambda + b) \ , \\ \delta \pi_{\bar{c}} &= \{Q_B, \pi_{\bar{c}}\} = 0 \ , \\ \delta b &= [Q_B, b] = 0 \ , \\ \delta p_b &= [Q_B, p_b] = -(\pi_{\bar{c}} - \dot{c}) \ . \end{aligned} \tag{5.6}$$

Here M, R_α, N_α are the quantities defined by (4.7). It will be seen below that with the use of the equations of motion $\delta\bar{c} = b$ and $\delta p_b = 0$.

We now add to the first-order Lagrangian L of (4.1) a trivially BRST-invariant gauge-fixing contribution, which we choose as

$$\begin{aligned} L_{\rm GF} &= -\delta[\bar{c}(\dot{\lambda} + hq + \tfrac{1}{2}b)] \\ &= -b(\dot{\lambda} + hq) - \tfrac{1}{2}b^2 + \dot{\bar{c}}\dot{c} - h\bar{c}c \ , \end{aligned} \tag{5.7}$$

where h is a constant with appropriate dimension, and in the last line we dropped a total time derivative after a partial integration in the action integral. Of course, one could make a different choice, e.g., $L_{\rm GF} = -\delta[\bar{c}(\dot{\lambda} - p + \frac{1}{2}b)]$, but then, in view of $\delta p = -Mc$, one would arrive at a much more complicated Euler-Lagrange equation for c, i.e., $\ddot{c} - cM = 0$, with further complications in the arguments below. In fact, the choice (5.7) which corresponds to the covariant gauge in QED results in free equations of motion for the ghost fields c and $\bar{c}$, which then allow one to build the physical states on a particular state of the free ghost sector.

It is now a simple matter to verify that the overall Lagrangian, i.e.,

$$\begin{aligned} L_B = p\dot{q} + p_\alpha \dot{Q}_\alpha - \tfrac{1}{2}p_\alpha p_\alpha - V - \lambda\varphi \\ -b(\dot{\lambda} + hq) - \tfrac{1}{2}b^2 + \dot{\bar{c}}\dot{c} - h\bar{c}c \ , \end{aligned} \tag{5.8}$$

is invariant under the BRST transformation (5.6); i.e.,

$$\delta L_B = 0 \ .$$

Applying the directional derivatives of (5.2), we obtain

$$\pi_c = \dot{\bar{c}}, \quad \pi_{\bar{c}} = \dot{c} \ , \tag{5.9}$$

so that, in (5.6),

$$\delta p_b = 0 \ .$$

From (5.8) we obtain the Euler-Lagrange equation of b; i.e.,

$$b = -(\dot{\lambda} + hq) \ , \tag{5.10}$$

so that

$$\begin{aligned} L_B = p\dot{q} + p_\alpha \dot{Q}_\alpha - \tfrac{1}{2}p_\alpha p_\alpha - V - \lambda\varphi \\ + \tfrac{1}{2}(\dot{\lambda} + hq)^2 + \dot{\bar{c}}\dot{c} - h\bar{c}c \ . \end{aligned} \tag{5.11}$$

From L_B we obtain

$$p_\lambda = \dot{\lambda} + hq = -b \ ,$$

so that, in (5.6),

$$\delta\bar{c} = b$$

and, in (5.1),

$$Q_B = -ic\varphi - i\dot{c}p_\lambda \ .$$

In (5.11) the gauge-fixing part $\frac{1}{2}(\dot{\lambda} + hq)^2$ clearly violates the invariance under the original gauge transformation (4.6). We also observe that the ghost sector completely decouples and its fields satisfy the free field equations

$$\ddot{c} + hc = 0, \quad \ddot{\bar{c}} + h\bar{c} = 0 \ . \tag{5.12}$$

We can therefore write the Heisenberg operators $c, \bar{c}$ as

$$\begin{aligned} c(t) &= e^{i\omega t}B + e^{-i\omega t}D \ , \\ \bar{c}(t) &= e^{-i\omega t}B^\dagger + e^{i\omega t}D^\dagger \ , \end{aligned} \tag{5.13}$$

with

$$h \equiv \omega^2 \ ,$$

where, of course, $B, D, B^\dagger, D^\dagger$ are again fermionic operators.

