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Exact $O(d,d)$ transformations in WZW models

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Using the algebraic Hamiltonian approach, we derive the exact to all orders (Abelian) $O(d,d)$ transformations of the metric and the dilaton field in WZW and WZW coset models for both compact and noncompact groups. This is obtained in spite of the fact that the algebraic Hamiltonian method does not enable us to derive the exact to all orders antisymmetric tensor (thus, we cannot use it to derive the exact transformations of the antisymmetric tensor). It is shown that under the exact $O(d)\times O(d)$ transformation only the leading order of the inverse metric G^{-1}, is transformed. The quantity $\sqrt{G}\exp(\Phi)$ is the same in all the dual models and in particular is independent of k. We also show that the exact metric and dilaton field that correspond to $G/U(1)^d$ WZW models can be obtained by applying the exact $O(d,d)$ transformations on the (ungauged) WZW model, a result that was known to one loop order only. As an example we give the O(2,2) transformations in the $SL(2,R)$ WZW model that transform to its dual exact models. These include also the exact three-dimensional (3D) black string and the exact 2D black hole with an extra U(1) free field.

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I. INTRODUCTION

$O(d,d)$ symmetries, originally discovered in flat d-dimensional toroidal compactification of closed string theories, provide the moduli space of the corresponding vacuum [1]. The global structure of the moduli space reveals the following property. Points corresponding to different backgrounds are related by a duality group of discrete symmetries, isomorphic to $O(d,d,\mathbb{Z})$ [2]. In other words, $O(d,d,\mathbb{Z})$ symmetries identify equivalent models in the moduli space. This is the generalization of the familiar $R\to 1/R$ symmetry in conformal field theory (CFT) [3].

These duality transformations are generalized to string theories with curved background. It was shown by Buscher [4] that duality symmetry of conformal backgrounds with one isometry transforms from one CFT to another (to one-loop order). Subsequently, it was noticed [5] that if we start with the low-energy effective action in string theory and restrict to space translation invariant but time-dependent background, the effective action exhibits an $O(d,d)$ symmetry. The generalization of the $O(d,d)$ symmetry in curved backgrounds with d isometries [6] and in the heterotic strings [7] was proven to be an exact symmetry of string theories by means of string field theory. The corresponding $O(d,d)$ transformations could be derived to one-loop order in α' only. Giveon and Rocek [8] have used the σ-model approach and showed the one-loop order $O(d,d)$ transformations for a general CFT background with d isometries. Duality symmetries from non-Abelian isometries were considered later [9] where it was found that this new duality transformation maps spaces with a non-Abelian duality to spaces that may have no isometries at all. Such symmetries were also discussed in [10].

$O(d,d)$ symmetries are powerful and intriguing. In addition to providing a better understanding of the moduli space of a given solution, $O(d,d)$ symmetry leads to striking cosmological consequences which we do not fully understand, such as those discussed in [11–14].

The duality symmetries of Wess-Zumino-Witten (WZW) and WZW coset models were discussed in [15]. The exact underlying symmetry responsible for semiclassical duality was identified with the symmetry under affine Weyl transformations. This identification shows that in the compact and unitary case they are exact symmetries of string theory to all orders in α'. Duality transformations were shown [16,17] to be equivalent to integrated marginal perturbations by bilinears in the chiral currents.

In a previous paper [18] we have shown the exact to all orders $O(d,d)$ transformations of the metric and the dilaton field for WZW coset models with d isometries which have a certain type of a background. The aim of this paper is to generalize this result to general WZW and WZW coset models with Abelian isometries.[1] We shall be using the algebraic Hamiltonian approach [20] to derive the exact metric and dilaton field. In spite of

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[1]In the case of $SL(2,R)/U(1)$ and $SU(2)/U(1)$ there is a regularization scheme where the semiclassical background receives no higher-order corrections in α' [19]. In such a case the semiclassical $O(d,d)$ symmetry transformations are also exact [13].

the fact that it is not possible to derive the exact antisymmetric tensor with this method (thus we cannot use it to derive the exact transformations of the antisymmetric tensor), it is shown that we can still derive the exact transformations of the metric and the dilaton field.

The paper is organized as follows. In Sec. II, we just give the expressions for the $O(d,d)$ transformations in one-loop order for the general case with d isometries. In Sec. III we derive an expression for the exact metric and dilaton field for $G/U(1)^d$ WZW coset models by using the algebraic Hamiltonian approach. In Sec. IV, we derive the exact $O(d)\times O(d)$ dual models of the coset models in Sec. III and compare with the one-loop order transformations. In Sec. V, we derive the exact dual models to ungauged WZW models. In particular, we show that the exact coset models can be obtained by exact duality from the ungauged WZW models. In Sec. VI we demonstrate the exact duality transformations in the case of the $SL(2,R)$ WZW model and get also familiar models, such as the exact three-dimensional (3D) black string. In Sec. VII we derive the transformations in the case of noncompact groups where n isometries correspond to compact coordinates and $d-n$ correspond to noncompact coordinates. The duality in this case is generated by $O(n,d-n)\times O(n,d-n)$ (in addition to constant coordinate transformations and a constant shift of the antisymmetric tensor). Section VIII is reserved for a summary and remarks.

