Applications of membranes in the Abelian Higgs model

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The membrane description is used to investigate small perturbations of vortices with a winding number bigger than one. The new kind of vortex excitations, looking like a bubble of arbitrary shape moving along the vortex with the speed of light, is found.

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I. INTRODUCTION

Recently there has been wide interest in the study of nonlinear field equations, motivated by the existence of vortex solutions which appeared to be connected to such phenomena as the superfluidity of helium described by the Goldstone model [1], superconductivity described by the Ginzburg-Landau model [2], and by the hypothesis that confinement in QCD can be understood in terms of QCD-strings which are represented by a kind of vortices called flux tubes [3]. Many conjectures have been made about possible astrophysical phenomena that may be explained by the presence of large cosmic strings. Massive strings, which appear in grand unification models, may have served as seeds for galaxy formation [4], and provide an explanation for quasars [5], and may act as gravitational lenses [6].

One of the widest known, for its very interesting features, is the Abelian Higgs model. Apart from being a textbook example of the Higgs phenomenon, it contains sectors with a nonzero topological charge which is the winding number of the Higgs field.

Nonlinearity is a reason for our poor knowledge about solutions of this model. Only a few exact solutions are known. The static cylindrical vortex solutions in this model were obtained by Nielsen and Olesen [7]. Although no closed-form solutions have been found, the solutions are known to be smooth and to approach vacuum values fast with a distance from the center of the vortex.

Nielsen and Olesen have proposed to view such solutions as stringlike structures [7]. The above statement is based on the property of the vortex solutions that when we approach some values of parameters of these solutions the width of the vortex (understood as the width of the region in space where the density of the energy significantly differs from its vacuum value) decreases, finally reaching zero width. In this limit, the vortex tube coincides with the line of zeros of the Higgs field. This is the reason for regarding the vortex as a string in the above mentioned limit. This idea has been generalized by Förster to describe the nontrivial time evolution of the vortices [8].

The string description, although not exact as it is in the zero width limit, is possible for vortices with a nonzero width. If we define a string, for example, as a line of zeros of the Higgs field, we will obtain and approximate string description of a vortex.

During the time evolution, the line of zeros of the Higgs field sweeps out a two-dimensional manifold which is called the world sheet of the vortex. Such a definition of the worldsheet has the same difficulty as the definition of the phase speed in the description of a moving wave. It has been shown that when two vortices interact with each other the speed of the lines of zeros of the Higgs field can exceed the speed of light [9]. As we know the physical aspects of a moving wave have a better description by the group speed which never exceeds the speed of light. If we define the worldsheet of the vortex as a surface swept out by a set of points where the density of the energy of the vortex has its maximal value $E_{\max}$, we will obtain characteristics similar to the group speed in the problem of a moving wave. Such a physical description works very good for one-quantum flux vortex solutions, but for solutions with a winding number bigger than one it can be saved only in the case when we define a string as a collection of "centers of mass" of the vortex [10]. This difficulty is a consequence of the fact that for a vortex with $n \geq 2$ the maximum of the density of the energy E does not lie on a line as it does for a vortex with $n = 1$ (see Fig. 1), but lies on the surface enclosing the line of zeros of the Higgs field [see Figs. 2(a) and 2(b)] [11].

Although the string description of the vortex solutions with $n \geq 2$ exists, it seems that in this case a more suitable and more precise description could be a membrane description. The higher precision in the case of vortices with a winding number bigger than one has a purely ge-

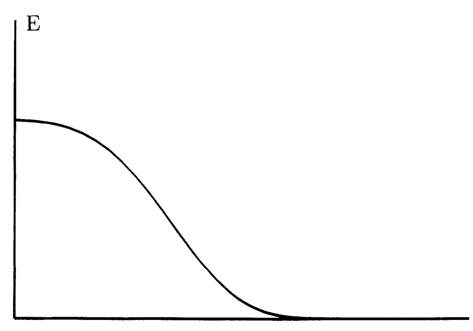

FIG. 1. The density of the energy E in the tangent section of the vortex with $n = 1$ which lies along the OZ axis.

ometrical nature—a membrane has a richer geometrical structure than a string.

I would like to point out that this paper does not aim at giving a comprehensive description of vortices with $n \geq 2$, but only gives a description of a special, although very important from a physical point of view, property of the field configuration; namely, the density of the energy. The membrane description proposed in this paper could be compared with a string description of vortices based on a surface of the zeros of the Higgs field. As is known, a surface of the zeros of the Higgs field, independently of the finite thickness of the vortex, is a well-defined geometrical object in space-time. Similarly, the hypersurface of the maximum of the density of the energy is also well defined. The difference between the surface of the zeros of the Higgs field and the hypersurface of the maximum of the density of the energy lies in the fact that only the latter has a well-defined physical meaning.

The description of arbitrarily deformed vortices by the membrane approximation is possible [12], but in this work we would like to use the membrane description of the field theoretical model only in the sector containing small perturbed straight-linear vortices to show on particular examples the usefulness of the membrane description.

The paper is organized as follows. In Sec. II we fix our notation and introduce the geometry of a three-dimensional hypersurface in four-dimensional space-time. The field theoretical model is defined in Sec. III. The same section contains a derivation of the effective membrane theory of slightly perturbed straight-linear vortex solutions. Section IV is devoted to considerations of small perturbations of the Abrikosov-Nielsen-Olesen-type solutions with $n \geq 2$ and its membrane representation. Section V contains remarks.

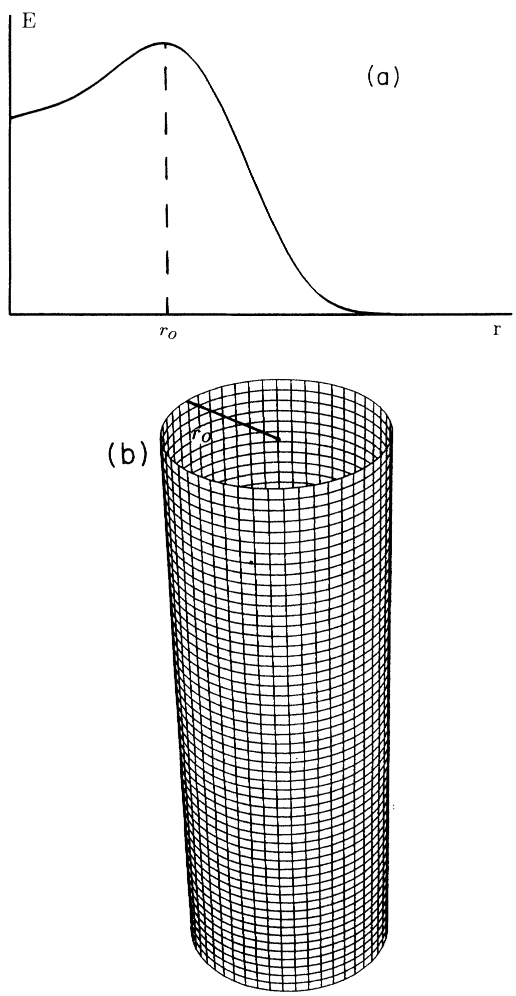

FIG. 2. (a) The density of the energy E in the tangent section of the vortex with $n \geq 2$ which lies along the OZ axis. (b) The surface $E = E_{\max}$ in some instant of time.