VI. THE HAMILTONIAN AND ITS DIAGONALIZATION

As in the standard procedure, we now pass from L_B to the corresponding Hamiltonian H_B by defining the latter as the complete Legendre transform of L_B; i.e.,

$$H_B = p\dot{q} + p_\alpha \dot{Q}_\alpha + p_\lambda \dot{\lambda} + \pi_c \dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - L_B \ . \tag{6.1}$$

Inserting L_B, we obtain

$$H_B = \tfrac{1}{2}p_\alpha p_\alpha + V + \lambda\varphi - hp_\lambda q + \tfrac{1}{2}p_\lambda^2 + \dot{\bar{c}}\dot{c} + h\bar{c}c \ . \tag{6.2}$$

With (5.13) we can check that H_B is Hermitian. In fact,

$$\dot{\bar{c}}\dot{c} + h\bar{c}c = 2\omega^2(B^\dagger B + D^\dagger D), \quad h \equiv \omega^2 > 0 \ , \tag{6.3}$$

because $\{D, D^\dagger\} = 1/2\omega$, ω real, so that $\omega^2 > 0$. We observe also that this expression is time independent.

Considered as independent canonical variables, c and $\bar{c}$ must satisfy the anticommutation relations

$$\{c,\bar{c}\} = 0 = \{\pi_c, \pi_{\bar{c}}\} = \{\dot{\bar{c}}, \dot{c}\} , \tag{6.4}$$

so that also

$$\frac{d}{dt}\{c,\bar{c}\} = 0, \quad \{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} . \tag{6.5}$$

Thus

$$\{\pi_c, c\} = -\{\pi_{\bar{c}}, \bar{c}\} .$$

This is consistent with (5.4) if

$$\{\dot{\bar{c}}, c\} = i, \quad \{\dot{c}, \bar{c}\} = -i . \tag{6.6}$$

We can also check the consistency of these relations with Hamilton's equations for ghost fields. Setting

$$\begin{aligned} H_{\text{ghost}} &= \dot{\bar{c}}\dot{c} + h\bar{c}c \\ &= \pi_c \dot{c} + h\bar{c}c , \end{aligned}$$

we have, with (5.12) and (5.9),

$$\frac{\partial H}{\partial \pi_c} = \dot{c}, \quad \frac{\partial H}{\partial c} = -h\bar{c} = \ddot{\bar{c}} = \dot{\pi}_c . \tag{6.7}$$

These are the equal-sign Hamilton equations for fermionic variables. One can also check the consistency of these equations with the Heisenberg equations: e.g.,

$$[H, c] = -\{\dot{\bar{c}}, c\}\dot{c} = -i\dot{c} . \tag{6.8}$$

We shall require the physical states $|\psi\rangle$ to be BRST invariant: i.e.,

$$Q_B|\psi\rangle = 0$$

or

$$-i[e^{i\omega t}B(\varphi + i\omega p_\lambda) + e^{-i\omega t}D(\varphi - i\omega p_\lambda)]|\psi\rangle = 0 .$$

The set of states $\{|\psi\rangle\}$ satisfying this condition contains states with $\varphi|\psi\rangle = 0$ and $p_\lambda|\psi = 0$, but also states $|\psi\rangle \equiv |0\rangle$, for which, instead,

$$B|0\rangle = 0 \quad \text{and} \quad D|0\rangle = 0$$

or

$$(B \pm D)|0\rangle = 0 ,$$

i.e.,

$$c(0)|0\rangle = 0, \quad \dot{c}(0)|0\rangle = 0 .$$

Since $c, \bar{c}, \dot{c}, \dot{\bar{c}}$ obey a number of relations, we can use these in order to obtain relations between the operators $B, D, B^\dagger, D^\dagger$ in (5.13). There are six different conditions. Imposing these at $t = 0$, we obtain

(1)$c^2 = 0$ giving
$$B^2 + \{B, D\} + D^2 = 0 ,$$

(2)$\dot{c}^2 = 0$ giving
$$B^2 - \{B, D\} + D^2 = 0 ,$$

(3)$c\dot{c} + \dot{c}c = 0$ giving
$$B^2 - D^2 = 0 ,$$

(4)$c\bar{c} + \bar{c}c = 0$ giving
$$\{B, B^\dagger\} + \{D, D^\dagger\} + \{B, D^\dagger\} + \{D, B^\dagger\} = 0 ,$$

(5)$\dot{c}\dot{\bar{c}} + \dot{\bar{c}}\dot{c} = 0$ giving
$$\{B, B^\dagger\} + \{D, D^\dagger\} - \{B, D^\dagger\} - \{D, B^\dagger\} = 0 ,$$

(6)$c\dot{\bar{c}} + \dot{\bar{c}}c = i$ giving
$$\{B, B^\dagger\} - \{D, D^\dagger\} - \{B, D^\dagger\} + \{D, B^\dagger\} = -\frac{1}{\omega} . \tag{6.9}$$

