Nonvacuum bounces and quantum tunneling at finite energy

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A bouncelike solution with nonzero energy is used for the explicit calculation of the decay rate of an excited state of the inverted double-well potential. Three negative eigenvalues are associated with the second variation of the action at the nonvacuum bounce which is different from the case of the vacuum bounce. The imaginary part of the energy results only from the symmetry of the bounce itself, and the additional negative eigenvalues do not present difficulties. In fact, it is shown explicitly that only one negative eigenvalue contributes to the tunneling. The calculated imaginary part of the energy of the excited state is in agreement with that of WKB calculations. The tunneling effects are investigated for high and low energies compared with the barrier height. Finally the Bogomolny-Fateyev relation is established, thus checking our results.

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I. INTRODUCTION

The instanton method is by now well known as a powerful tool for dealing with quantum tunneling phenomena. Most of the calculations which have been given in the literature are based on vacuum instantons, namely, kinks, with nontrivial topological charge, and these kinks are responsible for quantum tunneling between neighboring degenerate vacua [1] of the double-well or sine-Gordon potential. It was demonstrated long ago by Coleman and Callan [2] that the quantum tunneling process for the decay of a metastable ground state (or "false vacuum") as in the case of the inverted double-well potential, is, however, dominated by a nontopological pseudoparticle configuration named "bounce," which is not a minimum of the action but a saddle point. The second variational derivative of the Euclidean action at the bounce has one negative eigenvalue which leads to the imaginary part of the energy. An explicit calculation of the imaginary part of the energy based on such vacuum bounces has recently been given in [3].

Instanton transitions which relate to the possibility of baryon- and lepton-number violation in electroweak theory have attracted widespread attention [4]. It has gradually been realized that vacuum instantons and vacuum bounces which prescribe vacuum boundary conditions may not be adequate for the description of tunneling at finite, nonzero energy [5]. The investigation of quantum tunneling with a new type of instantonlike configurations which are characterized by nonzero energy and satisfy manifestly nonvacuum boundary conditions is therefore of great interest [6–11]. Previously we extended the calculation of quantum tunneling between vacuum states by means of periodic instantons; this was the case of the double-well potential in which the Hamiltonian is bounded from below and self-adjoint [8]. In this case the tunneling effect gives rise to the level splitting. In the present work we develop a procedure for dealing with quantum tunneling away from an excited state of the inverted double-well potential, this being the case in which the Hamiltonian is not bounded from below and is not self-adjoint, so that the energy eigenvalues are complex. In this case the tunneling is dominated by nonvacuum bounces and leads to the decay of the excited state instead of the metastable ground state as in the situation with vacuum bounces [3]. The periodic case to be considered here is that related to a finite-temperature process.

The case considered here differs from the vacuum case [3] in that the operator associated with the second variation of the Euclidean action at the nonvacuum bounce (which one can also call a periodic bounce) has more than one negative eigenvalue [11]. At a first glance this seems to present a difficulty. However, we show explicitly that the imaginary part of the energy results directly from characteristic properties of the bounce itself, namely, the antisymmetry of its first time derivative under time reversal. This is in agreement with a general argument given by Coleman [9] that if there exist two or more negative eigenvalues of the second variational derivative of the action at the bounce only one has to do with tunneling.

In Sec. II we recall from [11] the bounce with nonzero energy for an inverted double-well potential, and the associated equation of small fluctuations about it. In Secs. III and IV, we present the major part of the procedure for the path-integral calculation of the tunneling process. The tunneling behavior for the cases (a) of energy far below the barrier height, and (b) of an energy approaching the barrier height is studied in Sec. V. Finally in Sec. VI, the Bogomolny-Fateyev relation is established for quantum tunneling through the central barrier of the double-well potential.

II. BOUNCES WITH NONZERO ENERGY FOR THE INVERTED DOUBLE-WELL POTENTIAL

The Lagrangian for a scalar field $\phi(t)$ in one time and zero-space dimensions is

$$L = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi) \tag{2.1}$$

using unit mass and $\hbar = 1$ throughout. The potential $V(\phi)$ we use is the inverted double-well potential with a local minimum at $\phi = 0$ given by

$$V(\phi) = -\frac{\mu^2}{2a^2}(\phi^2 - a^2)^2 + \tfrac{1}{2}\mu^2 a^2 = \mu^2\phi^2 - \tfrac{1}{2}\frac{\mu^2}{a^2}\phi^4 ,$$

where μ and a are real parameters. The classical solution ϕ_c which minimizes the action with Euclidean time $\tau = it$, satisfies the equation

$$\frac{1}{2}\left(\frac{d\phi_c}{d\tau}\right)^2 - V(\phi_c(\tau)) = -E_{\rm cl} . \tag{2.2}$$

The solution ϕ_c is the trajectory of the classical pseudoparticle. The integration constant $E_{\rm cl} \geq 0$ can be regarded as the energy of the classical particle. If one wishes, one can look at the solution as describing the motion of a real particle with energy $-E_{\rm cl}$ in the potential $-V(\phi)$. It is convenient to set

$$E_{\rm cl} = \tfrac{1}{2}a^2\mu^2u^2, \quad u = \frac{1-k^2}{1+k^2}, \quad 0 \leq k \leq 1 .$$

This substitution defines the parameter k which varies from 0 to 1 as $E_{\rm cl}$ varies between its extreme values: i.e.,

$$-\tfrac{1}{2}\mu^2a^2 \leq -E_{\rm cl} \leq 0 .$$

We demand that $\phi_c(\tau)$ be periodic with period $\mathcal{T}$ so that

$$\phi_c(\tau) = \phi_c(\tau + \mathcal{T}) .$$

The solution of (2.2) is then given by

$$\phi_c(\tau) = s_+(k)dn[\beta(k)(\tau+\tau_0)|\gamma] , \tag{2.3}$$

where dnu (like snu, cnu) denotes a Jacobian elliptic function, τ_0, an integration constant, defines the position of the bounce, and γ is the modulus of the elliptic functions which is related to k by

$$\gamma^2 = \frac{4k}{(1+k)^2} ,$$

$$\gamma'^2 = 1 - \gamma^2 = \left(\frac{1-k}{1+k}\right)^2 = \frac{1-u'}{1+u'} ,$$

$$u'^2 = 1 - u^2 .$$

The other k-dependent parameters in (2.3) are given by

$$\beta(k) = \frac{\mu}{a}s_+(k) ,$$

$$s_+(k) = \frac{a(1+k)}{\sqrt{1+k^2}} = a\left(\frac{2}{1+\gamma'^2}\right)^{1/2} \equiv s_+(\gamma) .$$

The Jacobian elliptic function $dn[\beta(k)\tau|\gamma]$ has period

$$\beta(k)\mathcal{T} = n2\mathcal{K}(\gamma), \quad n = 1, \ldots , \tag{2.4}$$

where $\mathcal{K}(\gamma)$ is the elliptic quarter period or complete elliptic integral of the first kind. The pseudoparticle oscillates from turning point $\tilde{a}$ to $\tilde{a}'$ and back in the barrier as shown in Fig. 1. Setting $\mathcal{T} \equiv 2T$ and taking $n = 1$, we have

$$\beta(k)T = \mathcal{K}(\gamma)$$

where T is half the period of the motion of the pseudoparticle as indicated in Fig. 1. As the energy tends to zero with $k \to 1$, solution (2.3) reduces to the usual vacuum bounce: i.e.,

$$\phi_c(\tau) \to a\sqrt{2}{\rm sech}[\mu\sqrt{2}(\tau+\tau_0)] .$$

For the sake of a better distinction we dub the new solution of Eq. (2.3) a "nonvacuum bounce" or periodic bounce. On the other hand, as the energy approaches the top of the barrier, i.e., $E_{\rm cl} = \frac{1}{2}\mu^2a^2$, with $k \to 0$, the solution becomes the trivial configuration $\phi_c = a$ since [12] $dn[u|\gamma] = 1$ for $\gamma = 0$. This trivial solution is called a sphaleron [10,11]. The nonvacuum bounce thus interpolates between the vacuum bounce and this sphaleron.