II. GEOMETRY OF THE WORLD HYPERSURFACE OF THE VORTEX

The membrane we identify at each instant of time with a surface $E = E_{\max}$. By time evolution of this surface we obtain a three-dimensional manifold in four-dimensional space-time. A radius vector of the arbitrary point on the world hypersurface $X^{\mu}(\sigma^{a})$, where $\mu = 0, 1, 2, 3$ are Lorentzian indices, can be parametrized by the timelike parameter τ, lengthlike parameter σ, and the parameter θ which can be identified with an angle of cylindrical coordinates (see Fig. 3). The coordinates $(\tau, \sigma, \theta) = (\sigma^{a})$ are variables on the three-dimensional manifold. The coordinate on the straight line perpendicular to the world hypersurface Σ at the point $X^{\mu}(\tau, \sigma, \theta)$ we mark ξ. Having all the variables $(\zeta^{\alpha}) = (\sigma^{\alpha}, \xi)$ in the neighborhood of the world hypersurface Σ we can write

$$x^{\mu} = X^{\mu}(\tau, \sigma, \theta) + n^{\mu}\xi, \tag{1}$$

where n^{μ} is four-vector normal to the hypersurface Σ in the point $X^{\mu}(\sigma^{a})$. Tangent four-vectors $X^{\mu}{}_{,a} = \partial_{a} X^{\mu}$,

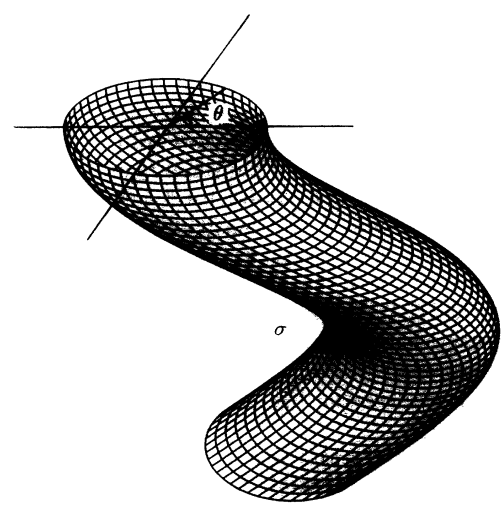

FIG. 3. A surface which in each instant of time has the shape of tube can be parametrized by an angle θ and the length of the tube σ.

and the normal four-vector $n_i{}^{\mu}$ to the hypersurface Σ obey the conditions

$$n^{\mu}n_{\mu} = -1, \quad n_{\mu}X^{\mu}{}_{,a} = 0, \quad g_{ab} = X^{\mu}{}_{,a}X_{\mu,b}, \tag{2}$$

where $\eta^{\mu\nu}$ is Minkowski metric with signature $(+,-,-,-)$. The last equation is a definition of the intrinsic metric on the world hypersurface Σ. When we move from one point of the hypersurface to another one in its neighborhood the normal and tangent vectors $(n^{\mu}, X^{\mu}{}_{,a})$ can change in the normal direction as well as in the tangent directions:

$$X^{\mu}{}_{,ab} = \partial_a X^{\mu}{}_{,b} = \Gamma^{c}{}_{ab}X^{\mu}{}_{,c} + K_{ab}n^{\mu}, \tag{3}$$

$$\partial_a n_{\mu} = K^{*}{}_{ac}X_{\mu}{}^{,c} + \omega_a n_{\mu}. \tag{4}$$

In principle the coefficients of the above decompositions could be completely independent, but the geometry of the hypersurface imposes on them some restrictions. To find these constraints we project [3] on n_{μ} and then use [2] to obtain the coefficients K_{ab}:

$$K_{ab} = -X^{\mu}{}_{,ab}n_{\mu}. \tag{5}$$

This relation can be transformed to the form

$$K_{ab} = \partial_b n_{\mu}X^{\mu}{}_{,a}, \tag{6}$$

where we used the relation of orthogonality n^{μ} and $X^{\mu}{}_{,a}$ (2) differentiated with respect to σ^a. In addition, from (6) it follows that K_{ab} are symmetric with respect to the change of indices $K_{ab} = K_{ba}$. On the other hand if we project Eq. (4) on a vector tangent to the hypersurface Σ and use (2), we will obtain

$$K^{*}{}_{ab} = \partial_b n_{\mu}X^{\mu}{}_{,a}. \tag{7}$$

Comparing (6) and (7) we conclude that $K^{*}{}_{ab} = K_{ab}$. The same procedure [projecting (4) on n_{μ} and using (2)] gives

$$\omega_a = -n^{\mu}\partial_a n_{\mu}. \tag{8}$$

Now if we come back to the normalization condition (2) for n_{μ} and differentiate it with respect to σ^a, we will obtain $n^{\mu}\partial_a n_{\mu} = 0$. Equation (8) together with $n^{\mu}\partial_a n_{\mu} = 0$ gives $\omega_a = 0$. Finally we have transformed (3) and (4) to the form

$$X^{\mu}{}_{,ab} = \Gamma^{c}{}_{ab}X^{\mu}{}_{,c} + K_{ab}n^{\mu}, \tag{9}$$

$$\partial_a n_{\mu} = K_{ac}X_{\mu}{}^{,c}, \tag{10}$$

where K_{ab} are called coefficients of extrinsic curvature. From formula (10) we can see that n^{μ} do not change in the normal direction to the hypersurface, which is a simple consequence of the fact that its length is constant. The coefficients $\Gamma^{c}{}_{ab}$ can be found by projecting Eq. (9) on $X^{\mu}{}_{,a}$, and they turn out to be defined completely by the metric tensor g_{ab}. Thus, $\Gamma^{c}{}_{ab}$ are the Christoffel symbols for metric tensor g_{ab}. Equations (9) and (10) correspond to the Gauss-Weingarten formulas for the two-dimensional surface.

It is easy to verify that K_{ab} obey some additional identities. Commutativity of derivatives $\partial_a\partial_b n_{\mu} = \partial_b\partial_a n_{\mu}$ together with the Gauss-Weingarten formulas (9) and (10) and orthonormality relations (2) for n^{μ} and $X^{\mu}{}_{,a}$ gives the identity

$$\nabla_a K_{bc} = \nabla_b K_{ac}, \tag{11}$$

which corresponds to the Peterson-Codazzi identities for two-dimensional surface. The symbol ∇_a denotes a covariant derivative with respect to the reparametrization of the hypersurface. In the same way from $\partial_a\partial_b X_{,c}{}^{\mu} = \partial_b\partial_a X_{,c}{}^{\mu}$ we get

$$R_{abcd} = K_{ac}K_{bd} - K_{bc}K_{ad}, \tag{12}$$

where $R^{d}{}_{abc} = \partial_b\Gamma^{d}{}_{ac} - \partial_c\Gamma^{d}{}_{ab} + \Gamma^{s}{}_{ac}\Gamma^{d}{}_{sb} - \Gamma^{s}{}_{ab}\Gamma^{d}{}_{sc}$ is the tensor of the intrinsic curvature. Equation (12) corresponds to the Gauss identity for surfaces.