Of these, (1)–(3) give

$$B^2 = D^2 = \{B, D\} = 0$$

and (4)–(6) give

$$\begin{aligned} &\{B, B^\dagger\} + \{D, D^\dagger\} = 0 , \\ &\{B, D^\dagger\} + \{D, B^\dagger\} = 0 , \\ &\{B, B^\dagger\} + \{D, B^\dagger\} = -\frac{1}{2\omega} , \\ &\{D, D^\dagger\} + \{B, D^\dagger\} = \frac{1}{2\omega} . \end{aligned} \tag{6.10}$$

The last set of equations has the solutions

$$\begin{aligned} &\{B, D^\dagger\} = 0 , \\ &\{B, B^\dagger\} = -\frac{1}{2\omega} , \\ &\{D, D^\dagger\} = \frac{1}{2\omega} . \end{aligned} \tag{6.11}$$

With $\omega > 0$ and $|0\rangle$ as the vacuum state, for which as above $c(0)|0\rangle = 0$ and $\dot{c}(0)|0\rangle = 0$, i.e.,

$$B|0\rangle = D|0\rangle = 0 , \tag{6.12}$$

we have

$$\begin{aligned} \langle 0|DD^\dagger|0\rangle &= \frac{1}{2\omega}\langle 0|0\rangle , \\ \langle 0|BB^\dagger|0\rangle &= -\frac{1}{2\omega}\langle 0|0\rangle . \end{aligned} \tag{6.13}$$

For $\langle 0|0\rangle$ positive, $D^\dagger|0\rangle$ is a state with positive norm and $B^\dagger|0\rangle$ one with negative norm. As a matter of convenience, we could take $\langle 0|0\rangle$ negative, in which case $D^\dagger|0\rangle$ is the lowest (negative norm) state. We also have, from (6.11),

$$\langle 0|BD^\dagger + D^\dagger B|0\rangle = 0$$

and so

$$0 = \langle 0|BD^\dagger|0\rangle = -\langle 0|D^\dagger B|0\rangle \ . \tag{6.14}$$

Thus $B^\dagger|0\rangle$ is orthogonal to $D^\dagger|0\rangle$. Moreover,

$$\begin{aligned}\langle 0|BH_{\text{ghost}}D^\dagger|0\rangle &= 2\omega^2\langle 0|B(B^\dagger B + D^\dagger D)D^\dagger|0\rangle \\ &= -2\omega^2\langle 0|D^\dagger BDD^\dagger|0\rangle \\ &= 0\ ,\end{aligned} \tag{6.15}$$

since $\langle 0|D^\dagger = 0$. Thus the ghost part of the Hamiltonian does not lead to transitions between the states $D^\dagger|0\rangle$ and $B^\dagger|0\rangle$.

The occurrence of the negative norm states here is quite similar to their occurrence in QED when a gauge-fixing term of the form $(\partial\mu A^\mu)^2$ is added. It is clear that since the ghost sector is free and completely decouples from the rest of the system its negative norm states lie in that part of Hilbert space which is orthogonal to the subspace of physical states. Thus, in view of (6.14), if we choose $D^\dagger|0\rangle$ at $t = 0$ we completely exclude the $B^\dagger|0\rangle$ states for all time t.

VII. ANTI-BRST TRANSFORMATION AND PHYSICAL STATES

We can verify that H_B of (6.2) is invariant under the BRST transformation; in fact,

$$\begin{aligned}\delta H_B &= \delta V - \dot{c}(\varphi - \dot{b}) - \lambda cM - \lambda\delta(f'\Gamma_{\alpha\beta}Q_\beta p_\alpha) \\ &= \delta V - \dot{c}(\varphi - \dot{b}) \\ &= 0\ ,\end{aligned} \tag{7.1}$$

since $\delta V = 0$ and $\varphi - \dot{b} = 0$ is the Euler-Lagrange equation which results from variation of L_B with respect to λ and on using (5.10). Thus here H_B is fully invariant, which is different from what we observed in the case of the gauge transformation. Hence

$$[Q_B, H_B] = 0 \tag{7.2}$$

and Q_B is conserved.