II. $O(d,d)$ TRANSFORMATIONS IN ONE-LOOP ORDER

Consider a general σ model with d isometries that correspond to a CFT:

$$\begin{aligned} S = \frac{k}{2\pi}\int d^2\sigma[&F_{\mu\nu}(X)\partial_+X^\mu\partial_-X^\nu + E_{\mu i}(X)\partial_-Y^i\partial_+X^\mu \\ &+\tilde{E}_{i\mu}(X)\partial_+Y^i\partial_-X^\mu + D_{ij}(X)\partial_+Y^i\partial_-Y^j] \\ &-\frac{1}{8\pi}\int d^2\sigma\sqrt{h}R^{(2)}\Phi(X)\ , \end{aligned} \tag{2.1}$$

where the background has D target-space dimensions with $\mu=1,\ldots,D-d$ and $i=1,\ldots,d$. $\sqrt{h}$ and $R^{(2)}$ are the determinant of the metric and the curvature in the Riemann surface, respectively, and Φ is the dilaton field. This background corresponds to the target-space metric G and the antisymmetric tensor B with

$$G = \frac{1}{2}\begin{pmatrix} F+F^T & E+\tilde{E}^T \\ E^T+\tilde{E} & D+D^T \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} F-F^T & E-\tilde{E}^T \\ -E^T+\tilde{E} & D-D^T \end{pmatrix}. \tag{2.2}$$

The background (2.1) exhibits the $O(d,d)$ symmetry whose generators correspond to the following symmetry transformations [8].

(a) $O(d)\times O(d)$ symmetry transformations, under which

$$D \to D' = [(O_1+O_2)D + (O_1-O_2)] \times[(O_1-O_2)D+(O_1+O_2)]^{-1}, \tag{2.3}$$

$$E \to E' = \tfrac{1}{2}[(O_1+O_2) - D'(O_1-O_2)]^{-1}E, \tag{2.4}$$

$$\tilde{E} \to \tilde{E}' = 2\tilde{E}[(O_1-O_2)D+(O_1+O_2)]^{-1}, \tag{2.5}$$

$$F \to F' = F - \tilde{E}[(O_1-O_2)D+(O_1+O_2)]^{-1} \times(O_1-O_2)E, \tag{2.6}$$

$$\Phi \to \Phi' = \Phi + \frac{1}{2}\ln\left(\frac{\det G}{\det G'}\right), \tag{2.7}$$

where G' is the transformed target space metric. In the above O_1 and O_2 are two constant $O(d)$ matrices. These symmetry transformations can be derived by gauging a $U(1)^d$ subgroup in a "larger" background with $D+2d$ target space dimensions with $2d$ isometries [8] and are correct to one-loop order only.

(b) Coordinate transformations: $Y^i \to A^i_jY^j$ where A is a $GL(d,R)$ constant matrix. The transformations with $AA^T=I$ are already included in (a).

(c) A constant shift of the antisymmetric tensor $B_{ij} \to B_{ij}+C_{ij}$, where C is a $d\times d$ antisymmetric matrix.

Notice that the transformations in (b) and (c) are exact to all orders in α' since the equations of motion to all orders are covariant and depend only on the torsion $H_{\alpha\beta\gamma}$.

III. EXACT METRIC AND DILATON IN WZW COSET MODELS

In order to find the exact to all orders metric and dilaton field we shall be using the algebraic Hamiltonian approach for G/H coset models. This approach was derived in [21] and systematized in [22].[2] However, this method is not helpful in calculating the exact antisymmetric tensor. Therefore, we shall not discuss the exact transformations of the antisymmetric tensor in this paper.

Let us first briefly describe the method for coset models G/H. (For a review see [20].) Denote by J^G_a ($a=1,\ldots,\dim G$) and J^H_i ($i=1,\ldots,\dim H$) the currents of the group G and its subgroup H, respectively and $J^G_{a,n}, J^H_{i,n}$ are their "Fourier" components in the Kac-Moody algebra. L_0 is the zero generator of the Virasoro algebra. Then the ground state T (the Tachyon) satisfies the conditions

$$(L_0+\bar{L}_0-2)T = 0, \tag{3.1}$$

[2] In the $SL(2,R)/U(1)$ mode it was shown [23,24] that the background obtained by this method satisfies the β-function equations at least up to the fourth order in α'. In the $SL(2,R)\times U(1)/U(1)$ coset it was shown [25] that the exact background satisfies the β-function equations at least up to second order.