The coordinates introduced in this section (1) are well defined in the region of space-time where there exists an inversion of the metric tensor in curvilinear coordinates ζ^{α}:

$$G_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha x^{\mu}\partial_\beta x^{\nu} = \begin{bmatrix} S_a{}^{c}S_{cb} & 0 \\ 0 & -1 \end{bmatrix}$$

$$G^{\alpha\beta} = \begin{bmatrix} (S^{-1})^{ac}(S^{-1})_c{}^{b} & 0 \\ 0 & -1 \end{bmatrix}, \tag{13}$$

where $S_{ab} = g_{ab} + \xi K_{ab}$. The inverse of $G_{\alpha\beta}$ exists if the matrix S_{ab} is invertible. The matrix S_{ab} is invertible in the region where $\det\|S_{ab}\| = g[1 + \xi K_a{}^{a} + \frac{1}{2}\xi^2(K_a{}^{a}K_b{}^{b} - K_a{}^{b}K_b{}^{a})]$ is not equal to zero (here $g = \det\|g_{ab}\|$). Thus the coordinates ζ^{α} are well defined only in the region constrained by the minimal $\xi_{\min}$ for which $\det\|S_{ab}\| = 0$. Now we are prepared for construction of an effective membrane theory of the vortices with a winding number $n \geq 2$.

III. EFFECTIVE MEMBRANE DESCRIPTION PERTURBATIONS OF THE VORTEX WITH $n \geq 2$ IN THE ABELIAN-HIGGS MODEL

We start from the Lagrangian density of an Abelian Higgs model:

$$L = (D_\mu\phi)^{*}(D^{\mu}\phi) - \frac{1}{4}\lambda\left(\phi^{*}\phi - \frac{2m^2}{\lambda}\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{14}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$ is covariant derivative, ϕ is a complex scalar field, and A_μ is a gauge potential. The corresponding gauge invariant energy-momentum tensor has the form

$$T_{\mu\nu} = \tfrac{1}{2}[(D_\mu\phi)^{*}(D_\nu\phi) + (D_\nu\phi)^{*}(D_\mu\phi)] - F_{\mu\lambda}F_\nu{}^{\lambda} - \eta_{\mu\nu}L. \tag{15}$$

The static solutions of the corresponding equations of motion which are invariant under translations along the third axis have the form [in cylindrical coordinates (t,r,θ,z)]

$$\phi = e^{-in\theta}F(r),\ A_\theta = A_\theta(r),\ A_t = A_z = A_r = 0, \quad (16)$$

where A_θ and $F(r)$ are not known in closed form but have been found numerically. They are called Abrikosov-Nielsen-Olesen solutions. The energy density $E = T_{00}$ for an Abrikosov-Nielsen-Olesen solution with $n \geq 2$ poses its maximum on a cylinder enclosing the line of zeros of the Higgs field [see Fig. 2(a)]. Thus the equation which defines this cylinder has the form $\partial_r E|_{r=r_0} = 0$. This equation allows us to find r_0. When we are concerned with more complicated vortex solutions which depend on t as well as on z and θ (for any finite r vortex solutions can depend on t,z,θ,r, only for $\to\infty$ does the function $F(r)$ approach its asymptotical value $\sqrt{2m^2/\lambda}$) we can take the condition $\partial_r E|_{r=r_0(t,z,\theta)} = 0$ as a definition of a membrane world hypersurface. In coordinates (1) this definition takes the from $\partial_\xi E|_{\xi=0} = 0$. Although the above proposed definition of a membrane world hypersurface seems to be reasonable, it contains one serious difficulty which is the lack of invariance with respect to the Lorentz transformations. We can avoid this difficulty, using instead of E a scalar function $w = g^{ab}X_{,a}{}^{\mu}X_{,b}{}^{\nu}T_{\mu\nu}$, which is projected on a membrane world hypersurface Σ energy-momentum tensor. The function w seems to be a natural Lorentz invariant generalization of E, which means that for Abrikosov-Nielsen-Olesen solutions the condition $\partial_\xi w|_{\xi=0} = 0$ ($T_{\mu\nu}$ projected on the cylinder) defines the same r_0-like condition $\partial_\xi E|_{\xi=0} = 0$ (w does not have the same values as E but the maximum of w lies on the same hypersurface Σ as the maximum of E). Moreover, for solutions which are slightly perturbed Abrikosov-Nielsen-Olesen solutions the difference between hypersurfaces defined by both conditions is also small.

Because we have decided to define the membrane world hypersurface by the condition $\partial_\xi w|_{\xi=0} = 0$, we need w for the Abelian Higgs model, i.e., for $T_{\mu\nu}$ given by (15). Using relations (2), (9), and (10) and the definition of w we obtain

$$w = -\tfrac{1}{4}G^{ab}[G^{cd}F_{ac}F_{bd} + 2F_{a\xi}F_{b\xi} + 2(D_a\phi)^*(D_b\phi)] + 3H_s, \quad (17)$$

where we have denoted $H_s = \frac{1}{2}(D_\xi\phi)^*(D_\xi\phi) + \frac{1}{4}\lambda(\phi^*\phi - 2m^2/\lambda)^2$. The field strength tensor $F_{ab} = \nabla_a A_b - \nabla_b A_a$ because of the symmetry of the Christoffel symbols reduces to the form $F_{ab} = \partial_a A_b - \partial_b A_a$. The covariant derivatives have the form $D_a\phi = \nabla_a\phi - ieA_a\phi$ and $D_\xi\phi = \partial_\xi\phi - ieA_\xi\phi$. Because ϕ is a scalar function with respect to the reparametrization $D_a\phi = \partial_a\phi - ieA_a\phi$.

We intend to construct a membrane theory describing the sector of the Abelian Higgs model containing small perturbations of any fixed vortex solution with $n \geq 2$. We will start from the Euler-Lagrange equations for the Lagrangian (14). A curved vortex is in a natural way obtained as a solution of the equations of motion in coordinates on the curved hypersurface (we assume the Lorentz gauge condition):

$$G^{ab}[\nabla_a\nabla_b\phi - \xi(S^{-1})^{cd}(\nabla_a K_{bd})\nabla_c\phi - S_{ac}K_b{}^c\partial_\xi\phi] - \partial_\xi{}^2\phi + 2ieG^{ab}A_a\nabla_b\phi$$
$$-2ieA_\xi\partial_\xi\phi - e^2G^{ab}A_aA_b\phi + e^2A_\xi A_\xi\phi + \frac{1}{2}\lambda\left(\phi^*\phi - \frac{2m^2}{\lambda}\right)\phi = 0, \quad (18)$$

$$\begin{aligned}G^{ab}\{&\nabla_a[\nabla_b A_h - \xi(S^{-1})^{cd}(\nabla_b K_{hd})A_c - S_{bc}K_h{}^cA_\xi] - \xi(S^{-1})^{ce}(\nabla_a K_{he})[\nabla_b A_c - \xi(S^{-1})^{gd}(\nabla_b K_{gc})A_d - S_{bd}K_c{}^dA_\xi]\\ &- \xi(S^{-1})^{ce}(\nabla_a K_{be})[\nabla_c A_h - \xi(S^{-1})^{gd}(\nabla_c K_{hg})A_d - S_{dc}K_h{}^dA_\xi]\\ &- S_{ae}K_h{}^e[\nabla_b A_\xi - K_{bd}(S^{-1})^{cd}A_c] - S_{ae}K_b{}^e[\partial_\xi A_h - K_{hg}(S^{-1})^{gd}A_d]\}\\ &- K_{he}(S^{-1})^{ce}[\partial_\xi A_c - K_{cg}(S^{-1})^{gd}A_d] - \partial_\xi[\partial_\xi A_h - K_{hd}(S^{-1})^{cd}A_c]\\ &+ie(\phi\nabla_h\phi^* - \phi^*\nabla_h\phi) + 2e^2A_h\phi\phi^* = 0,\end{aligned} \quad (19)$$