The small fluctuation equation about the classical solution $\phi_c(\tau)$ is

$$\hat{M}\psi = \omega^2\psi \tag{2.5}$$

with

$$\hat{M} = -\frac{d^2}{d\tau^2} + \mu^2\left(2 - \frac{6}{a^2}\phi_c^2\right)$$

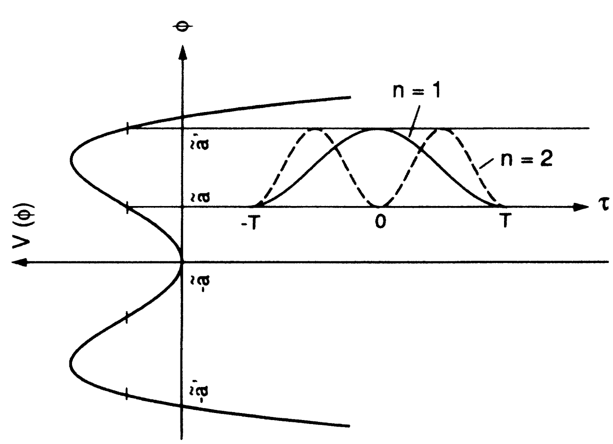

FIG. 1. Inverted double-well potential and trajectories of bounces for $n = 1$ and 2.

and can be reexpressed as the Lamé equation

$$\frac{d^2\psi}{dz^2} + \{\omega^2 - 6\gamma^2 sn^2[z|\gamma]\}\psi = 0 \ , \qquad (2.6)$$

where $z = \beta(k)\tau$. It will be seen later that $\hat{M}$ is simply the second variational derivative of the Euclidean action at the bounce ϕ_c. The discrete eigenmodes $\psi_1, \psi_2, \ldots, \psi_5$ of (2.6) are given in [11] as

$$sn[z|\gamma]cn[z|\gamma], sn[z|\gamma]dn[z|\gamma] \ ,$$

$$cn[z|\gamma]dn[z|\gamma] \ ,$$

and

$$sn^2[z|\gamma] - \frac{1}{3\gamma^2}[1 + \gamma^2 \pm \sqrt{1 - \gamma^2\gamma'^2}] \qquad (2.7a)$$

with eigenvalues (expressed in terms of k): $\omega^2 \rightarrow \omega_1^2, \omega_2^2, \ldots, \omega_5^2$ given by

$$0, -\frac{3\mu^2(1-k)^2}{1+k^2}, -\frac{3\mu^2(1+k)^2}{1+k^2}$$

and

$$-2\mu^2 \mp 2\mu^2 \frac{\sqrt{1+14k^2+k^4}}{1+k^2} \ , \qquad (2.7b)$$

respectively. For $k \in (0, 1)$ there are three negative eigenvalues, their respective eigenmodes having periods $\mathcal{T} = 4n\mathcal{K}(\gamma)/\beta(k)$. The topological charge of ϕ_c is zero (as a consequence of the periodicity of ϕ_c) and therefore ϕ_c is an unstable, nontopological pseudoparticle configuration.

III. QUANTUM TUNNELING FROM AN EXCITED STATE

We let $|E\rangle$ be an eigenstate of the Hamiltonian $\hat{H}$ with energy E. Because of tunneling and escape to infinity, E becomes a complex number in the case under discussion, with the imaginary part characterizing the rate of decay of the state. The quantity we wish to calculate is precisely this imaginary part of the energy E.

To begin with we consider the transition amplitude from the state $|E\rangle$ to itself due to quantum tunneling in Euclidean time $2T$. When there is no tunneling this is

$$A = \langle E|e^{-2\hat{H}T}|E\rangle = e^{-2ET}, \ \ \langle E|E\rangle = 1 \ . \qquad (3.1)$$

In general the amplitude can be calculated with the help of the path-integral method. We rewrite it

$$A = \int \psi_E^*(\phi_f)\psi_E(\phi_i)K(\phi_f, \tau_f; \phi_i, \tau_i)d\phi_f d\phi_i \qquad (3.2)$$

with $\phi_f = \phi(\tau_f)$, $\phi_i = \phi(\tau_i)$, and $\tau_f - \tau_i = 2T$.

Thus ϕ_i and ϕ_f denote the end points of the bounce motion which tend to the turning point a with $\phi(\tau_i) \equiv \phi_i = \phi(\tau_f) \equiv \phi_f \rightarrow a$. The Feynman propagator from ϕ_i to ϕ_f resulting from the bounce motion is defined by

$$K(\phi_f, \tau_f; \phi_i, \tau_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\{\phi\} \exp(-S) \ , \qquad (3.3)$$

where

$$S = \int_{\tau_i}^{\tau_f} \left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + V(\phi)\right] d\tau \qquad (3.4)$$

is the classical action. Here, i.e., in (3.2),

$$\psi_E(\phi_f) \equiv \langle\phi_f|E\rangle \ \text{ and } \ \psi_E(\phi_i) \equiv \langle\phi_i|E\rangle$$

are wave functions to be specified later.

IV. THE IMAGINARY PART OF THE ENERGY

We derive the imaginary part of the energy by considering the amplitude A as the sum of contributions from any number of bounces. The first contribution $A^{(0)}$ in the sum is that of no bounces, and so of no tunneling. In this approximation the shoulders of the inverted double-well potential are infinitely high and the eigenenergy E of (3.1) is the energy of that excited state of the harmonic oscillator with minimum at $\phi = 0$, which is closest to the classical energy $E_{\rm cl}$. Remembering that we assume unit mass of the pseudoparticle and $\hbar \equiv 1$ this implies

$$A^{(0)} = e^{-2E_{\rm cl}T}$$

with $E_{\rm cl} \simeq (2n+1)(\mu/\sqrt{2})$, $n = 0, 1, \ldots$. Next we consider the various bounce contributions.