$$(J_0^H + \bar{J}_0^H)T = 0, \tag{3.2}$$

$$J_n^G T = \bar{J}_n^G T = 0, \quad n \geq 1. \tag{3.3}$$

Here

$$L_0 = \frac{\Delta_G}{k - \tilde{c}_G} - \frac{\Delta_H}{k - \tilde{c}_H}, \quad \bar{L}_0 = \frac{\bar{\Delta}_G}{k - \tilde{c}_G} - \frac{\bar{\Delta}_H}{k - \tilde{c}_H}, \tag{3.4}$$

and Δ_G, Δ_H are the Casimir operators in G and in H, i.e., $\Delta_G = J^G \cdot J^G$, $\bar{\Delta}_G = \bar{J}^G \cdot \bar{J}^G$, $\Delta_H = J^H \cdot J^H$, $\bar{\Delta}_H = \bar{J}^H \cdot \bar{J}^H$, and $\tilde{c}_G, \tilde{c}_H$ are the coexter of G, H respectively. In the language of the group elements g and the left and right gauged subgroup elements h_L and h_R, condition (3.2) is a remnant of the gauge invariance $T(h_L g h_R^{-1}) = T(g)$ which demands that the tachyon is a singlet under the action of the subgroup H. Now we parametrize the group elements of G with the coordinates X_μ, $\mu = 1, \ldots, N = \dim G$, and express the currents in terms of first-order differential operators of X_μ which satisfy the Lie algebra of the group (so that L and L_0 become a differential operator). Finally we define gauge-invariant coordinates $\tilde{X}_\mu$, $\mu = 1, \ldots, D = \dim G - \dim H$, so that when the tachyon is $T = T(\tilde{X})$ it automatically satisfies the condition (3.2). Now we write the Casimir operators in terms of $\tilde{X}_\mu$, by simply using the chain rule. As is well known, the effective action for the tachyon is

$$S(T) = \int d^D X \sqrt{-G} e^{\Phi} [G^{\mu\nu} \partial_\mu T \partial_\nu T - V(T)], \tag{3.5}$$

where Φ is the dilaton field and $V(T)$ is the tachyon potential. On the other hand, since the tachyon is completely defined through the action of the zero modes, its action is equivalent to

$$S(T) = \int d^D x \sqrt{-G} e^{\Phi} [T(L_0 + \bar{L}_0)T - V(T)]. \tag{3.6}$$

From actions (3.5) and (3.6), expressed in terms of $\tilde{X}_\mu$, we obtain

$$L_0 + \bar{L}_0 = -e^{-\Phi} \frac{1}{\sqrt{-G}} \partial_\mu (e^{\Phi} \sqrt{-G} G^{\mu\nu} \partial_\nu) \tag{3.7}$$

from which we find the exact metric and the exact dilaton field.

This approach agrees with the fact that (ungauged) WZW are exact to all order in $1/k$ up to a shift of the level $k \to k - \tilde{c}_G$. To get the one-loop order background one should put $\tilde{c}_G = \tilde{c}_H = 0$ in $L_0 + \bar{L}_0$ of the gauged model, which is equivalent to taking astronomically large k and neglecting $\tilde{c}_G$ and $\tilde{c}_H$.

Now, consider a general $(D + d)$-dimensional target space that corresponds to a (ungauged) WZW model based on a group G with level k which has a subgroup $U(1)^d$. We pick a basis T^i, $i = 1, \ldots, d$ in the Cartan subalgebra with $[T^i, T^j] = 0$ and $\mathrm{tr} T^i T^j = \delta^{ij}$. Now we parametrize the group elements as

$$g = \exp\left(i \sum_{i=1}^d \theta_1^i T^i\right) \tilde{g}(X) \exp\left(i \sum_{i=1}^d \theta_2^i T^i\right) \tag{3.8}$$

so the action can be written as

$$\begin{aligned} S = \frac{k}{2\pi} \int d^2\sigma [&\partial_+ \theta_1^i \partial_- \theta_1^i + \partial_+ \theta_2^i \partial_- \theta_2^i \\ &+ 2M_{ij}(X) \partial_- \theta_1^i \partial_+ \theta_2^j + 2N^1_{\mu i}(X) \partial_+ X^\mu \partial_- \theta_1^i \\ &+ 2N^2_{\mu i}(X) \partial_- X^\mu \partial_+ \theta_2^i + F_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu], \end{aligned} \tag{3.9}$$

with $\mu = 1, \ldots, D - d$ and $i = 1, \ldots, d$. This action (3.9) is invariant under $U(1)_L^d \times U(1)_R^d$ symmetry generated by the conserved holomorphic currents that correspond to translational symmetry in θ_1^i and θ_2^i:

$$J^i = \partial_+ \theta_1^i + M_{ij} \partial_+ \theta_2^j + N^1_{\mu i} \partial_+ X^\mu, \tag{3.10}$$

$$\bar{J}^i = \partial_- \theta_2^i + M_{ji} \partial_- \theta_1^j + N^2_{\mu i} \partial_- X^\mu. \tag{3.11}$$

Now we want to describe the coset model $G/U(1)^d$ in the algebraic Hamiltonian approach. The group G is parametrized so that the left and the right $U(1)$ currents are the commuting differential operators

$$J_j = i\partial_{\theta_1^j}, \bar{J}_j = i\partial_{\theta_2^j}, \quad j = 1, \ldots, d. \tag{3.12}$$