$$\begin{aligned}G^{ab}\{&\nabla_a[\nabla_b A_\xi - K_{bd}(S^{-1})^{cd}A_c] - K_{ad}(S^{-1})^{cd}[\nabla_b A_c - \xi(S^{-1})^{ge}(\nabla_b K_{cg})A_e - S_{bg}K_c{}^gA_\xi]\\ &-\xi(S^{-1})^{ed}(\nabla_a K_{be})[\nabla_d A_\xi - K_{gd}(S^{-1})^{cg}A_c] - S_{ae}K_b{}^e\partial_\xi A_\xi\}\\ &-\partial_\xi{}^2A_\xi + ie(\phi\partial_\xi\phi^* - \phi^*\partial_\xi\phi) + 2e^2A_\xi\phi\phi^* = 0.\end{aligned} \quad (20)$$

The metric $g_{ab} + G_{ab}|_{\xi=0}$ describes the hypersurface swept out by a static cylinder. The static cylinder is an example of a flat hypersurface (in the sense of vanishing coefficients of internal curvature $R^d{}_{abc} = 0$) but the embedding into space-time described by the coefficients of extrinsic curvature K_{ab} is not trivial (i.e., not all K_{ab} vanish). The membrane world hypersurface for a curve vortex is given by the equation $\partial_\xi w = 0$, which in the case of w given by (17) reads

$$\tfrac{1}{2}K^{ab}[g^{cd}F_{ac}F_{bd} + 2F_{a\xi}F_{b\xi} + 2((D_a\phi)^*(D_b\phi))]|_{\xi=0}$$
$$-\tfrac{1}{4}g^{ab}[-2K^{cd}F_{ac}F_{bd} + g^{cd}\partial_\xi(F_{ac}F_{bd}) + 2\partial_\xi(F_{a\xi}F_{b\xi}) + 2\partial_\xi((D_a\phi)^*(D_b\phi))]|_{\xi=0} + 3\partial_\xi H_s|_{\xi=0} = 0. \quad (21)$$

We base the construction of membrane theory on the assumption that we know the straight-linear vortex solution, which we can consider in arbitrary curvilinear coordinates defined around the cylinder of the maximum of the function w. Then we allow for small perturbations of straight-linear solutions to obtain membrane theory of a curve vortex. We will obtain the geometric constraints which describe how the straight-linear vortex can be deformed for the perturbed function to be a solution of Eqs. (18)–(20). Let us assume that

$$\begin{aligned} \phi &= \phi[\tau,\sigma,\theta,r_0(\tau,\sigma,\theta)+\xi] \underset{\xi\to\infty}{\sim} e^{-in\theta}F(\xi),\\ A_a &= A_a(\tau,\sigma,\theta,r_0(\tau,\sigma,\theta)+\xi),\\ A_\xi &= A_\xi(\tau,\sigma,\theta,r_0(\tau,\sigma,\theta)+\xi) \end{aligned} \tag{22}$$

are unknown vortex solutions (with $n \geq 2$) of Eqs. (18)–(20). The radius $r_0(\tau,\sigma,\theta)$ of a membrane $\partial_\xi w|_{\xi=0}=0$ [which for the above solutions is defined by the Eq. (21)] can be split into the radius of cylindrical membrane r_0 (describing the straight-linear vortex) and the small perturbation $\varepsilon(\tau,\sigma,\theta)$, i.e., $r_0(\tau,\sigma,\theta)=r_0+\varepsilon(\tau,\sigma,\theta)$. In the next step, assuming that a small perturbation of functions causes the same order perturbation of the shape of the cylinder (i.e., coefficients of extrinsic curvature K_{ab}), we expand Eqs. (18)–(21) with respect to powers of ε. Before we start the perturbative calculations we rescale the mass $m\to m/M$ and length $x^\mu \to Mx^\mu$ by some mass factor (rescaling $t\to Mt$ and $z\to Mz$ causes rescaling $\tau\to M\tau$ and $\sigma\to M\sigma$). According to formula (1) rescaling the length causes a rescaling of $\xi\to M\xi$ and $X^\mu\to MX^\mu$. The rescaling operation we made enables us to consider our theory as completely dimensionless (particularly coefficients of expansion ε and k_{ab} are dimensionless). In the calculations we will use the expansions

$$\begin{aligned} &K_{ab}=K^{\rm cyl}{}_{ab}+k_{ab},\quad S_{ab}=S^{\rm cyl}{}_{ab}+\xi k_{ab},\\ &(S^{-1})^{ab}=(S^{-1})_{\rm cyl}{}^{ab}-\xi(S^{-1})_{\rm cyl}{}^{ac}(S^{-1})_{\rm cyl}{}^{be}k_{ce}+\cdots,\\ &G^{ab}=G_{\rm cyl}{}^{ab}-\xi(S^{-1})_{\rm cyl}{}^{ac}(S^{-1})_{\rm cyl}{}^{bd}[(S^{-1})^{\rm cyl}{}_d{}^e k_{ec}+(S^{-1})^{\rm cyl}{}_c{}^e k_{ed}]+\cdots,\\ &\phi(r_0+\varepsilon+\xi)=\phi(r_0+\xi)+\phi'(r_0+\xi)\varepsilon+\cdots,\\ &A_\alpha(r_0+\varepsilon+\xi)=A_\alpha(r_0+\xi)+A'_\alpha(r_0+\xi)\varepsilon+\cdots, \end{aligned} \tag{23}$$

where $(\alpha)=(a,\xi)$ and k_{ab} describes the small (of order ε) deformations of the cylinder. A zero order perturbation of Eqs. (18)–(21) gives

$$\begin{aligned} &G_{\rm cyl}{}^{ab}[\nabla_a\nabla_b\phi-\xi(S^{-1})_{\rm cyl}{}^{cd}(\nabla_a K^{\rm cyl}{}_{bd})\nabla_c\phi-S^{\rm cyl}{}_{ac}K^{\rm cyl}{}_b{}^c\partial_\xi\phi]-\partial_\xi{}^2\phi\\ &\qquad+2ieG_{\rm cyl}{}^{ab}A_a\nabla_b\phi-2ieA_\xi\partial_\xi\phi-e^2G_{\rm cyl}{}^{ab}A_aA_b\phi+e^2A_\xi A_\xi\phi+\frac{1}{2}\lambda\left(\phi^*\phi-\frac{2m^2}{\lambda}\right)\phi=0, \end{aligned} \tag{24}$$