We require physical states $|\psi\rangle$ to be BRST invariant: i.e.,

$$Q_B|\psi\rangle = 0$$

or

$$\{-ie^{i\omega t}B(\varphi + i\omega p_\lambda) - ie^{-i\omega t}D(\varphi - i\omega p_\lambda)\}|\psi\rangle = 0\ . \tag{7.3}$$

This condition is obviously not only satisfied by $\varphi|\psi\rangle = 0$ and $p_\lambda|\psi\rangle = 0$, but also, as mentioned earlier, by states built from $|0\rangle$, for which

$$B|\psi\rangle = 0, \quad D|\psi\rangle = 0\ ,$$

i.e., states $|0\rangle$ multiplied by any functions of the bosonic variables $q, p, Q_\alpha, p_\alpha, \lambda, p_\lambda$.

However, the Hamiltonian is also invariant under the anti-BRST transformation generated by the antighostlike generator

$$\begin{aligned}\bar{Q}_B :&= i\bar{c}(t)\varphi + i\dot{\bar{c}}(t)p_\lambda \\ &= ie^{-i\omega t}B^\dagger(\varphi - i\omega p_\lambda) + ie^{i\omega t}D^\dagger(\varphi + i\omega p_\lambda)\ ,\end{aligned} \tag{7.4}$$

which generates the variations

$$\begin{aligned}\bar{\delta}q &= \bar{c}\ ,\\ \bar{\delta}p &= \bar{c}M\ ,\\ \bar{\delta}Q_\alpha &= -\bar{c}R_\alpha\ ,\\ \bar{\delta}p_\alpha &= \bar{c}N_\alpha\ ,\\ \bar{\delta}\lambda &= \dot{\bar{c}}\ ,\\ \bar{\delta}p_\lambda &= 0\ ,\\ \bar{\delta}c &= b\ ,\\ \bar{\delta}\pi_c &= 0\ ,\\ \bar{\delta}\bar{c} &= 0\ ,\\ \bar{\delta}\pi_{\bar{c}} &= 0\ .\end{aligned} \tag{7.5}$$

Again, one can show that H_B is anti-BRST invariant so that $\bar{Q}_B$ is conserved, i.e.,

$$\bar{\delta}H_B = 0, \quad [H_B, \bar{Q}_B] = 0\ . \tag{7.6}$$

One can easily verify that the trivially BRST-invariant gauge-fixing term L_{GF} of (5.7) is, in fact, also anti-BRST invariant, i.e.,

$$\begin{aligned}\bar{\delta}L_{\text{GF}} &= \bar{\delta}[-b(\dot{\lambda} + hq) - \tfrac{1}{2}b^2 - h\bar{c}c + \dot{\bar{c}}\dot{c}] \\ &= -b(\bar{\delta}\dot{\lambda} + h\bar{\delta}q) + h\bar{c}\bar{\delta}c - \dot{\bar{c}}\bar{\delta}\dot{c} \\ &= -b\ddot{\bar{c}} + \ddot{\bar{c}}b \\ &= 0\ .\end{aligned}$$

The operator $\bar{Q}_B$ is, of course, the adjoint of Q_B. Hence we have to demand not only (7.3) for physical states $|\psi\rangle$, but also

$$\{ie^{-i\omega t}B^\dagger(\varphi - i\omega p_\lambda) + ie^{+i\omega t}D^\dagger(\varphi + i\omega p_\lambda)\}|\psi\rangle = 0\ . \tag{7.7}$$

Since the states $B|\psi\rangle = 0$, $D|\psi\rangle = 0$ obviously do not satisfy this condition, the only way both (7.3) and (7.7) can be satisfied is by states projected out by the constraints, i.e., those satisfying

$$\varphi|\psi\rangle = 0 \quad\text{and}\quad p_\lambda|\psi\rangle = 0\ .$$

Hence the additional anti-BRST symmetry is needed here in order to recover only the physical states projected out by the constraints.

VIII. COMMENTS ON GAUGE-FIXING CHOICES

A particular choice of the gauge-fixing term which has occasionally been favored in the literature [9]–[11] is described as gauge fixing the collective coordinate to zero. An advantage of such a choice would seem to be that when it is inserted into the appropriate partition functional, it is easy to recover the original theory after integrating out the auxiliary field b. The gauge-fixing part of the Lagrangian is taken to be

$$\tilde{L}_{\rm GF} = -\delta(\bar{c}q) = -(bq + \bar{c}c) \ . \tag{8.1}$$

The BRST-invariant Lagrangian is

$$\tilde{L}_B = L - (bq + \bar{c}c) \ , \tag{8.2}$$

where L is given by (4.1). However, it is necessary to reexamine the BRST transformation itself. Thus we now consider the BRST generator in the form

$$\tilde{Q}_B := -ic\varphi - i\dot{c}p_\lambda + i\pi_{\bar{c}}b \ . \tag{8.3}$$