We can choose the rest of the generators to correspond to action (3.9) (since the WZW is exact). Now we gauge the axial symmetry. In the gauged model,

$$\begin{aligned} L_0 + \bar{L}_0 = -\frac{1}{k - \tilde{c}_G} [&\Sigma^\mu(X) \partial_{X^\mu} + \Gamma_1^i(X) \partial_{\theta_1^i} + \Gamma_2^i(X) \partial_{\theta_2^i} + G^{\mu\nu}(X) \partial_{X^\mu} \partial_{X^\nu} + 2G_1^{\mu i}(X) \partial_{X^\mu} \partial_{\theta_1^i} \\ &+ 2G_2^{\mu i}(X) \partial_{X^\mu} \partial_{\theta_2^i} + L_1^{ij}(X) \partial_{\theta_1^i} \partial_{\theta_1^j} + L_2^{ij}(X) \partial_{\theta_2^i} \partial_{\theta_2^j} + 2P^{ij}(X) \partial_{\theta_1^i} \partial_{\theta_2^j}] + \frac{1}{k} \sum_{i=1}^d (\partial^2_{\theta_1^i} + \partial^2_{\theta_2^i}), \end{aligned} \tag{3.13}$$

where $G^{\mu\nu}$, $G_1^{\mu i}$, $G_2^{\mu i}$, L_1^{ij}, L_2^{ij}, and P^{ij} are the components of the inverse of the metric in the WZW action, i.e.,

$$G^{-1} = \begin{pmatrix} G & G_1 & G_2 \\ G_1^T & L_1 & P \\ G_2^T & P^T & L_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(F + F^T) & N^1 & N^2 \\ {N^1}^T & I & M \\ N_2^T & M^T & I \end{pmatrix}^{-1} \tag{3.14}$$

and $\Sigma, \Gamma_1, \Gamma_2$ are obtained from (3.7) with $\Phi = 0$. (Notice that we have written all matrices with upper indices for convenience. This has nothing to do with covariant raising indices.)

Now we should define d gauge-invariant coordinates which we denote by Y^i and in addition to X^μ, which are not transformed, the target space of the gauged action is D dimensional. In the axial gauge the gauge-invariant coordinates should satisfy d conditions:

$$(J_i + \bar{J}_i)Y^j = 0, \quad i,j = 1,\ldots,d. \tag{3.15}$$

Taking

$$Y^i = \theta_1^i - \theta_2^i \tag{3.16}$$

we obtain

$$L_0 + \bar{L}_0 = -\frac{1}{k - \tilde{c}_G}\left(\Sigma_\mu(X)\partial_{X^\mu} + (\Gamma_i^1 - \Gamma_i^2)\partial_{Y^i} + G^{\mu\nu}\partial_{X^\mu}\partial_{X^\nu} + 2(G_1^{\mu i} - G_2^{\mu i})\partial_{X^\mu}\partial_{Y^i}\right.$$
$$\left. + (L_1^{ij} + L_2^{ij} - 2\delta^{ij} - 2P^{ij})\partial_{Y^i}\partial_{Y^j} + \frac{2\tilde{c}_G}{k}\delta^{ij}\partial_{Y^i}\partial_{Y^j}\right). \tag{3.17}$$

Define $\tilde{I}$ to be the $D \times D$ matrix,

$$\tilde{I} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{3.18}$$

and I is the $d \times d$ unit matrix. From expression (3.17) we see that

$$G^{-1}_{\text{exact}} = \frac{1}{k - \tilde{c}_G}\left(G^{-1}_{\text{classical}} + \frac{2\tilde{c}_G}{k}\tilde{I}\right). \tag{3.19}$$

Here we omitted the prefactor k in front of $G_{\text{classical}}$. We could, of course, choose different gauge-invariant coordinates $Y^i = C^i_j(\theta_1^j - \theta_2^j)$, where C is a constant GL(d) matrix, but this is simply a coordinate transformation of the Y^i coordinates. In this case the $1/k$ correction to the semiclassial metric becomes $(C^T C)\tilde{c}_G/k$ in the (ij) directions.

IV. EXACT DUALITY TRANSFORMATIONS

All the O(d)×O(d) dual models to the gauged model in the previous section can be obtained by all the different anomaly free gaugings (generators) of the U(1)^d subgroup. The condition for the anomaly cancellation is [26]

$$\text{tr}T_L^a T_L^b = \text{tr}T_R^a T_R^b, \quad a,b = 1,\ldots,\text{dim}H, \tag{4.1}$$

where T_L^a, T_R^a are the generators of the left and right gauged subgroups. Instead of looking for all the anomaly-free gaugings we shall use the following method, which is an application of the idea used in [8] for the σ-model action. The ungauged action (3.9), or equivalently, the Casimir operator of the ungauged model, is invariant under the transformation