$$\begin{aligned} &G_{\rm cyl}{}^{ab}\{\nabla_a[\nabla_b A_h-\xi(S^{-1})_{\rm cyl}{}^{cd}(\nabla_b K^{\rm cyl}{}_{hd})A_c-S^{\rm cyl}{}_{bc}K^{\rm cyl}{}_h{}^c A_\xi]\\ &\qquad-\xi(S^{-1})_{\rm cyl}{}^{ce}(\nabla_a K^{\rm cyl}{}_{he}]\nabla_b A_c-\xi(S^{-1})_{\rm cyl}{}^{gd}(\nabla_b K^{\rm cyl}{}_{gc})A_d-S^{\rm cyl}{}_{bd}K^{\rm cyl}{}_c{}^d A_\xi]\\ &\qquad-\xi(S^{-1})_{\rm cyl}{}^{ce}(\nabla_a K^{\rm cyl}{}_{be})[\nabla_c A_h-\xi(S^{-1})_{\rm cyl}{}^{gd}(\nabla_c K^{\rm cyl}{}_{hg})A_d-S^{\rm cyl}{}_{dc}K^{\rm cyl}{}_h{}^d A_\xi]\\ &\qquad-S^{\rm cyl}{}_{ae}K^{\rm cyl}{}_h{}^e[\nabla_b A_\xi-K^{\rm cyl}{}_{bd}(S^{-1})_{\rm cyl}{}^{cd}A_c]\\ &\quad-S^{\rm cyl}{}_{ae}K^{\rm cyl}{}_b{}^e[\partial_\xi A_h-K^{\rm cyl}{}_{hg}(S^{-1})_{\rm cyl}{}^{gd}A_d]\}-K^{\rm cyl}{}_{he}(S^{-1})_{\rm cyl}{}^{ce}[\partial_\xi A_c-K^{\rm cyl}{}_{cg}(S^{-1})_{\rm cyl}{}^{gd}A_d]\\ &\qquad-\partial_\xi[\partial_\xi A_h-K^{\rm cyl}{}_{hd}(S^{-1})_{\rm cyl}{}^{cd}A_c]+ie(\phi\nabla_h\phi^*-\phi^*\nabla_h\phi)+2e^2A_h\phi\phi^*=0, \end{aligned} \tag{25}$$

$$\begin{aligned} &G_{\rm cyl}{}^{ab}\{\nabla_a[\nabla_b A_\xi-K^{\rm cyl}{}_{bd}(S^{-1})_{\rm cyl}{}^{cd}A_c]\\ &\quad-K^{\rm cyl}{}_{ad}(S^{-1})_{\rm cyl}{}^{cd}[\nabla_b A_c-\xi(S^{-1})_{\rm cyl}{}^{ge}(\nabla_b K^{\rm cyl}{}_{cg})A_e\xi(S^{-1})_{\rm cyl}{}^{ge}(\nabla_b K^{\rm cyl}{}_{cg})A_e-S^{\rm cyl}{}_{bg}K^{\rm cyl}{}_c{}^g A_\xi]\\ &\qquad-\xi(S^{-1})_{\rm cyl}{}^{ed}(\nabla_a K^{\rm cyl}{}_{be})[\nabla_d A_\xi-K^{\rm cyl}{}_{gd}(S^{-1})_{\rm cyl}{}^{cg}A_c]-S^{\rm cyl}{}_{ae}K^{\rm cyl}{}_b{}^e\partial_\xi A_\xi\}\\ &\qquad-\partial_\xi{}^2A_\xi+ie(\phi\partial_\xi\phi^*-\phi^*\partial_\xi\phi)+2e^2A_\xi\phi\phi^*=0. \end{aligned} \tag{26}$$

$$\begin{aligned} &\tfrac{1}{2}K_{\rm cyl}{}^{ab}[g^{cd}F_{ac}F_{bd}+2F_{a\xi}F_{b\xi}+2((D_a\phi)^*(D_b\phi))]|_{\xi=0}-\tfrac{1}{4}g^{ab}[-2K_{\rm cyl}{}^{cd}F_{ac}F_{bd}+g^{cd}\partial_\xi(F_{ac}F_{bd})+2\partial_\xi(F_{a\xi}F_{b\xi})\\ &\qquad+2\partial_\xi((D_a\phi)^*(D_b\phi))]|_{\xi=0}+3\partial_\xi H_s|_{\xi=0}=0. \end{aligned} \tag{27}$$

By assumption the above equations are satisfied by the functions (22) for $r_0(\tau,\sigma,\theta) = r_0 = \text{const}$.

The first-order equations generally have a much more complicated form then the zero-order Eqs. (24)–(27). Although first-order (in ε) equations are satisfied in the whole space-time [in the region of space-time where coordinates (σ^a,ξ) are well defined], in order to make the analysis simpler we contract them to hypersurface $\xi = 0$. Contraction of complete first order in ε field theory to the theory on the hypersurface $\xi = 0$ means that in expansion of these equations in powers of ξ we keep only zero-order terms as an approximation of the complete theory. If we want to have a better approximation of the field theoretical model (in sector localized around some particular vortex solutions) by the membrane theory we should consider higher-order terms, but the purpose of this paper is to show the advantages of using a membrane description in the Abelian Higgs model, not a detailed study of an effective membrane theory.

The first-order equations on the hypersurface $\xi = 0$ take the form

$$\partial_\xi\phi|_{\xi=0}k_a{}^a = 0, \tag{28}$$

$$\{g^{ab}\nabla_a(k_b{}^cA_c) - 2k^{ab}K^{\text{cyl}}{}_abA_\xi\}|_{\xi=0} = 0, \tag{29}$$

$$\{g^{ab}\nabla_a(k_{bh}A_\xi) - \partial_\xi(k_h{}^dA_d) + k_h{}^cF_{c\xi} + [2k_{hc}K_{\text{cyl}}{}^{cd} + k_h{}^dK_{\text{cyl}_c}{}^c + 2K^{ad}K^{\text{cyl}}{}_{ah}]A_d\}|_{\xi=0} = 0, \tag{30}$$

$$k^{ab}[g^{cd}F_{ac}F_{bd} + F_{a\xi}F_{b\xi} + (D_a\phi)^*(D_b\phi)]|_{\xi=0} = 0, \tag{31}$$

where we used (24)–(27) on hypersurface $\xi = 0$ to make them simpler. If we expand A_ξ, A_a, and ϕ in powers of ξ,

$$\begin{aligned}\phi &= \phi^0 + \xi\phi^1 + \xi^2\phi^2 + \cdots,\\ A_a &= A^0{}_a + \xi A^1{}_a + \xi^2A^2{}_a + \cdots,\\ A_\xi &= A^0{}_\xi + \xi A^1{}_\xi + \xi^2A^2{}_\xi + \cdots,\end{aligned} \tag{32}$$

[where coefficients of expansion are functions of (σ^a)] and put them into (28)–(31) we will see that in these equations appear only coefficients $A^0{}_\xi, A^0{}_a, A^1{}_a, \phi^0$, and ϕ^1:

$$\phi^1k_a{}^a = 0, \tag{33}$$

$$g^{ab}\nabla_a(k_b{}^cA^0{}_c) - 2k^{ab}K^{\text{cyl}}{}_{ab}A^0{}_\xi = 0, \tag{34}$$

$$g_{ab}\nabla_a(k_{bh}A^0{}_\xi) - k_h{}^dA^1{}_d + k_h{}^c(\partial_cA^0{}_\xi - A^1{}_c) + [2k_{hc}K_{\text{cyl}}{}^{cd} + k_h{}^dK_{\text{cyl}_c}{}^c + 2k^{ad}K^{\text{cyl}}{}_{ah}]A^0{}_d = 0, \tag{35}$$

$$\begin{aligned}k^{ab}[g^{cd}&(\partial_aA^0{}_c - \partial_cA^0{}_a)(\partial_bA^0{}_d - \partial_dA^0{}_b)\\ &+(\partial_aA^0{}_\xi - A^1{}_a)(\partial_bA^0{}_\xi - A^1{}_b)\\ &+(\partial_a\phi^0 - ieA^0{}_a\phi^0)^*(\partial_b\phi^0 - ieA^0{}_b\phi^0)] = 0.\end{aligned} \tag{36}$$

Equations (33)–(36) define the effective membrane theory which has internal degrees of freedom given by the $A^0{}_\xi$, $A^0{}_a$, $A^1{}_a$, ϕ^0, ϕ^1. If vortex solutions for which $A^0{}_\xi = A^0{}_a = A^1{}_a = \phi^0 = 0$ exists then the effective theory is a simple Nambu-Goto membrane $K_a{}^a = 0$.