A. The one bounce contribution

The Feynman propagator or kernel K defined by (3.3) can be evaluated with the standard path-integral method. Considering fluctuations about the bounce $\phi_c(\tau)$ we set

$$\phi(\tau) = \phi_c(\tau) + \chi(\tau) \ , \qquad (4.1)$$

where $\chi(\tau)$ denotes the small deviation of ϕ from the classical trajectory with end points held fixed. Thus necessary boundary conditions for $\chi(\tau)$ are

$$\chi(\tau_f) = \chi(\tau_i) = 0 \ . \qquad (4.2)$$

Substituting (4.1) for $\phi(\tau)$ in Eq. (3.3), we obtain

$$K = \exp[-S_c(\phi)] \cdot I \ , \qquad (4.3a)$$

where I is the functional integral

$$I = \int_{\chi(\tau_i)=0}^{\chi(\tau_f)=0} \mathcal{D}\{\chi\} \exp[-\delta S] \ . \qquad (4.3b)$$

Here

$$S_c(\phi)=\int_{\tau_i}^{\tau_f}d\tau\left[\frac{1}{2}\left(\frac{d\phi_c}{d\tau}\right)^2+\mu^2\phi_c^2-\frac{1}{2}\frac{\mu^2}{a^2}\phi_c^4\right]$$
$$=\int_{\tau_i}^{\tau_f}d\tau\left[\left(\frac{d\phi_c}{d\tau}\right)^2+E_{\rm cl}\right] \qquad (4.4)$$

and up to terms of $O(\chi^2)$ (for weak coupling) δS is given by

$$\delta S=\frac{1}{2}\int_{\tau_i}^{\tau_f}d\tau\left[-\chi\frac{d^2\chi}{d\tau^2}+2\chi\left(\mu^2-\frac{3\mu^2}{a^2}\phi_c^2\right)\chi\right]$$
$$\equiv\frac{1}{2}\int_{\tau_i}^{\tau_f}d\tau\,\chi\hat{M}\chi\ , \qquad (4.5)$$

where $\hat{M}$ is the operator defined by (2.5). Choosing the bounce position $\tau_0=0$ and inserting into (4.4) the derivative of $\phi_c(\tau)$, i.e.,

$$\frac{d\phi_c}{d\tau}=-s_+(k)\beta(k)\gamma^2 sn[\beta(k)\tau|\gamma]cn[\beta(k)\tau|\gamma]\ , \qquad (4.6)$$

we can evaluate (4.4) with the help of tables of integrals [12] and find with $\tau_i=-T$, $\tau_f=+T$, and $T=\mathcal{K}(\gamma)/\beta(k)$

$$S_c(\phi)=W(\phi(\tau_f),\phi(\tau_i),E_{\rm cl})+2E_{\rm cl}T\ , \qquad (4.7)$$

where for $\phi(\tau_i)\to\tilde{a}$, $\phi(\tau_f)\to\tilde{a}$ (cf. Fig. 1)

$$W(\phi(\tau_f),\phi(\tau_i),E_{\rm cl})\to\gamma^4 s_+^2(k)\beta(k)\int_{-\mathcal{K}(\gamma)}^{\mathcal{K}(\gamma)}du\,sn^2[u|\gamma]cn^2[u|\gamma]$$
$$=\tfrac{2}{3}s_+^2(k)\beta(k)\{(2-\gamma^2)E[\mathcal{K}(\gamma)]-2\gamma'^2\mathcal{K}(\gamma)\}\ . \qquad (4.8)$$

Here $E[\mathcal{K}(\gamma)]$ is the complete elliptic integral of the second kind.

The traditional way of deriving the imaginary part of the energy as described by Coleman and Callan [2] is to expand χ in terms of the eigenfunctions of $\hat{M}$, i.e., those of (2.5). The evaluation of I then leads to a divergent Gaussian integral as a result of an associated negative eigenvalue. By continuation into the complex plane the integral acquires an imaginary part. One therefore needs one and only one negative eigenmode. In our case, however, there are three negative eigenmodes. Nonetheless, this does not lead to a difficulty here, since, as will be seen, the boundary condition (4.2) selects precisely one of these modes, namely the third of (2.7a), $cn[z|\gamma]dn[z|\gamma]$, which together with the zero mode $sn[z|\gamma]cn[z|\gamma]$ contributes to the path integral.

Instead of expanding $\chi(\tau)$ in terms of the eigenmodes of $\hat{M}$, we resort to an alternative method in evaluating the functional integral I for our purposes. We perform the transformation to $y(\tau)$ given by

$$\chi(\tau)=y(\tau)+N(\tau)\int_{\tau_i}^{\tau}\frac{\dot{N}(\tau')}{N^2(\tau')}y(\tau')d\tau'\ , \qquad (4.9)$$

where

$$N(\tau)=\frac{d\phi_c(\tau)}{d\tau}$$

is the unnormalized zero mode of the small fluctuation equation (2.5). The evaluation of the path-integral I of Eq. (4.3) can then be carried out by direct integration [8,13,14]. The result is (as we show in Appendix B)

$$I=\left[\frac{1}{2\pi}\right]^{1/2}\left[\frac{1}{N(\tau_f)N(\tau_i)}\right]^{1/2}\left[\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]^{-1/2} \qquad (4.10)$$

with $\tau_i=-T,\tau_f=T$.

We observe that the bounce solution ϕ_c given by (2.3) is symmetric under time reversal [since $dn(u)=dn(-u)$] whereas its derivative $N(\tau)=d\phi_c/d\tau$ given by (4.6) is antisymmetric [since $sn(-u)=-sn(u)$ and $cn(-u)=cn(u)$]. The square root of $N(T)N(-T)$ therefore implies that I is imaginary. In Appendix A, we show that in fact only one of the three eigenmodes with negative eigenvalue actually contributes to the quantum tunneling.

The propagator (4.3a) for motion from the turning point $\phi=\tilde{a}$ and back to $\tilde{a}$ (i.e., $\tau=\pm T$) is divergent because of the vanishing velocity at turning points [i.e., $cn(\pm\mathcal{K})=0$ in (4.6)]. This is unlike the case of vacuum bounces which can reach the turning points asymptotically.

The transition amplitude, of course, has to be finite, and hence the singularity of the propagator has to be smoothed out by the end-point integrations of $d\phi_i$ and $d\phi_f$. To this end we use the following relations established in Appendix C:

$$I=\left[\frac{1}{2\pi}\right]^{1/2}\left[\frac{N(\phi_f)}{N(\phi_i)}\right]^{1/2}\left[\frac{\partial^2 S_c(\phi_f,\phi_i;T)}{\partial\phi_f^2}\right]^{1/2}$$
$$\times\Delta(\phi_f,\phi_i)\ , \qquad (4.11a)$$

where

$$\frac{\partial^2 S_c(\phi_f,\phi_i;T)}{\partial\phi_f^2}$$
$$=\frac{1}{N(\phi_f)}\left[\frac{\partial^2\phi_f}{\partial\tau^2}+\frac{1}{N(\phi_f)\int_{\tau_i}^{\tau_f}[d\tau/N^2(\tau)]}\right] \qquad (4.11b)$$

and

$\Delta(\phi_f,\phi_i)$

$$= \left[\frac{\partial^2 S_c(\phi_f,\phi_i;T)}{\partial\phi_f^2}N^2(\phi_f)\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]^{-1/2} \quad (4.12a)$$

with

$$\Delta \equiv \lim_{\substack{\phi_f\to\tilde{a}\\ \phi_i\to\tilde{a}}} \Delta(\phi_f,\phi_i) = -i\,. \quad (4.12b)$$

Equation (4.11a) follows, of course, from (4.10), (4.11b), and (4.12a).