$$\begin{aligned} \theta_1 \to \theta_1' = O_1\theta_1, \quad \theta_2 \to \theta_2' = O_2\theta_2, \\ L_1 \to L_1' = O_1 L_1 O_1^T, \quad L_2 \to L_2' = O_2 L_2 O_2^T, \\ P \to P' = O_1 P O_2^T, \\ G_1 \to G_1' = G_1 O_1^T, \quad G_2 \to G_2' = G_2 O_2^T, \\ \Gamma_1 \to \Gamma_1' = \Gamma_1 O_1^T, \quad \Gamma_2 \to \Gamma_2' = \Gamma_2 O_2^T, \end{aligned} \tag{4.2}$$

where O_1, O_2 are two O(d) matrices [or O($d,\mathbb{Z}$) if we want to preserve periodicity]. Now we write $L_0 + \bar{L}_0$ in the WZW action with respect to the transformed coordinates θ_1', θ_2' and gauge the U(1)^d subgroup generated by the currents $J_i = i\partial_{\theta_1'^i}$ and $\bar{J}_i = i\partial_{\theta_2'^i}$, which is still an anomaly-free gauging. Using $\text{tr}T^iT^j = \delta^{ij}$ it is easy to see that the new generators, which are linear combinations of the T^i's satisfy the condition (4.1). The result (in the θ' coordinates) is a rotation of the Casimir operator of the ungauged action while the gauged subgroup is unchanged. Therefore, we get an expression for all the O(d)×O(d) dual models:

$$G^{-1} = \frac{1}{k - \tilde{c}_G}\begin{pmatrix} G & G_1 O_1^T - G_2 O_2^T \\ O_1 G_1^T - O_2 G_2^T & O_1 L_1 O_1^T + O_2 L_2 O_2^T - O_1 P O_2^T - O_2 P^T O_1^T - 2I + \frac{2\tilde{c}_G}{k} I \end{pmatrix} \tag{4.3}$$

for two general O(d) matrices. The important point to notice is that the $1/k$ correction to the inverse metric with respect to the semiclassical one is invariant under the O(d)×O(d) transformations [also when it takes the form $C^T C(\tilde{c}_G/k)$]. Finally, we recall that all the one-loop order metrics that are obtained by O(d)×O(d) transformations can also be obtained by different gaugings of the ungauged WZW model. Therefore, under the O(d)×O(d)

transformations the exact inverse metric transforms as

$$G^{-1}_{\rm exact} \to G'^{-1}_{\rm exact} = \frac{1}{k-\tilde{c}_G}\left(G'^{-1}_{\rm classical} + \frac{2\tilde{c}_G}{k}\tilde{I}\right), \quad (4.4)$$

where $\tilde{I}$ is given in (3.18) and $G'_{\rm classical}$ is the semiclassical metric obtained after the one-loop $O(d)\times O(d)$ transformations, as is written in (2.3)–(2.6). Thus, to calculate the exact inverse metrics of the models related by $O(d)\times O(d)$ duality we need only to transform the leading order of the original metric, which involves only the leading order of both the metric and the antisymmetric tensor. Hence, we reach the following conclusion. Given an exact metric that corresponds to a coset model, in order to find the metrics of all the exact $O(d)\times O(d)$ dual models it is enough to know the antisymmetric tensor to one-loop order.

The transformation of the exact dilaton under the $O(d)\times O(d)$ can be derived as well. Since the matrix G and the vector Σ are invariant under the transformation, using (3.7) it is easy to see that the quantity $\sqrt{G}e^{\Phi}$ is invariant under the transformation. Therefore under the transformation $G_{\rm exact} \to G'_{\rm exact}$ we obtain

$$\Phi_{\rm exact} \to \Phi'_{\rm exact} = \Phi_{\rm exact} + \frac{1}{2}\ln\left(\frac{\det G}{\det G'}\right), \quad (4.5)$$

where, G, G' correspond to the exact metric and the transformed exact metric.

In particular, we can construct the axial-vector duality. This duality was revealed in the σ-model approach [27] and was proven to be an exact symmetry of the CFT [13]. If we take $O_1 = -O_2 = I$ we interchange the axial and vector gaugings.

V. EXACT O(d,d) TRANSFORMATIONS IN (UNGAUGED) WZW MODELS

The method described above can be used to construct the exact $O(d,d)$ transformations also in ungauged WZW models. Consider a WZW model based on a group G which has a $U(1)^d$ subgroup. This means that the σ model has $2d$ isometries. Although the ungauged action is exact to all orders up to a shift of the level (which is just a prefactor), the exact $O(d)\times O(d)$ transformations introduce $1/k$ corrections in the dual models. To apply the duality transformations we should consider an equivalent model: $G\times U(1)^{2d}_k/U(1)^{2d}_k$. We use the notations in (3.13) for the (ungauged) WZW model and parametrize the extra $U(1)$ generators by the differential operators $i\partial_{\varphi_1^i}$ and $i\partial_{\varphi_2^i}$ taken with level k. This introduces the term $-(2/k)\sum_{i=1}^d(\partial^2_{\varphi_1^i}+\partial^2_{\varphi_2^i})$ to $L_0+\bar{L}_0$ of the ungauged model. Now we gauge the axial $U(1)^{2d}_L\times U(1)^{2d}_R$ subgroup generated by

$$\begin{aligned}
\mathcal{J}_i &= i\partial_{\theta_1^i} \ \text{ for } i=1,\dots,d,\\
\mathcal{J}_i &= i\partial_{\varphi_1^{i-d}} \ \text{ for } i=d+1,\dots,2d,\\
\bar{\mathcal{J}}_i &= i\partial_{\theta_2^i} \ \text{ for } i=1,\dots,d,\\
\bar{\mathcal{J}}_i &= i\partial_{\varphi_2^{i-d}} \ \text{ for } i=d+1,\dots,2d.
\end{aligned} \quad (5.1)$$