The construction presented in this section works equally well for a sector of the field theory localized around the straight-linear vortices as for a sector localized, for example, around torus shape vortices (the only change we need to do is put $K_{\text{tor}}{}^{ab}$ instead of $K_{\text{cyl}}{}^{ab}$).

In the next section we will find the correspondence between membranes and strings for the simple Abrikosov-Nielsen-Olesen solutions.

IV. EXAMPLES

From (33)–(36) we can see that the shape and evolution of a membrane depends, in a very strong way, on an analytical form of the solutions A_ξ, A_a, ϕ. In the zero order (with respect to ξ) approximation the evolution of a membrane depends only on the first few coefficients $(A^0{}_\xi, A^0{}_a, A^1{}_a, \phi^0, \phi^1)$ of expansion in powers of ξ.

Let us consider a particular vortex solution

$$\phi = e^{-in\theta}F(r_0+\xi), \quad A_\theta = (r_0+\xi)^2A(r_0+\xi),$$

$$A_i = A_\xi = 0, \tag{37}$$

where $r_0 = \text{const}$, $F(r_0+\xi)$, and $A(r_0+\xi)$ are some functions of ξ, and $(\sigma^i) = (\tau,\sigma)$ are stringlike coordinates. The above solutions are called Abrikosov-Nielsen-Olesen solutions. Although the particular choice of solutions makes Eqs. (33)–(36) not look covariant with respect to the reparametrization, it allows us to find important features of the model. We will start the analysis of this particular case from straight Abrikosov-Nielsen-Olesen vortex solutions.

A. Straight vortex solutions

The membrane equation for these solutions has the form

$$\{K_{\text{cyl}}{}^{\theta\theta}h - \tfrac{1}{2}g^{\theta\theta}\partial_\xi h + 3\partial_\xi H_s\}|_{\xi=0} = 0, \tag{38}$$

where $h = (\partial_\xi A_\theta)^2 + (n+eA_\theta)^2F^2$ and $H_s = \frac{1}{2}(\partial_\xi F)^2 + \lambda(F^2 - 2m^2/\lambda)^2$. On the other hand from considerations of Sec. I (see Fig. 2) we know that the maximum of the function w lies in each instant of time on a cylinder of radius $r_0 = \text{const}$. If we parameterize the cylinder by parameters (t,z,θ) we will obtain that the $g^{\theta\theta}$ component of the metric on a cylinder of radius r_0 is equal to $-1/r_0{}^2$ and $K^{\text{cyl}}{}_{\theta\theta} = -r_0(K_{\text{cyl}}{}^{\theta\theta} = -1/r_0{}^3)$. Thus from (38) we have

$$\{-2h + r_0\partial_\xi h + 6r_0{}^3\partial_\xi H\}|_{\xi=0} = 0. \tag{39}$$

If the maximum of the function w is close to the line of zeros of the Higgs field we can presume asymptotic behavior of F and A_θ, i.e., $F \approx \bar{a}(r_0+\xi)^n$ and $A_\theta \approx \bar{b}(r_0+\xi)^2$. In the opposite regime, i.e., when $r_0+\xi$ is very large, functions A_θ and F approach their asymptotic values $A_\theta \to -n/e$ and $F \to \sqrt{2m^2/\lambda} \sim m\sqrt{\lambda}$. Thus, in a regime where $e \sim \sqrt{\lambda}$, we have $A_\theta \sim 1/e$ and $F \sim m/e$. Assuming that dependence on constants m and e is the same for any $r_0+\xi$ (dependence on inversion of coupling constant is typical for any soliton solution) we introduce new constants by the formulas $\bar{a}=(m/e)a$ and $\bar{b}=(1/e)b$. If we had had the theory before rescaling of mass and length it would have been better to have dimensionless constants a and b defined by the formulas $\bar{a}=(m/e)am^n$ and $\bar{b}=(1/e)bm^2$. Because we rescaled mass and length we have all fields and constants dimensionless. If we put $F \approx (m/e)a(r_0+\xi)^n$ and $A_\theta \approx (1/e)b(r_0+\xi)^2$ into (39) we will obtain

$$\left[3\frac{\lambda}{e^2}m^2na^4\right]x^{n+1}+[b^2a^2(n+1)]x^2+[2na^2(nb-3m^2)]x$$
$$+[2n^2(n-2)a^2]=0, \quad (40)$$

where the new variable is $x={r_0}^2$. To approximate roughly the dependence of the solutions of Eq. (40) on the winding number we assume that self-interaction of the Higgs fields is much weaker than the interaction of the Higgs field with electromagnetic fields (λ/e^2 is very small). In this limit Eq. (40) simplifies to a simple square equation which has two solutions:

$$x=\frac{n}{(n+1)b^2}[3m^2$$
$$-nb \pm \sqrt{9m^4-6nbm^2+n^2b^2-2(n+1)(n-2)b^2}]. \quad (41)$$

If m^2 is large but much smaller than λ/e^2 (i.e., λ/e^2 approaches zero value faster then m^2 approaches infinity so as $(\lambda/e^2)m^2$ approaches the zero value) we have

$$x \approx \frac{6m^2}{b^2}\left(\frac{n}{n+1}\right). \quad (42)$$

Thus in considered limit the radius of a cylinder r_0 groves with n from $2m/b$ for $n=2$ to $\sqrt{6}m/b$ for $n\to\infty$.

B. Curved vortices

We considered the connection between parameters characterizing the field theoretical model and the radius of the membrane defined by the straight Abrikosov-Nielsen-Olesen solutions. Let us turn to consideration of slightly curved Abrikosov-Nielsen-Olesen-type vortex solutions. If we put functions of type (37) into Eqs. (28)–(31) and parameterize the cylinder ($\partial_\xi w|_{\xi=0}=0, \varepsilon=0$) by (t,z,θ) [in this parametrization ${K^{\rm cyl}}_{i\theta}={K^{\rm cyl}}_{ij}=0, {K^{\rm cyl}}_{\theta\theta}=-r_0$ and $g_{ij}=\eta_{ij}, g_{\theta\theta}=-1/{r_0}^2$ where $(x^i)=(t,z)$] we will obtain

$$k_{\theta\theta}=k_{i\theta}=0, \quad k_{tt}=k_{zz}. \quad (43)$$

Conditions (43) mean that the hypersurface obtained by deformation of the cylinder ($\partial_\xi w|_{\xi=0}=0, \varepsilon=0$) which is defined by $K_{ab}={K^{\rm cyl}}_{ab}+k_{ab}$ has to obey the conditions

$$K_{\theta\theta}=-r_0, \quad K_{i\theta}=0, \quad K_{tt}=K_{zz}. \quad (44)$$

Let us find some hypersurfaces satisfying constraints (44).