Then

$$\begin{aligned}\tilde{\delta}\bar{c} = \{\tilde{Q}_B, \bar{c}\} &= -i\{\dot{c}, \bar{c}\}p_\lambda + i\{\pi_{\bar{c}}, \bar{c}\}b \\ &= -i\{\dot{c}, \bar{c}\}p_\lambda + b \ .\end{aligned} \tag{8.4}$$

We cannot identify $\dot{c}$ with $\pi_{\bar{c}}$ since now

$$\pi_{\bar{c}} = 0, \quad \pi_c = 0 \ .$$

From the equations of motion of $c, \bar{c}$, we obtain

$$c = 0, \quad \bar{c} = 0 \ ,$$

and so

$$\dot{c} = 0, \quad \dot{\bar{c}} = 0 \ .$$

Thus c and $\bar{c}$ are not even dynamical variables, but as before

$$\tilde{\delta}\bar{c} = b \ .$$

We also have

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0$$

and now

$$\begin{aligned}\tilde{\delta}\lambda = [\tilde{Q}_B, \lambda] &= -i\dot{c}[p_\lambda, \lambda] \\ &= -\dot{c} \\ &= 0 \ .\end{aligned} \tag{8.5}$$

Further,

$$\tilde{\delta}q = [\tilde{Q}_B, q] = -i[p, q]c = -c = 0$$

and similarly $\tilde{\delta}p = 0 = \tilde{\delta}p_\alpha = \tilde{\delta}Q_\alpha$. Thus, without the dynamics of velocities of c and $\bar{c}$ in $\tilde{L}_{\rm GF}$, the transformation becomes trivial. The choice (8.1) therefore is meaningless in the present example.

Another interesting choice of the gauge-fixing part of the Lagrangian is

$$\begin{aligned}\tilde{L}_{\rm GF} &= -\delta[\bar{c}(\dot{\lambda} + Q_1 + \tfrac{1}{2}b)] \\ &= -b(\dot{\lambda} + Q_1) - \tfrac{1}{2}b^2 + \dot{\bar{c}}\dot{c} + S_1\bar{c}c \ ,\end{aligned} \tag{8.6}$$

where

$$S_1 = f'\Gamma_{1\alpha}Q_\alpha = -\frac{\partial\varphi}{\partial p_1} = \{\varphi, \chi\}_{\rm Poisson} \ . \tag{8.7}$$

This expression is equal to the Poisson bracket of φ and $\chi = Q_1 = 0$. Thus it corresponds to the gauge-fixing condition $Q_1 = 0$. Then

$$\begin{aligned}\tilde{L}_B &= L + \tilde{L}_{\rm GF} \\ &= p\dot{q} + p_\alpha\dot{Q}_\alpha - \tfrac{1}{2}p_\alpha p_\alpha - V - \lambda\varphi \\ &\quad -b(\dot{\lambda} + Q_1) - \tfrac{1}{2}b^2 + \dot{\bar{c}}\dot{c} + \bar{c}S_1c \ .\end{aligned} \tag{8.8}$$

The BRST transformation is again given by (5.6). Now

$$\pi_{\bar{c}} = \dot{c}, \quad \pi_c = \dot{\bar{c}} \ .$$

Also, $\partial\tilde{L}_B/\partial b = 0$, and so

$$b = -(\dot{\lambda} + Q_1) \ . \tag{8.9}$$

The equations of motion of c and $\bar{c}$ are

$$\begin{aligned}\ddot{c} - S_1c = 0 \ , \\ \ddot{\bar{c}} - S_1\bar{c} = 0 \ .\end{aligned} \tag{8.10}$$

These equations express a coupling of the ghost and antighost fields to Q_α and q. Thus the ghost sector does not decouple as in the earlier case. One can now quantize the theory in terms of the free part of the Hamiltonian and consider the effects of interactions perturbatively. Then one begins from an eigenstate of the free part of the ghost Hamiltonian and builds all states $|\psi\rangle$ with $\tilde{Q}_B|\psi\rangle = 0$ by tensoring this with arbitrary functions of $q, p, Q_\alpha, p_\alpha, \lambda, p_\lambda$ as described in Ref. [2].