Notice that $1/k$ is simply a redefinition of the $U(1)$ free fields but it ensures that the condition for anomaly cancellation (4.1) is satisfied after the rotation we shall make. Thus, we have now

$$L_0 + L_0 = -\frac{1}{k-\tilde{c}_G}[\Sigma^\mu(X)\partial_{X^\mu} + \Gamma_1^i(X)\partial_{\theta_1^i} + \Gamma_2^i(X)\partial_{\theta_2^i} + G^{\mu\nu}\partial_{X^\mu}\partial_{X^\nu} + 2G_1^{\mu i}\partial_{X^\mu}\partial_{\theta_1^i} + 2G_2^{\mu i}\partial_{X^\mu}\partial_{\theta_2^i}$$
$$+L_1^{ij}\partial_{\theta_1^i}\partial_{\theta_1^j} + L_2^{ij}\partial_{\theta_2^i}\partial_{\theta_2^j} + 2P^{ij}\partial_{\theta_1^i}\partial_{\theta_2^j}] - \frac{2}{k}\sum_{i=1}^d(\partial^2_{\varphi_1^i}+\partial^2_{\varphi_2^i}) + \frac{1}{k}\sum_{i=1}^d(\partial^2_{\theta_1^i}+\partial^2_{\theta_2^i}+\partial^2_{\varphi_1^i}+\partial^2_{\varphi_2^i}). \quad (5.2)$$

To obtain the metric we define the gauge-invariant coordinates

$$Y_1^i = \theta_1^i - \theta_2^i,\ \ Y_2^i = \varphi_1^i - \varphi_2^i,\ \ i=1,\dots,d \quad (5.3)$$

and substitute in $L_0 + \bar{L}_0$.

Now, in order to see the $O(2d)\times O(2d)$ duality we define the $2d$-dimensional vector ϕ_1 with $\phi_1^i = \theta_1^i$ for $i=1,\dots,d$ and $\phi_1^i = \varphi_1^{i-d}$ for $i=d+1,\dots,2d$ and similarly we define the vector ϕ_2 with $\phi_2^i = \theta_2^i$ for $i=1,\dots,d$ and $\phi_2^i = \varphi_2^{i-d}$ for $i=d+1,\dots,2d$. Now we rewrite $L_0+\bar{L}_0$ in (5.2) as

$$L_0 + \bar{L}_0 = -\frac{1}{k-\tilde{c}_G}[\Sigma^\mu(X)\partial_{X^\mu} + \tilde{\Gamma}_1^i(X)\partial_{\phi_1^i} + \tilde{\Gamma}_2^i(X)\partial_{\phi_2^i} + G^{\mu\nu}\partial_{X^\mu}\partial_{X^\nu} + 2\tilde{G}_1^{\mu i}\partial_{X^\mu}\partial_{\phi_1^i}$$
$$+2\tilde{G}_2^{\mu i}\partial_{X^\mu}\partial_{\phi_2^i} + \tilde{L}_1^{ij}\partial_{\phi_1^i}\partial_{\phi_1^j} + \tilde{L}_2^{ij}\partial_{\phi_2^i}\partial_{\phi_2^j} + 2\tilde{P}^{ij}\partial_{\phi_1^i}\partial_{\phi_2^j}] + \frac{1}{k}\sum_{i=1}^{2d}(\partial^2_{\phi_1^i}+\partial^2_{\phi_2^i}), \quad (5.4)$$

where $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ are the $2d$-dimensional vectors

$$\tilde{\Gamma}_1 = (\Gamma_1, \mathbf{0}),\ \ \tilde{\Gamma}_2 = (\Gamma_2, \mathbf{0}), \quad (5.5)$$

$\tilde{G}_1, \tilde{G}_2$ are $(D-d)\times 2d$ matrices

$$\tilde{G}_1 = (G_1 \ \mathbf{0}), \tilde{G}_2 = (G_2 \ \mathbf{0}), \tag{5.6}$$

and, P, L_1, L_2 are $2d \times 2d$ matrices

$$\tilde{P} = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \ \tilde{L}_1 = \begin{pmatrix} L_1 & \mathbf{0} \\ \mathbf{0} & 2\left(1 - \frac{\tilde{c}_G}{k}\right) I \end{pmatrix}, \ \tilde{L}_2 = \begin{pmatrix} L_2 & \mathbf{0} \\ \mathbf{0} & 2\left(1 - \frac{\tilde{c}_G}{k}\right) I \end{pmatrix}. \tag{5.7}$$