We evaluate the amplitude A of (3.2) by replacing the wave functions $\psi_E(\phi_f)$, $\psi_E(\phi_i)$ by their leading WKB approximations, and by expanding the action S in (4.12) (which is a function of the endpoints ϕ_i and ϕ_f) in powers of $\phi_f - \phi_c(T)$ up to the second power for the Gaussian or one-loop approximation. Thus we write

$$\begin{aligned} S_c(\phi) &\equiv S_c(\phi_f,\phi_i,T) \\ &= S_c[\phi(T),\phi(-T),T] \\ &\quad + \frac{1}{2}\frac{\partial^2 S}{\partial\phi(T)^2}[\phi_f-\phi(T)]^2+\cdots\,. \end{aligned} \quad (4.13)$$

We define

$$\tilde{a} \equiv \phi(-T) = \phi(T), \quad \tilde{a}' \equiv \phi(0)$$

as turning points. Inserting (2.3) here we obtain

$$\tilde{a} = \frac{a(1-k)}{\sqrt{1+k^2}}, \quad \tilde{a}' = \frac{a(1+k)}{\sqrt{1+k^2}}\,. \quad (4.14)$$

These turning points are shown in Fig. 1. The WKB approximations of the wave functions are given by

$$\begin{aligned} \psi_E(\phi_f) &= C\frac{\exp\left(-\int_{\tilde{a}}^{\phi_f}\dot{\phi}d\phi\right)}{\sqrt{\dot{\phi}_f}}\,, \\ \psi_E(\phi_i) &= C\frac{\exp\left(-\int_{\tilde{a}}^{\phi_i}\dot{\phi}d\phi\right)}{\sqrt{\dot{\phi}_i}}\,. \end{aligned} \quad (4.15)$$

The normalization constant C is defined, as usual, by

$$C = \left[\frac{1/2}{\int_{-\tilde{a}}^{\tilde{a}}\frac{d\phi}{\sqrt{2[E-V(\phi)]}}}\right]^{1/2}, \quad (4.16)$$

where the integration extends from turning point to turning point across the nontunneling domain (i.e., the region of the harmonic-oscillator approximation). Evaluating C one obtains

$$C = \left[\frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')}\right]^{1/2} \quad (4.17)$$

[using integrals given in [12] and the relation $F[\pi/2,\gamma'] = \mathcal{K}(\gamma')$]. The end-point integrations in Eq. (3.2) can now be carried out for the one-bounce contribution $A^{(1)}$. We have, for $\tau_f\to T$, $\tau_i\to -T$, using (4.3), (4.11), (4.12), and (4.15) and writing $d\phi_i = \dot{\phi}_i d\tau$,

$$\begin{aligned} A^{(1)} &= \int \psi_E^*(\phi_f)\psi_E(\phi_i)K(\phi_f,\tau_f;\phi_i,\tau_i)d\phi_f d\phi_i \\ &= \int \psi_E^*(\phi_f)\psi_E(\phi_i)I\exp[-S_c(\phi)]d\phi_f d\phi_i \\ &\simeq \frac{C^2}{\sqrt{2\pi}}\int\frac{d\phi_f d\phi_i\Delta}{\dot{\phi}_i}\left(\frac{\partial^2 S_c}{\partial\phi(T)^2}\right)^{1/2}\exp[-S_c(\phi)] \\ &\simeq \frac{C^2}{\sqrt{2\pi}}\int_{-T}^{T}d\tau\int\Delta\left(\frac{\partial^2 S_c}{\partial\phi(T)^2}\right)^{1/2}d\phi_f e^{-S_c[\phi(T),\phi(-T),T]}\exp\left[-\frac{1}{2}\frac{\partial^2 S_c}{\partial\phi(T)^2}[\phi_f-\phi(T)]^2\right] \\ &\simeq -i2TC^2e^{-W}e^{-2E_{\rm cl}T}\,, \end{aligned} \quad (4.18)$$

where in the last step we used (4.12b) and (4.7). Inserting (4.17) this becomes

$$A^{(1)} = (-i)2Te^{-W}e^{-2E_{\rm cl}T}\frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')}\,. \quad (4.19)$$

B. Summing over an infinite number of bounces

The path integral implies a sum over all possible paths. The one-bounce contribution is that of the classical configuration (2.3) with period $\mathcal{T} = n2\mathcal{K}(\gamma)/\beta(k)$ [Eq. (2.4)] with $n=1$. For $n=2$ we have (since $\mathcal{T}=2T$) $\beta(k)T = 2\mathcal{K}(\gamma)$, and so there are two bounces moving from $-T$ to $+T$ with "positions" $\tau_0 = \pm\mathcal{K}(\gamma)/\beta(k)$ as shown in Fig. 1. The contribution $A^{(2)}$ to the transition amplitude arising from two bounces can be calculated in analogy to the contribution of an instanton–anti-instanton pair to the leading instanton contribution as discussed in [1]. Thus

$$A^{(2)} = \int_{-T}^{T} d\tau \int_{-T}^{\tau} d\tau_1 (-i)^2 e^{-2W} e^{-2E_{\rm cl}T} \left(\frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')} \right)^2$$
$$= (-i)^2 \frac{(2T)^2}{2!} \left[\frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')} \right]^2 e^{-2W} e^{-2E_{\rm cl}T} . \tag{4.20}$$

The generalization to n bounces is now straightforward. We have

$$A^{(n)} = (-i)^n \frac{(2T)^n}{n!} \left[\frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')} \right]^n e^{-nW} e^{-2E_{\rm cl}T} . \tag{4.21}$$

The total transition amplitude A which results from quantum tunneling dominated by bounces is obtained by summation: i.e.,

$$A = \sum_n A^{(n)} = e^{-2E_{\rm cl}T} \exp\left\{ -i2T \frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')} e^{-W} \right\} . \tag{4.22}$$

Comparing this expression with (3.1) we see that the imaginary part of the energy is given by

$$\mathrm{Im}E = \frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')} e^{-W} ,$$

where W is given by (4.8): i.e.,

$$\mathrm{Im}E = \frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')} \exp\left[-\frac{2^{5/2}}{3} \frac{\mu a^2}{(1+\gamma'^2)^{3/2}} \{(2-\gamma^2)E[\mathcal{K}(\gamma)] - 2\gamma'^2\mathcal{K}(\gamma)\} \right] . \tag{4.23}$$

This expression is nothing but a WKB formula which is similar to the well-known level-splitting formula [8,15,16] for a double-well potential.

V. LOW- AND HIGH-ENERGY LIMITS

It is interesting to investigate the low- and high-energy limits of Eq. (4.23). By "low" and "high" energies we mean energies far below or near the barrier maximum, respectively.

(a) For the energy far below the barrier maximum $E_{\rm cl} \ll \frac{1}{2}a^2\mu^2$ and therefore $k^2 \to 1$ or $\gamma^2 \to 1$. For $\gamma \to 1$ the quarter period $\mathcal{K}(\gamma) \to \infty$ and [12] $E(\mathcal{K} = \infty) = E(k=1)$. The expansions of $E(\gamma)$ and $\mathcal{K}(\gamma)$ in this domain (or correspondingly for $\gamma'^2 \to 0$) are [12]

$$E(\gamma) = 1 + \frac{1}{2}\left\{ \ln\left(\frac{4}{\gamma'}\right) - \frac{1}{2} \right\}\gamma'^2 + \frac{3}{16}\left\{ \ln\left(\frac{4}{\gamma'}\right) - \frac{13}{12} \right\}\gamma'^4 + \cdots \tag{5.1}$$

and

$$\mathcal{K}(\gamma) = \ln\left(\frac{4}{\gamma'}\right) + \frac{1}{4}\left[\ln\left(\frac{4}{\gamma'}\right) - 1 \right]\gamma'^2 + \cdots . \tag{5.2}$$

The parameters k or γ parametrize the energy $E_{\rm cl}$. The latter is quantized either by reference to the harmonic oscillator at the central well (with mass and $\hbar = 1$) or by using the Bohr-Sommerfeld condition

$$\int_{-\tilde{a}}^{\tilde{a}} \frac{d\phi}{dt} d\phi = (n + \tfrac{1}{2})\pi$$

and keeping only terms up to those of $O(\gamma'^2)$. In either case one finds

$$E_{\rm cl} = (n + \tfrac{1}{2})\omega, \quad \omega = \mu\sqrt{2} \tag{5.3}$$

and $n = 0, 1, \ldots$.