For perturbation theory the gauge-fixing condition $Q_1 = 0$ is not the most convenient. In that case the calculations become easier and more transparent if we use the gauge-fixing condition

$$\chi := Q_1 - r_1 = \eta_1 = 0 \ , \tag{8.11}$$

where

$$r_1 = \mathbf{r}\cdot\mathbf{r}_f/\sqrt{\mathbf{r}_f^2} \ .$$

We set

$$\mathbf{Q} = \mathbf{r} + \eta$$

and treat η as small perturbative fluctuations about $\mathbf{r}$. The derivative $\mathbf{r}_f \equiv d\mathbf{r}/df$ is the zero mode of the problem as can be seen by differentiating the classical equation

$$\mathbf{r}_{ff} = -\nabla V(\mathbf{r}) \ ,$$

which gives

$$\left[\frac{d^2}{df^2}\delta_{ij} + \left(\frac{\partial^2 V}{\partial R_i \partial R_j}\right)_{\mathbf{r}}\right](\mathbf{r}_f)_j = 0 \ . \tag{8.12}$$

We choose the normalization

$$f'\mathbf{r}_f^2 = 1 \ ,$$

so that

$$\mathbf{r}_f \cdot \mathbf{r}_{ff} = 0 \ . \tag{8.13}$$

For the Poisson bracket $\{\varphi, \chi\}$ which determines the mass of the ghosts c and $\bar{c}$, we now obtain (from differentiations with respect to q, p and Q_1, p_1)

$$\{\varphi, \chi\}_{\text{Poisson}} = -f'(\mathbf{r}_{ff} \cdot \mathbf{n}_a)(\mathbf{n}_a \cdot \mathbf{r}) + f'\frac{d}{df}(\mathbf{r} \cdot \mathbf{r}_f) + O(\eta) \ . \tag{8.14}$$

Using (8.13) and the completeness relation $|\mathbf{n}_a\rangle\langle\mathbf{n}_a| = 1 - |\mathbf{r}_f\rangle\langle\mathbf{r}_f|$, we obtain

$$\{\varphi, \chi\}_{\text{Poisson}} = f'\mathbf{r}_f^2 + O(\eta) = 1 + O(\eta) \ , \tag{8.15}$$

with appropriate normalization. Thus the effective mass of the ghosts is determined by the lowest-order approximation of the Poisson bracket $\{\varphi, \chi\}$ and this is, effectively, the Faddeev-Popov determinant. In lowest order this determinant is given by the normalization of the associated zero mode as is well known in the context of soliton considerations. This normalization, of course, can also be looked at as the finite kinetic energy of the classical particle with zero total energy. The gauge-fixing condition (8.11), i.e., $\eta \cdot \mathbf{r}_f = 0$, means that the fluctuations η are orthogonal to the zero mode. This, of course, is precisely the condition for the existence of the Green's function required for the perturbation expansion. We can conclude from the above that if we consider the functional integral

$$Z = \int [dq\, dp][dQ_\alpha dp_\alpha][d\lambda\, dp_\lambda][dc\, d\pi_c][d\bar{c}\, d\pi_{\bar{c}}] \exp\left[\frac{i}{\hbar}S_B\right]$$

and if we write

$$L_{\text{GF}} = -\delta[\bar{c}(\dot{\lambda} + \chi)] \ ,$$

the integration with respect to λ gives $\delta(\varphi)$, the integration with respect to p_λ gives $\delta(\bar{\chi})$, $\bar{\chi} = \dot{\lambda} + \chi$, the integrations with respect to c and $\bar{c}$ give $\det\{\bar{\chi}, \varphi\} = \det\{\chi, \varphi\}$, and the integrations with respect to π_c and $\pi_{\bar{c}}$ give a constant. One then obtains the well-known form [8]

$$Z = \int [dq\, dp][dQ_\alpha dp_\alpha]\delta(\bar{\chi})\det\{\bar{\chi}, \varphi\}\delta(\varphi)\exp\left[\frac{i}{\hbar}S\right] \ .$$

IX. RESIDUAL BRST INVARIANCE AND SCHWINGER-DYSON EQUATIONS

Schwinger-Dyson equations are equations which (when supplemented by appropriate boundary conditions) are identities for arbitrary functionals (specifically Green's functions) which provide quantum mechanically exact statements about a theory, so that the theory is given equivalently completely by the solutions of these equations. Schwinger-Dyson equations as the Ward identities associated with BRST symmetry have been considered previously in Refs. [9–11]. Here our approach is somewhat different, and of course, we apply it to our present example.