Now we can repeat the steps in Sec. IV. Taking O_1 and O_2 to be two O($2d$) matrices, the Casimir operator of the ungauged model is invariant under the O($2d$)×O($2d$) transformations

$$\begin{gathered} \phi_1 \to \phi_1' = O_1\phi_1, \quad \phi_2 \to \phi_2' = O_2\phi_2, \\ \tilde{L}_1 \to \tilde{L}_1' = O_1\tilde{L}_1O_1^T, \quad \tilde{L}_2 \to \tilde{L}_2' = O_2\tilde{L}_2O_2^T, \quad \tilde{P} \to \tilde{P}' = O_1\tilde{P}O_2^T, \\ \tilde{G}_1 \to \tilde{G}_1' = \tilde{G}_1O_1^T, \quad \tilde{G}_2 \to \tilde{G}_2' = \tilde{G}_2O_2^T, \\ \tilde{\Gamma}_1 \to \tilde{\Gamma}_1' = \tilde{\Gamma}_1O_1^T, \quad \tilde{\Gamma}_2 \to \tilde{\Gamma}_2' = \tilde{\Gamma}_2O_2^T. \end{gathered} \tag{5.8}$$

So all the O($2d$)×O($2d$) dual models to the (ungauged) WZW model have the metrics

$$G^{-1} = \frac{1}{k - \tilde{c}_G} \begin{pmatrix} G & \tilde{G}_1O_1^T - \tilde{G}_2O_2^T \\ O_1\tilde{G}_1^T - O_2\tilde{G}_2^T & O_1\tilde{L}_1O_1^T + O_2\tilde{L}_2O_2^T - O_1\tilde{P}O_2^T - O_2\tilde{P}^TO_1^T - 2I + \frac{2\tilde{c}_G}{k}I \end{pmatrix}. \tag{5.9}$$

Thus, although the ungauged WZW model is exact to all orders (up to a shift in k), dual models receive nontrivial $1/k$ corrections with respect to the semiclassical backgrounds. To get the dilaton fields we simply need to calculate the determinant of the dual metric, divide by the determinant of the metric in the WZW action and take a log. Since $\sqrt{G}e^{\Phi}$ is the same in all the dual models as in the ungauged WZW, this quantity is independent of k to all orders. In particular, the coset models with d extra free U(1) fields are dual to the ungauged WZW and therefore this property holds also for the coset models. The coset models that were derived in Sec. III are obtained by taking $O_1 = O_2 = I$ for the axial gauge and $O_1 = -O_2 = I$ for the vector gauge. If we take

$$O_1 = I, \ O_2 = -\begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}, \tag{5.10}$$

we obtain the metric of the ungauged WZW in (3.14). Thus we have produced a result known to one-loop order [28,29], that the backgrounds that correspond to $G/\mathrm{U}(1)^d$ WZW can be obtained by applying O(d,d) transformations on the ungauged WZW.

VI. EXAMPLE: EXACT DUALITY IN SL(2, R) WZW

To demonstrate the above, we shall consider here the exact dual models to the SL($2,R$) WZW. This model has raised some interest recently with connection to the 3D black hole [30]. In one-loop order it is known [31] that by duality transformations it can be brought to the 3D black string [32] and to the 2D black hole with an extra free U(1) field [28,29].

In the case of SL($2,R$) WZW there are two isometries. We shall demonstrate the O(2)×O(2) duality. First let us parametrize the currents of the SL($2,R$) WZW as

$$\begin{aligned} J_3 &= i\partial_{\theta_L}, \quad \bar{J}_3 = i\partial_{\theta_R}, \\ J_\pm &= ie^{\pm\theta_L}\left(\frac{1}{2}\partial_r \pm \frac{1}{\sinh 2r}(\partial_{\theta_R} - \cosh 2r\partial_{\theta_L})\right), \\ \bar{J}_\pm &= ie^{\pm\theta_R}\left(\frac{1}{2}\partial_r \pm \frac{1}{\sinh 2r}(\partial_{\theta_L} - \cosh 2r\partial_{\theta_R})\right). \end{aligned} \tag{6.1}$$