Inserting $E(\gamma)$ and $\mathcal{K}(\gamma)$ into the exponent of (4.23) we obtain, for W,

$$W = \frac{2^{5/2}\mu a^2}{3}\left[1 - \frac{3}{4}\gamma'^2 - \frac{3}{2}\gamma'^2 \ln\left(\frac{4}{\gamma'}\right) \right] . \tag{5.4}$$

Setting

$$g^2 \equiv \frac{1}{\mu a^2}$$

(a dimensionless coupling constant) then, since

$$\gamma^2 = \frac{1-u'}{1+u'} \simeq \frac{u^2}{4} ,$$

we obtain in the domain of weak coupling, i.e., for $g^2 \ll 1$,

$$W = \frac{4\sqrt{2}}{3g^2}\left[1 - \frac{3}{16}u^2 - \frac{3}{8}u^2 \ln\left(\frac{8}{u}\right) \right] . \tag{5.5}$$

Since $E_{\rm cl} = \frac{1}{2}a^2\mu^2u^2$, we obtain, with (5.3),

$$u^2 = 2\sqrt{2}(n + \tfrac{1}{2})g^2 . \tag{5.6}$$

Inserting this into (5.5) implies

$$W=\frac{4\sqrt{2}}{3g^2}-(n+\tfrac{1}{2})-(n+\tfrac{1}{2})\ln\left[\frac{2^{9/2}}{g^2(n+\frac{1}{2})}\right] . \quad (5.7)$$

Inserting this into (5.7) [where in the domain under discussion $k\simeq 1$ and $\mathcal{K}(\gamma')\simeq\mathcal{K}(0)=\frac{1}{2}\pi$] we obtain

$$\mathrm{Im}E=\frac{\mu}{\pi\sqrt{2}}e^{-4\sqrt{2}/3g^2}e^{n+1/2}\left[\frac{16\sqrt{2}}{g^2(n+\frac{1}{2})}\right]^{n+1/2} . \quad (5.8)$$

The exponential suppression factor is seen to be identical with that for vacuum bounces [3]. An interesting observation is that for the case of weak coupling under discussion the tunneling effect described by (5.8) indeed grows with energy (i.e., n) exponentially, of course in the domain of validity of the expansions used in deriving (5.8), i.e., for

$$2\sqrt{2}g^2(n+\tfrac{1}{2})=u\ll 1 .$$

This observation is in agreement with common belief [17] in the analysis of baryon- and lepton-number violation at high energies or high temperature in models possessing instantons. Using Stirling's formula in the form

$$\left(\frac{e}{n+\frac{1}{2}}\right)^{n+1/2}\simeq\frac{\sqrt{2\pi}}{n!}$$

[obtained with the help of $e^z=\lim_{n\to\infty}(n+z/n)^n$] we can rewrite (5.8) as

$$\mathrm{Im}E=\frac{\mu}{\sqrt{\pi}n!}\left[\frac{16\sqrt{2}}{g^2}\right]^{n+1/2}e^{-4\sqrt{2}/3g^2} . \quad (5.9)$$

This result agrees with the complex energy eigenvalue of the Schrödinger equation for the inverted double-well potential obtained by Bender and Wu [18] using a WKB analysis, and by others [19] with an alternative method. It also reduces to the result obtained from a calculation with vacuum bounces for the metastable ground state [3]. (In the comparison with the literature, e.g., [19], it must be remembered that here we take the mass of the particle to be one, whereas in quantum-mechanics calculations one frequently takes one half; thus in the comparison with [19] E there is $2E$ here, h^2 there is $2\sqrt{2}\mu$ here, and C^2 is $2\mu^2/a^2$ here.)

(b) If the energy approaches the barrier height, i.e., $E\to\frac{1}{2}a^2\mu^2$ with k or $\gamma\to 0$, the complete elliptic integrals $E(\gamma)$ and $\mathcal{K}(\gamma)$ in (4.23) have to be expressed as power series in ascending powers of γ, i.e.,

$$E[\mathcal{K}(\gamma)]=\frac{\pi}{2}[1-\tfrac{1}{4}\gamma^2+\cdots] ,$$

$$\mathcal{K}(\gamma)=\frac{\pi}{2}[1+\tfrac{1}{4}\gamma^2+\cdots] ,$$

and the argument of the exponential in (4.23) is

$$-W=0 \quad\text{with } \gamma^2\to 0 . \quad (5.10)$$

Looking at (4.22) one might conclude that the transition amplitude is then not suppressed by the typical vacuum tunneling factor $\exp[-4\sqrt{2}/3g^2]$. However, the prefactor in (4.23), i.e.,

$$\frac{\mu(1+k)}{4\sqrt{1+k^2}\mathcal{K}(\gamma')}\simeq\frac{\mu}{4\ln(4/\gamma)} . \quad (5.11)$$

also approaches zero as k or $\gamma\to 0$, and thus the transition amplitude is again suppressed (though not by the vacuum tunneling factor). This phenomenon might appear as a new observation for quantum tunneling at high energy (where, one could think naively, the tunneling is not suppressed), but this interpretation originates obviously from a semiclassical point of view. When the energy is very high the effect of anharmonic oscillations becomes important for the inverted double-well potential, and the effective frequency, namely the number of impacts per unit time at the turning points approaches zero.

We close this section with a comment on the relation of the present work in the high-energy limit to that of [17]. We consider here principally the inverted double-well potential. The uninverted double-well potential was discussed in [8]. There we considered N to N transitions between the wells whereas the authors of [17] consider vacuum (0) to N transitions so that their penetration length tends to half of that for the vacuum to vacuum transitions as the energy approaches the barrier height.

VI. THE BOGOMOLNY-FATEYEV RELATION

In discussions of the large-order behavior of perturbation theory in gauge theories Bogomolny and Fateyev [20] estimated the behavior of the coefficients of the perturbation expansion around a nonunique vacuum state. It is pointed out that in general the classical vacuum state does not coincide with the true quantum-mechanical ground state. Therefore although the exact ground state is stable, the perturbation theory vacuum is only a metastable one due to the possibility of tunneling to the other vacuum state. If one considers perturbation theory around a local minimum, the perturbation theory state is metastable. The imaginary part of the energy of the metastable ground state can be calculated by the method of steepest descent. For the double-well potential

$$V(\phi)=\tfrac{1}{2}\lambda^2\left[\phi^2-\frac{1}{\lambda^2}\right]^2 . \quad (6.1)$$