(a) (i) Straight cylinder solution. Conditions (44) have obviously the trivial solution which is a cylindrical hypersurface [see Fig. 4(a)]:

$$[X^\mu]=[t, r_0\cos(\theta), r_0\sin(\theta), z]. \quad (45)$$

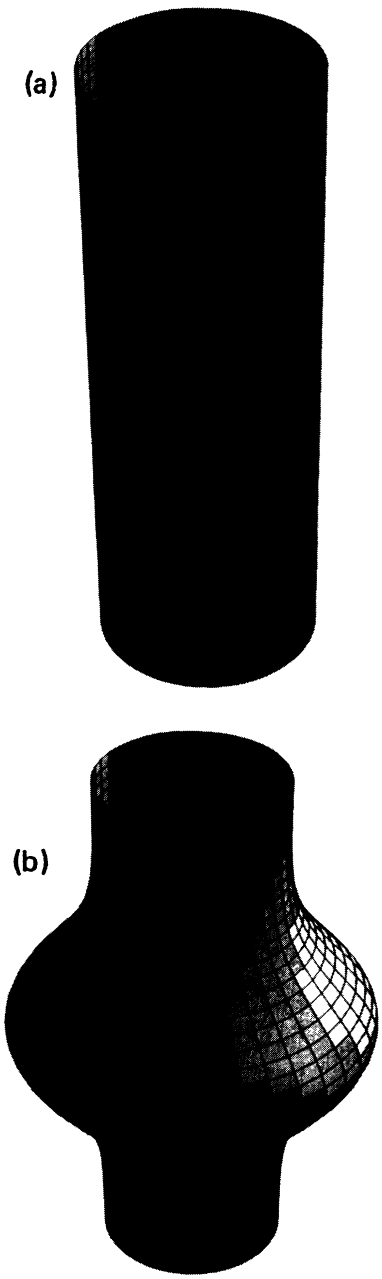

FIG. 4. Straight-linear vortex configurations (a) and (b) are indistinguishable for the string description.

The coefficients of extrinsic curvature for the cylinder are equal $K_{\theta\theta} = -r_0, K_{i\theta} = K_{ij} = 0$.

(ii) Vachaspati solution. The other solutions are those found by Vachaspati and Vachaspati [13]; they are defined by the hypersurface

$$[X^{\mu}] = [t, r_0\cos(\theta) + \chi(t+x), r_0\sin(\theta) + \psi(t+z), z], \tag{46}$$

which is a deformed cylinder of radius r_0 [see Fig. 5(a)]. The deformation of this type moves with the speed of light in the Z direction The coefficients of extrinsic curvature for this hypersurface are $K_{\theta\theta} = -r_0, K_{i\theta} = 0, K_{tt} = K_{zz} = K_{tz} = \chi''\cos(\theta) + \psi''\sin(\theta)$ where the prime denotes the derivative with respect to t or equivalently z.

(b) For solutions (a) (i) and (a) (ii) we have complete correspondence between the membrane and string descriptions (axis of constant radius membrane is just the Nambu-Goto string $K_i{}^i = 0$). At this point we will present a new type of vortex excitations. The radius of membranes describing these solutions is not just constant as it was for the solutions presented in point (a), but it has its own dynamics (can change in space and time). The existence of such solutions is a good reason for involving a complete membrane theory approximately describing Abelian Higgs model [12].

(i) Bubble moving solution. The interesting solution (because it shows usefulness of the membrane description) is a bubble of arbitrary shape moving along the cylinder with the speed of light. This solution is presented on Fig. 4(b) and is given by

$$[X^{\mu}] = [t, r_0(t+z)\cos(\theta), r_0(t+z)\sin(\theta), z]. \tag{47}$$

The extrinsic curvature constants are equal to $K_{\theta\theta} = -r_0(t+z), K_{i\theta} = 0, K_{tt} = K_{tz} = K_{zz} = r_0''$.

(ii) Bubble excited Vachaspati-Vachaspati solutions. There exists possibility to excite Vachaspati-Vachaspati solutions in the way presented in point (b) (i) [see Fig. 5(b)]. Solutions of this type have the form

$$[X^{\mu}] = [t, r_0(t+z)\cos(\theta) + \chi(t+z), \times r_0(t+z)\sin(\theta) + \psi(t+z), z], \tag{48}$$

and their extrinsic curvature coefficients have the form $K_{\theta\theta} = -r_0(t+z), K_{i\theta} = 0, K_{tt} = K_{tz} = K_{zz} = r_0'' + \chi''\cos(\theta) + \psi''\sin(\theta)$.

Thus the deformed Abrikosov-Nielsen-Olesen-type functions are solutions of the field Eqs. (18)–(20) (in consider approximation) if they are deformed in a way described by the membrane equations (33)–(36). Con-

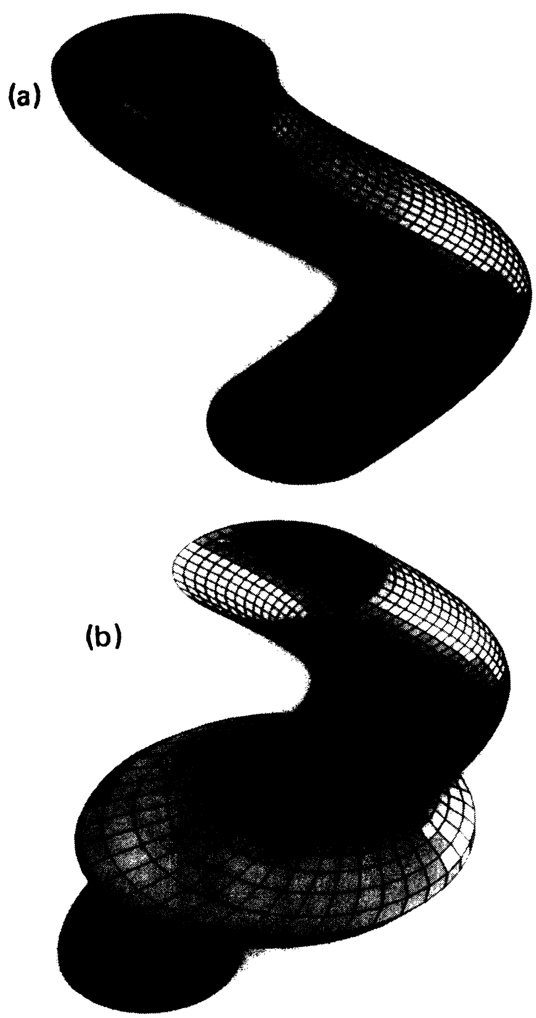

FIG. 5. Curved vortex configurations (a) and (b) are indistinguishable for the string description.