We started with variables (Q_α, p_α), $\alpha = 1, \ldots, N$. Introducing the collective variables (q, p), we increased the number of degrees of freedom to $2N+2$. Considering the Lagrange multiplier λ and its canonical momentum p_λ as dynamical variables, the number of degrees of freedom becomes $2N + 4$. The four ghosts $(c, \pi_c; \bar{c}, \pi_{\bar{c}})$ precisely cancel the spurious degrees of freedom so that the original number of $2N$ is recovered. Thus the canonical dynamical variables are $(q, p; Q_\alpha, p_\alpha; \lambda, p_\lambda; c, \pi_c; \bar{c}, \pi_{\bar{c}})$. The auxiliary variable b is related to p_λ (as we shall see below), so that db does not appear in the path integral.

The path integral or partition functional for the theory with first-order Lagrangian (5.8) is given by

$$\begin{aligned} &Z[q, p; Q_\alpha, p_\alpha; \lambda, p_\lambda; c, \pi_c; \bar{c}, \pi_{\bar{c}}] \\ &= \int [dq\, dp][dQ_\alpha dp_\alpha][d\lambda\, dp_\lambda][dc\, d\pi_c][d\bar{c}\, d\pi_{\bar{c}}] \\ &\quad \times \exp\left\{\frac{i}{\hbar}S + \frac{i}{\hbar}\int dt[-b(\dot{\lambda} + hq) - \tfrac{1}{2}b^2 + \dot{\bar{c}}\dot{c} - h\bar{c}c]\right\} \ , \end{aligned} \tag{9.1}$$

where

$$S = \int dt\, L, \quad L = p\dot{q} + p_\alpha \dot{Q}_\alpha - H(p_\alpha, Q_\alpha) - \lambda\varphi \ . \tag{9.2}$$

The expression $L_B = L + L_{\text{GF}}$, where L_{GF} is the gauge-fixing part given in (9.1), is the first-order Lagrangian expressed in terms of coordinates $q, Q_\alpha, \ldots$ and momenta $p, p_\alpha, \ldots$. The velocities appearing in L_B are to be replaced by the expressions given by Hamilton's equations: e.g.,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{Q}_\alpha = \frac{\partial H}{\partial p_\alpha} \ . \tag{9.3}$$

The Hamiltonian H_B associated with the first-order lagrangian L_B is defined by the appropriate Legendre transform: i.e.,

$$H_B = p\dot{q} + p_\alpha \dot{Q}_\alpha + p_\lambda \dot{\lambda} + \pi_c \dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - L_B \ . \tag{9.4}$$

Here we first replace $\dot{\lambda}$ by b and p_λ using

$$p_\lambda = \frac{\partial L_B}{\partial \dot{\lambda}} = -b, \quad 0 = \frac{\partial L_B}{\partial b} = -b - (\dot{\lambda} + hq) \ , \tag{9.5}$$

$$p_b = \frac{\partial L}{\partial \dot{b}} = 0 \ .$$

Then

$$H_B = H + \lambda\varphi + \tfrac{1}{2}p_\lambda^2 - p_\lambda hq + \dot{\bar{c}}\dot{c} + h\bar{c}c \ . \tag{9.6}$$

Thus, with (9.3),

$$\dot{q} = \lambda \ \text{ and } \dot{Q}_\alpha = p_\alpha - \lambda f' \Gamma_{\alpha\beta} Q_\beta \ . \tag{9.7}$$

Inserting these expressions into L_B, we obtain

$$Z = \int [dq\, dp][dQ_\alpha dp_\alpha][d\lambda\, dp_\lambda][dc\, d\pi_c][d\bar{c}\, d\pi_{\bar{c}}] \times \exp\left[\frac{i}{\hbar}\int dt\{\tfrac{1}{2}p_\alpha^2 - V + \tfrac{1}{2}p_\lambda^2 + \dot{\bar{c}}\dot{c} - h\bar{c}c\}\right] \ . \tag{9.8}$$

With Wick rotations and various integrations and dropping the appropriate phase space volume factors, we obtain

$$Z[Q_\alpha, c, \bar{c}] = \int [dQ_\alpha][dc][d\bar{c}] \exp\left[\frac{i}{\hbar} S_{\text{ext}}\right] , \tag{9.9}$$

where (with $S_{\text{eff}}[Q_\alpha] = \int dt\{-V\}$)

$$S_{\text{ext}} = S_{\text{eff}}[Q_\alpha] - h \int dt\, \bar{c}c \ . \tag{9.10}$$

Under the BRST variation (5.6) we obtain

$$\delta S_{\text{ext}} = \frac{\delta S_{\text{ext}}}{\delta Q_\alpha}\delta Q_\alpha + \frac{\delta S_{\text{ext}}}{\delta \bar{c}}\delta\bar{c} = \frac{\delta S_{\text{ext}}}{\delta Q_\alpha} f' \Gamma_{\alpha\beta} Q_\beta c - hbc \ . \tag{9.11}$$