Bogomolny and Fateyev [21] find the relation

$$\Delta E=2\pi i(\delta E)^2 , \quad (6.2)$$

where ΔE is the discontinuity of the ground-state energy at the cut $\lambda^2\geq 0$, while δE is the instanton contribution to the real part of the ground-state energy, namely, the level shift due to quantum tunneling [8]. Relation (6.2) has been reconsidered [19] recently by solving Schrödinger equations using modified WKB methods. Since there is no real decay in the system, the calculated imaginary part is the probability of tunneling away from one minimum only. Therefore the potential is considered in two forms in [19]. The first form considered, (6.1), leads to the level splitting δE, while the second, shifted form

$$V(\phi) = \begin{cases} \frac{1}{2}\lambda^2\left(\phi^2 - \frac{1}{\lambda^2}\right)^2 & \text{for } \phi \le \frac{1}{\lambda} \ , \\ -\frac{1}{2}\lambda^2\left(\phi^2 - \frac{1}{\lambda^2}\right)^2 & \text{for } \phi > \frac{1}{\lambda} \ , \end{cases} \tag{6.3}$$

as shown in Fig. 2, results in the imaginary part of the energy, ImE, for a "real" metastable ground state. Since $\Delta E = 2i\,\text{Im}E$, one has the equivalent relation

$$\text{Im}E = \pi(\delta E)^2 \tag{6.4}$$

which has been verified and extended to excited states (low-energy case) in [19] by comparing ImE and δE for the two systems. Formula (6.4) serves as a crucial test of the validity of calculating quantum tunneling effects with nonvacuum instantons [8] and bounces. In our earlier paper [8] the level splitting for the excited states of potential (6.1) was obtained with nonvacuum instantons which we also called periodic instantons. The classical solution which extremizes the Euclidean action is [11]

$$\phi_c(\tau) = \frac{kb(k)}{\lambda} sn[b(k)(\tau + \tau_0)|k] \ , \tag{6.5}$$

where

$$k^2 = \frac{1-u}{1+u}, \quad u = \lambda\sqrt{2E}, \quad b(k) = \left[\frac{2}{1+k^2}\right]^{1/2} .$$

The Jacobian elliptic function $sn[z|k]$ has period

$$\mathcal{T} = 4n\mathcal{K}(k) \ . \tag{6.6}$$

The small fluctuation equation about $\phi_c(\tau)$ of (6.5) is also a Lamé equation and the eigenmodes are the same as for nonvacuum bounces (cf. Sec. II) but with different eigenvalues [11]. In the calculation of the level splitting the solution for a half period is regarded as an instanton configuration, whereas the solution for a full period is a pair of instanton and anti-instanton configurations. The trajectory [for $n = 1$ and $\tau_0 = 0$ in Eq. (6.5)] of one

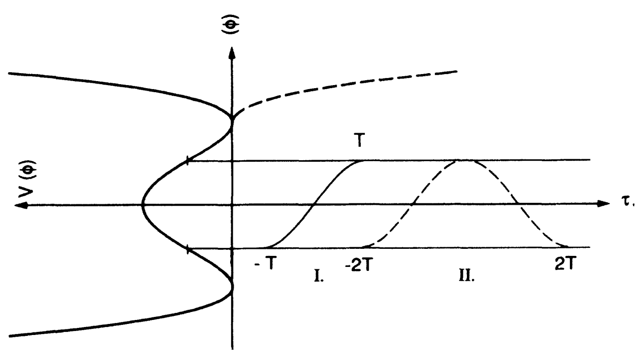

FIG. 2. Motion of a single nonvacuum instanton (I) and a single nonvacuum bounce (II) in the central barrier of the double-well potential.

nonvacuum instanton (or periodic instanton in [8]) is indicated by the solid line in Fig. 2. The level shift (i.e. half of the level splitting defined in [8]) is obtained as

$$\delta E = B \exp[-W'] \ , \tag{6.7}$$

where the prefactor B is given by

$$B = \frac{[1+u]^{1/2}}{2\mathcal{K}(k')}$$

and W' by

$$W' = \frac{4}{3\lambda^2}(1+u)^{1/2}[E(k) - u\mathcal{K}(k)] \ . \tag{6.8}$$

If we regard the configuration over the full period as a bounce configuration which returns to its original position as indicated by the dotted line in Fig. 2 (for $n = 1$ and $\tau_0 = 0$) we can write it

$$\tilde{\phi}_c(\tau) = \frac{kb(k)}{\lambda} sn[b(k)(\tau + \tau_0) + \mathcal{K}(k)] \ . \tag{6.9}$$

This motion is allowed for a physical system with potential (6.3). The imaginary part of the energy is obtained the way we obtained it in this paper. We then have

$$\text{Im}E = B \exp(-2W') \tag{6.10}$$

and so

$$\text{Im}E = \frac{1}{B}(\delta E)^2 \ . \tag{6.11}$$

In the low-energy limit $1/B = \pi$, and the Bogmolny-Fateyev relation holds exactly. In conclusion we point out that many of the considerations given here could be repeated with the method of complex paths [21–24] as our agreement with WKB results affirms.

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APPENDIX A

Here we show that of the three discrete fluctuation modes with negative eigenvalues, only one contributes to the tunneling. The five discrete eigenmodes ψ_i of the fluctuation equation and their respective eigenvalues are given by (2.7a) and (2.7b). The boundary conditions which the fluctuations field χ has to satisfy are

$$\chi(\tau = \pm T) = 0 \ \text{ or } \chi[z = \pm\mathcal{K}(\gamma)] = 0 \ . \tag{A1}$$

We set

$$\chi = \sum_{n=1}^{5} C_n \psi_n \ . \tag{A2}$$

Using $sn[-u|\gamma] = -sn[u|\gamma]$, $sn[\mathcal{K}(\gamma)|\gamma] = 1$, $cn[\mathcal{K}(\gamma)|\gamma] = 0$, and $dn[\mathcal{K}(\gamma)|\gamma] = \gamma'$ the two conditions (A1) imply

$$-C_2\gamma' + C_4\left(1-\frac{\Delta_1+\Delta_2}{3\gamma^2}\right) + C_5\left(1-\frac{\Delta_1-\Delta_2}{3\gamma^2}\right) = 0\,, \tag{A3}$$
$$+C_2\gamma' + C_4\left(1-\frac{\Delta_1+\Delta_2}{3\gamma^2}\right) + C_5\left(1-\frac{\Delta_1-\Delta_2}{3\gamma^2}\right) = 0\,,$$

where

$$\Delta_1 = 1+\gamma^2, \quad \Delta_2 = \sqrt{1-\gamma^2\gamma'^2}\,.$$

These conditions immediately imply $C_2 = 0$ and

$$C_4\left(1-\frac{\Delta_1+\Delta_2}{3\gamma^2}\right) = -C_5\left(1-\frac{\Delta_1-\Delta_2}{3\gamma^2}\right)\,. \tag{A4}$$

Now

$$\begin{aligned} C_4 &= \int \chi\psi_4 dz \\ &= \int_{-\mathcal{K}}^{+\mathcal{K}} \chi sn^2[z|\gamma]dz - \frac{\Delta_1+\Delta_2}{3\gamma^2}\int_{-\mathcal{K}}^{+\mathcal{K}} \chi dz \\ &= \int_{-\mathcal{K}}^{+\mathcal{K}} \chi sn^2[z|\gamma]dz \end{aligned} \tag{A5}$$

since $\int_{-\mathcal{K}}^{+\mathcal{K}} \chi dz = 0$ as a consequence of the periodicity of χ. Similarly C_5 is found to be equal to the same result (A5). Thus, $C_4 = C_5$. From (A4) we therefore conclude that

$$C_4 = C_5 = 0\,. \tag{A6}$$

Thus only ψ_1 with $E_1 = 0$ and ψ_3 with $E_3 = -3\mu^2(1+k)^2/(1+k^2)$ contribute to the tunneling.