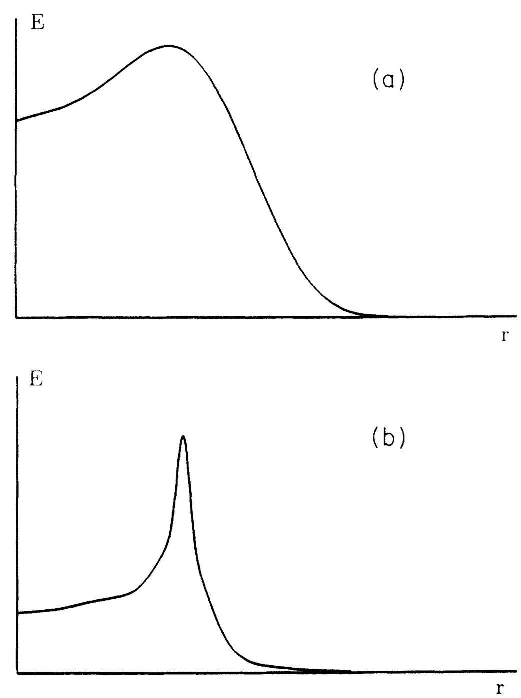

FIG. 6. The arbitrary order membrane approximation of a field theoretical model is better for field configuration (b) which has sharper peak of the maximum of the function w.

straints (44) we can understand as necessary conditions for the existence of deformed Abrikosov-Neilsen-Olesen type solutions (for example, Vachaspati or bubble moving type) of equations (18)–(21). The membrane theory (44) defines, in the case of deformed Abrikosov-Nielsen-Olesen–type functions, the Vachaspatis solutions as well as new bubble moving solutions.

V. REMARKS

Based on the existence of the maximum density of the energy for nonzero distances from the line of the zeros of the Higgs field $r_0 > 0$ we have obtained for vortex solutions with $n \geq 2$ the effective membrane theory. The membrane description makes sense only in the case when vortices with $n \geq 2$ are stable. In the Abelian Higgs model, vortex solutions with $n \geq 2$ are stable in the sector where the range of the matter self-interaction exceeds that of the electromagnetic one $\lambda < 2e^2$, in this case vortices with $n = 1$ attract each other forming vortices with $n > 1$. In the opposite case $\lambda > 2e^2$, it is found that the system is unstable against decay into separated vortices and that vortices $n = 1$ repeal each other [14].

We have considered only the stable vortices with $n \geq 2$ (e.g., $\lambda < 2e^2$). The membrane theory obtained in Sec. III is a zero order approximation of a starting field theoretical model, but proposed method allows us to obtain arbitrary order approximation. The arbitrary order approximation is a better one if the maximum of the density of the scalar function w is sharp and thin (see Fig. 6).

We could see on the simple example of Abrikosov-Nielsen-Olesen solutions correspondence between membrane and string description of vortices. For Abrikosov-Nielsen-Olesen solutions the membrane is a cylinder which axis is bent like Nambu-Goto string ($K_i{}^i = 0$), but the radius of the hypersurface obtained by the deformation of a straight cylinder has it own dynamics. The radius of the cylinder have been estimated for straight vortex. It appears that this radius is larger for solutions characterized by a larger winding number which agrees qualitatively with the figure content in Ref. [11]. Nontriviality of the membrane description can appear in this sense that string description does not distinguish between situation presented on Fig. 4(a) and on the Fig. 4(b) which is a bubble excited vortex. Simple Vachaspati-Vachaspati modes [13] [see Fig. 5(a)] and bubblelike excited Vachaspati-Vachaspati vortices [see Fig. 5(b)] in the string description are indistinguishable also. It seems that the excitation found in [15], because of the existence of nonvanishing electromagnetic fields A_i, can provide even more nontrivial examples of membrane solutions.

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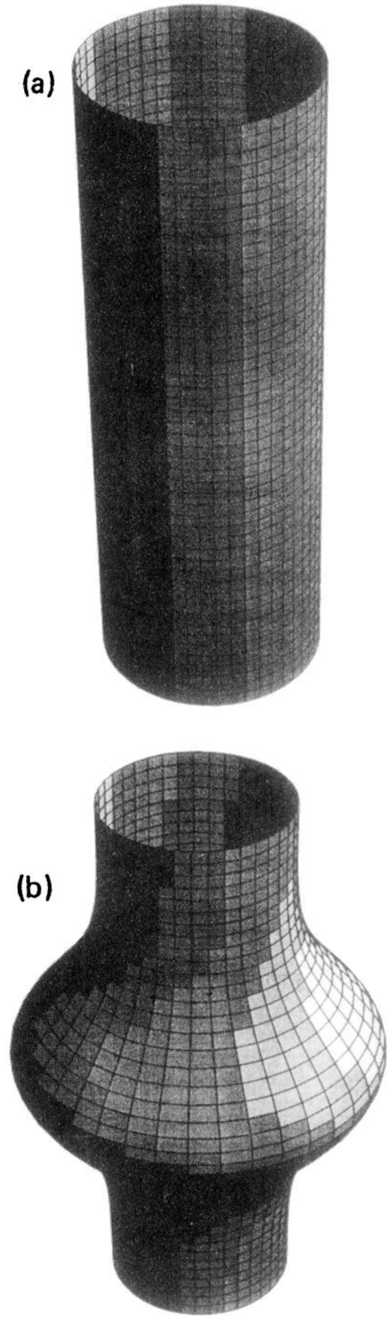

FIG. 4. Straight-linear vortex configurations (a) and (b) are indistinguishable for the string description.

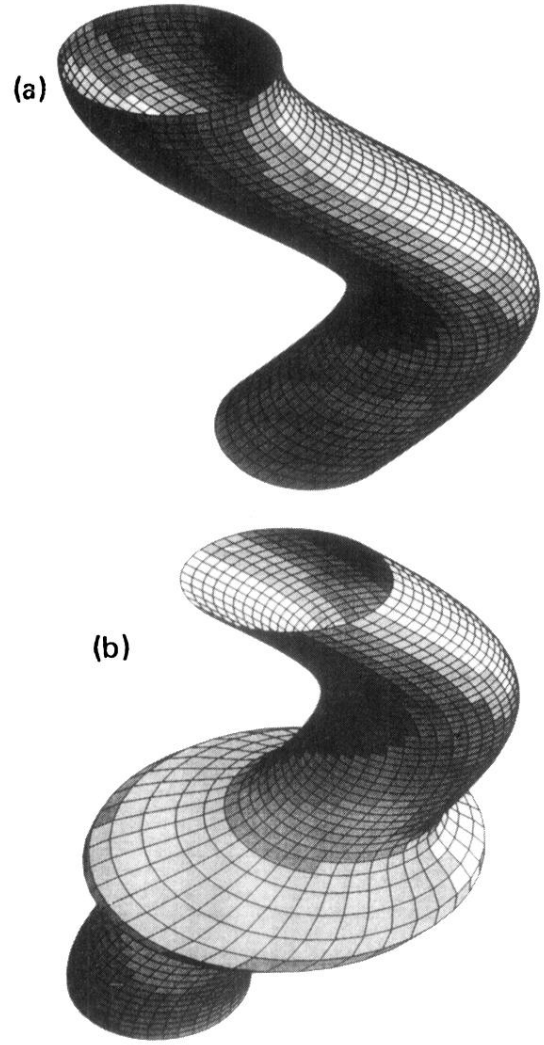

FIG. 5. Curved vortex configurations (a) and (b) are indistinguishable for the string description.