This vanishes and demonstrates the residual BRST invariance after the integrations if

$$b = \frac{1}{h}\frac{\delta S_{\text{ext}}}{\delta Q_\alpha} f' \Gamma_{\alpha\beta} Q_\beta \ . \tag{9.12}$$

In order to obtain such an equation, we consider the invariance of the action of (9.8) under variation of the independent variable q: i.e.,

$$\frac{\delta S}{\delta q} - bh = 0 \ ,$$

i.e.,

$$b = \frac{1}{h}\frac{\delta S}{\delta q} = \frac{1}{h}\frac{\delta S_{\text{eff}}}{\delta q} = \frac{1}{h}\frac{\delta S_{\text{eff}}}{\delta Q_\alpha}\frac{\delta Q_\alpha}{\delta q} \ .$$

Since, from (2.12), $\partial Q_\alpha / \partial q = f' \Gamma_{\alpha\beta} Q_\beta$, this verifies (9.12). In view of the assumed flat measure of the path integral (i.e., its variance under arbitrary local shifts of the variables), we see that the entire path integral is invariant under the residual BRST symmetry. In the present case this result may be trivial, but the reasoning is applicable to more complicated field theory examples.

The residual BRST algebra is given by the variations

$$\delta Q_\alpha = f' \Gamma_{\alpha\beta} Q_\beta c, \quad \delta c = 0 \ , \qquad \delta\bar{c} = b = \frac{1}{h}\frac{\delta S_{\text{eff}}}{\delta Q_\alpha} f' \Gamma_{\alpha\beta} Q_\beta \ . \tag{9.13}$$

The Ward identities which follow from this symmetry are the Schwinger-Dyson equations. Thus, for averages $\langle\cdots\rangle$ with respect to the path integral (9.9), the most general Schwinger-Dyson equation for an arbitrary functional $F[Q_\alpha]$ can be taken to be given by

$$0 = \langle \delta[F(Q_\alpha)\bar{c}] \rangle \ , \tag{9.14}$$

where δ means the variation (9.13). Inserting the variations (9.13), integrating out the ghosts, and remembering that δQ_α are independent since the collective coordinate has been integrated out, we obtain the Schwinger-Dyson equation

$$\left\langle \frac{\delta F(Q_\alpha)}{\delta Q_\alpha} + \frac{1}{h} F(Q_\alpha)\frac{\delta S_{\text{ext}}}{\delta Q_\alpha} \right\rangle = 0 \ . \tag{9.15}$$

X. CONCLUDING REMARKS

In the above we considered a quantum mechanical example in order to demonstrate in a relatively simple context how a theory with constraints, specifically field theories in the neighborhood of some classical configuration, may be quantized in a way which is very similar to methods applied to theories with gauge fields. The constraints of the theory which result in a singular Lagrangian determine the generators of a gauge group under which the first-order Lagrangian is invariant (it is essential to consider the first-order Lagrangian in terms of coordinates and momenta and not the second-order Lagrangian with the momenta integrated out since the singularity of the Lagrangian has its source in a constraint of the momenta). This shows clearly that a theory with constraints becomes a theory with a gauge symmetry. The BRST extension of phase space preserves the invariance of the Lagrangian, but in addition allows the Hamiltonian to become invariant. One can then diagonalize the Hamiltonian and demonstrate that with the help of the anti-BRST symmetry it is only the constraints which project out the physical (BRST and anti-BRST invariant) states of the theory.

The considerations presented here can be applied to theories with classical configurations with finite energy. Thus, in the (1+1)-dimensional scalar field theory for a double-well potential, one has the well-known static kink solution $\phi_c(x - x_0)$. Here x_0 is the position of the kink which becomes a dynamical collective coordinate if we allow it to depend on time t. Thus the transformation

$$\phi(x,t) \to \phi_c(x - x_0(t)) + \eta(x,t) \ ,$$

where η is the fluctuation is a transformation to a larger number of degrees of freedom. This transformation leads to a constraint very similar to (2.11); for details, we refer to Ref. [12]. In the case of a vortex theory in 1+2 dimensions, one has three collective coordinates and hence three constraints; for details, we refer to Ref. [13]. It is clear that all such theories including those with Skyrme interactions or with topologically unstable sphaleron configurations can, in principle, be quantized in a way analogous to the method developed here.

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