APPENDIX B

Here we evaluate the functional integral I defined by (4.3b) and derive the result (4.10). From (4.2) and (4.9) we see that $\dot{y}(\tau_i) = 0, \dot{y}(\tau_f) = 0$ and

$$\begin{aligned} &y(\tau_i) = 0\,, \\ &y(\tau_f) + N(\tau_f)\int_{\tau_i}^{\tau_f} \frac{\dot{N}(\tau')}{N^2(\tau')}y(\tau')d\tau' = 0\,. \end{aligned} \tag{B1}$$

Substitution of (4.9) into (4.5) yields

$$\begin{aligned} \delta S &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\chi\hat{M}\chi \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\left\{y + N\int_{\tau_i}^{\tau}\frac{\dot{N}(\tau')}{N^2(\tau')}y(\tau')d\tau'\right\}\hat{M} \\ &\quad\times\left\{y + N\int_{\tau_i}^{\tau}\frac{\dot{N}(\tau'')}{N^2(\tau'')}y(\tau'')d\tau''\right\}\,. \end{aligned} \tag{B2}$$

Using

$$\hat{M} := -\frac{d^2}{d\tau^2} + g(\tau), \quad g(\tau) = 2\left(\mu^2 - \frac{3\mu^2}{a^2}\phi_c^2\right)\,, \tag{B3}$$

and

$$\hat{M}N = 0, \quad \ddot{N} = gN \tag{B4}$$

and partial integrations, and dropping total derivative contributions which vanish because $\dot{N}(\tau_i) = 0 = \dot{N}(\tau_f)$, we have

$$\begin{aligned} \delta S &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\left\{y + N\int\cdots\right\}\left[-\ddot{y} + gy - \frac{2\dot{N}^2}{N^2}y - N\frac{d}{d\tau}\left(\frac{\dot{N}y}{N^2}\right)\right] \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\left[-y\ddot{y} - \ddot{y}N\int\cdots + y^2g + Ngy\int\cdots - \frac{\dot{N}^2y^2}{N^2} - \frac{d}{d\tau}\left(\dot{N}y\int\cdots\right) - yN\frac{d}{d\tau}\left(\frac{\dot{N}y}{N^2}\right)\right] \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\left[-y\ddot{y} - \dot{y}\left(\dot{N}\int\cdots + \frac{\dot{N}y}{N}\right) + y^2g + Ngy\int\cdots - \frac{\dot{N}^2y^2}{N^2} - y^2N\frac{d}{d\tau}\left(\frac{\dot{N}}{N^2}\right) - y\frac{\dot{N}}{N}\dot{y}\right] \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\left[-y\ddot{y} - y\ddot{N}\int\cdots - y^2\frac{\dot{N}^2}{N^2} + y^2g + Ngy\int\cdots - \frac{\dot{N}^2}{N^2}y^2 - y^2N\frac{d}{d\tau}\left(\frac{\dot{N}}{N^2}\right)\right] \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\left[-y\ddot{y} - y^2\left\{2\frac{\dot{N}^2}{N^2} - \frac{\ddot{N}}{N} + N\frac{d}{d\tau}\left(\frac{\dot{N}}{N^2}\right)\right\}\right] \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau[-y\ddot{y}] \\ &= \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau[\dot{y}^2]\,. \end{aligned} \tag{B5}$$

Thus, the corresponding path-integral reduces to that of a free-particle propagator subject to the constraints (B1).

We insert (B5) into I defined by (4.3b) with the help of the identity

$$1 = \int dy_f \delta(y_f + f(\tau_f)) = \int dy_f \frac{1}{2\pi} \int d\alpha \, \exp\{-i\alpha[y_f + f(\tau_f)]\} \; , \tag{B6}$$

where

$$y_f \equiv y(\tau_f)$$

and (B7)

$$f(\tau_f) \equiv N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{\dot{N}(\tau')}{N^2(\tau')} y(\tau') d\tau' \; .$$

With a partial integration we can write

$$y_f + f(\tau_f) = N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{1}{N(\tau)} \frac{dy}{d\tau} d\tau \; . \tag{B8}$$

We can therefore write I of (4.3b) in the form

$$I = \frac{1}{2\pi} \int dy_f \int_{y(\tau_i)=0}^{y(\tau_f)} \mathcal{D}\{y\} \left|\frac{\partial \chi}{\partial y}\right| \int_{-\infty}^{\infty} d\alpha \exp\left[-\int_{\tau_i}^{\tau_f} \frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 d\tau\right] \exp\{-i\alpha[y(\tau_f) + f(\tau_f)]\}$$
$$= \int \frac{dy_f}{2\pi} \left|\frac{\partial \chi}{\partial y}\right| \int_{y(\tau_i)=0}^{y(\tau_f)} \mathcal{D}\{y\} \exp\left[-\frac{1}{2}\int_{\tau_i}^{\tau_f} \left\{\frac{dy}{d\tau} + i\alpha \frac{N(\tau_f)}{N(\tau)}\right\}^2 d\tau\right] \int \exp\left[-\frac{\alpha^2}{2} N^2(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)}\right] d\alpha \; . \tag{B9}$$

We can write this

$$I = \int \frac{dy_f}{2\pi} \left|\frac{\partial \chi}{\partial y}\right| \mathcal{K}_0(y_f, \tau_f; y(\tau_i) = 0, \tau_i) \times \int \exp\left[-\frac{\alpha^2}{2} N^2(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)}\right] d\alpha \; , \tag{B10}$$

where $\mathcal{K}_0$ is the configuration space representation of the free nonrelativistic Green's function with

$$\int \mathcal{K}_0(y_f, \tau_f; y(\tau_i) = 0, \tau_i) dy_f = 1$$

[recall that for a free particle with energy $E = \frac{1}{2}mv^2$, momentum $p = \hbar k$, confined between walls a distance a apart, the eigenvalues are given by $k = 2n\pi/a$ so that $ET/\hbar = 2n\pi$ and the Green's function $\sum_n e^{iE_nT/\hbar}\psi_n(y_f)\psi_n(y_i) = \delta(y_f - y_i)$]. Hence

$$I = \frac{1}{\sqrt{2\pi}} \left|\frac{\partial \chi}{\partial y}\right| \left[N^2(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)}\right]^{-1/2} \; . \tag{B11}$$

In the literature [14] it is shown that the Jacobian of the transformation (4.9) is given by

$$\left|\frac{\partial x}{\partial y}\right| = \left(\frac{N(\tau_f)}{N(\tau_i)}\right)^{1/2} \; . \tag{B12}$$

Thus,

$$I = \frac{1}{\sqrt{2\pi}} \left[N(\tau_i) N(\tau_f) \int_{\tau_i}^{\tau_f} \frac{d\tau}{N^2(\tau)}\right]^{-1/2} \; . \tag{B13}$$

This establishes (4.10) of the text.

APPENDIX C

Here we establish relations (4.11b) and (4.12b). Our derivation is based on the following formula which we therefore prove first: i.e.,

$$\frac{\partial^2 S_c(\phi_f, \phi_i, T)}{\partial \phi_f^2} = \frac{1}{N(\phi_f)} \left[\left(\frac{d^2\phi}{d\tau^2}\right)_{\tau_f} + \frac{1}{N(\phi_f) \int_{\tau_i}^{\tau_f} [d\tau/N^2(\tau)]}\right] \; . \tag{C1}$$

For $\tau_i, \tau_f = \mp T \to \mp\infty$ the corresponding formula was derived in the Appendix of [8].

In demonstrating (C1) we start from

$$S(\phi_f, \phi_i; T) = W(\phi_f, \phi_i; E) + 2ET \; , \tag{C2}$$

where

$$E \equiv E(\phi_f, \phi_i; T) \; . \tag{C3}$$

From (C2) we obtain

$$0 = \frac{\partial S}{\partial E} = \frac{\partial W}{\partial E} + 2T \tag{C4}$$

and

$$\frac{\partial S}{\partial \phi_f} = \frac{\partial W}{\partial \phi_f} + \frac{\partial W}{\partial E}\frac{\partial E}{\partial \phi_f} + 2T\frac{\partial E}{\partial \phi_f} = \frac{\partial W(\phi_f, \phi_i; E)}{\partial \phi_f} \tag{C5}$$

with the help of (C4). Further,

$$\frac{\partial^2 S}{\partial \phi_f^2} = \frac{\partial^2 W}{\partial \phi_f^2} + \frac{\partial^2 W}{\partial E \partial \phi_f}\frac{\partial E}{\partial \phi_f} . \tag{C6}$$

Now from (C2) we see that

$$E = \frac{\partial S}{\partial (2T)}$$

so that, with (C5),

$$\begin{aligned}\frac{\partial E}{\partial \phi_f} &= \frac{\partial}{\partial \phi_f}\left(\frac{\partial S}{\partial (2T)}\right) \\ &= \frac{\partial}{\partial (2T)}\left(\frac{\partial S}{\partial \phi_f}\right) \\ &= \frac{\partial}{\partial (2T)}\left(\frac{\partial W}{\partial \phi_f}\right) \\ &= \frac{\partial}{\partial E}\left(\frac{\partial W}{\partial \phi_f}\right)\frac{\partial E}{\partial (2T)} \\ &= \frac{\partial^2 W}{\partial E \partial \phi_f}\frac{1}{\partial (2T)/\partial E} .\end{aligned} \tag{C7}$$

Using (C4) we can rewrite this

$$\frac{\partial E}{\partial \phi_f} = -\frac{\partial^2 W}{\partial E \partial \phi_f}\bigg/\frac{\partial^2 W}{\partial E^2} . \tag{C8}$$

Inserting this into (C6) we obtain

$$\frac{\partial^2 S}{\partial \phi_f^2} = \frac{\partial^2 W}{\partial \phi_f^2} - \left(\frac{\partial^2 W}{\partial \phi_f \partial E}\right)^2 \bigg/ \frac{\partial^2 W}{\partial E^2} . \tag{C9}$$

Now

$$W = \int_{\tau_i}^{\tau_f}\left(\frac{d\phi}{d\tau}\right)^2 d\tau = \int_{\phi_i}^{\phi_f} d\phi \frac{d\phi}{d\tau} \tag{C10}$$

so that

$$\frac{\partial W}{\partial \phi_f} = (\dot{\phi})_{\phi_f} = \sqrt{2(V-E)}|_{\phi_f} . \tag{C11}$$

It follows that

$$\frac{\partial^2 W}{\partial \phi_f \partial E} = -\frac{1}{\dot{\phi}}\bigg|_{\phi_f} , \tag{C12}$$

$$\frac{\partial^2 W}{\partial E^2} = -\int_{\tau_i}^{\tau_f}\frac{d\tau}{\dot{\phi}^2} , \tag{C13}$$

$$\frac{\partial^2 W}{\partial \phi_f^2} = \frac{(\partial V/\partial \phi)_{\phi_f}}{\sqrt{2[V(\phi_f)-E]}} = \frac{\ddot{\phi}}{\dot{\phi}}\bigg|_{\phi_f} , \tag{C14}$$

where in the last step we used the equation of motion. Inserting (C12)–(C14) into (C9) and setting $\dot{\phi} = N(\tau)$, $\phi_f = \phi(\tau_f)$, we obtain (C1).

We now consider the relation (4.11) which we rewrite in the form

$$\Delta = \lim_{\substack{\phi_i \to \phi(-T) \\ \phi_f \to \phi(T)}} \left[1 + \frac{d\phi_f}{d\tau}\frac{d^2\phi_f}{d\tau^2}\int_{\tau_i}^{\tau_f}\frac{d\tau}{N^2(\tau)}\right]^{-1/2} . \tag{C15}$$

From (4.6) we obtain

$$\begin{aligned}\frac{d\phi_c}{d\tau} &= -s_+(k)\beta(k)\gamma^2 sn[\beta(k)\tau|\gamma]cn[\beta(k)\tau|\gamma] \\ \frac{d^2\phi}{d\tau^2} &= -s_+(k)\beta^2(k)\gamma^2\{cn^2[\beta(k)\tau|\gamma] - sn^2[\beta(k)\tau|\gamma]\} \\ &\quad \times dn[\beta(k)\tau|\gamma] .\end{aligned} \tag{C16}$$

Using formula 361.10 of [12] we obtain

$$\begin{aligned}\int_{\tau_i}^{\tau_f}\frac{d\tau}{(d\phi_c/d\tau)^2} &= \frac{1}{s_+^2(k)\beta^3(k)\gamma^4}\int_{\beta\tau_i}^{\beta\tau_f} du\, ns^2[u|\gamma]nc^2[u|\gamma] \\ &= \frac{1}{s_+^2(k)\beta^3(k)\gamma^4\gamma'^2}[2\gamma'^2 u - (1+\gamma'^2)E(u) + dn[u|\gamma]\{tn[u|\gamma] - \gamma'^2 cs[u|\gamma]\}]_{\beta\tau_i}^{\beta\tau_f} ,\end{aligned} \tag{C17}$$

where $tn = sn/cn$ and $cs = cn/sn$. In the limits $\tau_i \to -T$, $\tau_f \to +T$, i.e., $\beta\tau_i \to -\mathcal{K}(\gamma)$, $\beta\tau_f \to +\mathcal{K}(\gamma)$ the contribution of $tn[u|\gamma]$ diverges owing to the property $cn[\mathcal{K}(\gamma)|\gamma] = 0$. However, the limit Δ is finite and since in this limit

$$\int_{\tau_i}^{\tau_f}\frac{d\tau}{(d\phi_c/d\tau)^2} \to \frac{2dn[\mathcal{K}(\gamma)|\gamma]sn[\mathcal{K}(\gamma)|\gamma]}{s_+^2(k)\beta^3(k)\gamma^4\gamma'^2 cn[\mathcal{K}(\gamma)|\gamma]} \tag{C18}$$

and we obtain

$$\Delta = \left[1 + s_+^2(k)\beta^3(k)\gamma^4 sn[u|\gamma]cn[u|\gamma]dn[u|\gamma]\frac{(cn^2[u|\gamma] - sn^2[u|\gamma])2sn[u|\gamma]}{cn[u|\gamma]s_+^2(k)\beta^3(k)\gamma^4\gamma'^2}\right]^{-1/2}\Bigg|_{u=\mathcal{K}(\gamma)}$$
$$= [1-2]^{-1/2} = -i \tag{C19}$$

since $dn[\mathcal{K}] = \gamma'$ and $sn[\mathcal{K}|\gamma] = 1$.

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