Now we shall follow the procedure suggested in Sec. V. We introduce two additional U(1) fields, denoted by φ_L, φ_R and the corresponding U(1) currents (with level k) are $K = i\partial_{\varphi_1}$ and $\bar{K} = i\partial_{\varphi_2}$. Next we gauge the currents $J_3, \bar{J}_3, K, \bar{K}$. Then, in the model SL($2,R$)×U(1)^2/U(1)^2, we have

$$\begin{aligned} L_0 + \bar{L}_0 = &-\frac{2}{k-2}\left(\frac{1}{4}\partial_r^2 + \frac{1}{2}\coth 2r\partial_r - \frac{1}{\sinh^2 2r}(\partial_{\theta_L}^2 - 2\cosh 2r\partial_{\theta_L}\partial_{\theta_R} + \partial_{\theta_R}^2)\right) \\ &-\frac{2}{k}(\partial_{\varphi_1}^2 + \partial_{\varphi_2}^2) + \frac{1}{k}(\partial_{\theta_L}^2 + \partial_{\theta_R}^2 + \partial_{\varphi_1}^2 + \partial_{\varphi_2}^2) \end{aligned} \tag{6.2}$$

where we used $\tilde{c}_G = 2$ for the SL($2,R$) group. We rewrite this expression as

$$\begin{aligned} L_0 + \bar{L}_0 = &-\frac{1}{k-2}(\tfrac{1}{2}\partial_r^2 + \coth 2r\partial_r) \\ &-\left[(\partial_{\theta_L}, \partial_{\varphi_1})A\begin{pmatrix} \partial_{\theta_L} \\ \partial_{\varphi_1} \end{pmatrix} + (\partial_{\theta_L}, \partial_{\varphi_1})B\begin{pmatrix} \partial_{\theta_R} \\ \partial_{\varphi_2} \end{pmatrix} + (\partial_{\theta_R}, \partial_{\varphi_2})A\begin{pmatrix} \partial_{\theta_R} \\ \partial_{\varphi_2} \end{pmatrix}\right] + \frac{1}{k}(\partial_{\theta_L}^2 + \partial_{\theta_R}^2 + \partial_{\varphi_1}^2 + \partial_{\varphi_2}^2), \end{aligned} \tag{6.3}$$

where

$$A = \begin{pmatrix} -\frac{2}{(k-2)\sinh^2 2r} & 0 \\ 0 & \frac{2}{k} \end{pmatrix},$$
$$B = \begin{pmatrix} \frac{4\cosh 2r}{(k-2)\sinh^2 2r} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.4}$$

In the axial gauge we define the gauge-invariant coordinates as

$$Y_1 = \theta_1 - \theta_R, \quad Y_2 = \varphi_1 - \varphi_2. \tag{6.5}$$

Then the dual models can be described with

$$L_0 + \bar{L}_0 = -\frac{1}{k-2}(\tfrac{1}{2}\partial_r^2 + \cosh 2r \partial_r) - (\partial_{Y_1}, \partial_{Y_2})\left(O_1 A O_1^T + O_2 A O_2^T - O_1 B O_2^T - \frac{1}{k} I\right)\begin{pmatrix} \partial_{Y_1} \\ \partial_{Y_2} \end{pmatrix}, \tag{6.6}$$

where O_1, O_2 are two $O(2)$ matrices. Finally, let us take

$$O_1 = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$
$$O_2 = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}. \tag{6.7}$$

Substituting these matrices in (6.6), we obtain a general expression for the inverse metric in the dual models:

$$G^{rr} = \frac{1}{2(k-2)},$$
$$G^{11} = -\frac{2}{k-2}\left[(\cos^2\alpha + \cos^2\beta)\frac{1}{\sinh^2 2r} - \left(1 - \frac{2}{k}\right)(\sin^2\alpha + \sin^2\beta - 1) + \cos\alpha\cos\beta\frac{2\cosh 2r}{\sinh^2 2r}\right],$$
$$G^{22} = -\frac{2}{k-2}\left[(\sin^2\alpha + \sin^2\beta)\frac{1}{\sinh^2 2r} - \left(1 - \frac{2}{k}\right)(\cos^2\alpha + \cos^2\beta - 1) + \sin\alpha\sin\beta\frac{2\cosh 2r}{\sinh^2 2r}\right], \tag{6.8}$$
$$G^{12} = \frac{2}{k-2}\left[(\sin\alpha\cos\alpha + \sin\beta\cos\beta) \times \left(\frac{1}{\sinh^2 2r} + 1 - \frac{2}{k}\right) + \sin(\alpha+\beta)\frac{\cosh 2r}{\sinh^2 2r}\right].$$

Let us now observe this metric. For $\sin\beta = \cos\alpha = 0$ we obtain the original ungauged WZW.

For $\cos\alpha = \cos\beta = 1$ we obtain the exact metric of the 2D black hole and an extra free U(1) field (this is the Lorentzian version of the solution in [21]). Taking $t = 2Y_1$ for the time and $X = 2Y_1$ the metric with a prefactor 2 is

$$dS^2 = (k-2)dr^2 - (k-2)\left(\coth^2 r - \frac{2}{k}\right)^{-1} dt^2 + k dX^2,$$
$$\Phi = \ln(\sinh 2r) + \frac{1}{2}\ln\left(\coth^2 r - \frac{2}{k}\right). \tag{6.9}$$

For $\cos\alpha = 1$ and $\cos\beta = -1$ we obtain the dual metric of the 2D black hole (that correspond to the vector gauging):

$$dS^2 = (k-2)dr^2 - (k-2)\left(\tanh^2 r - \frac{2}{k}\right)^{-1} dt^2 + k dX^2,$$
$$\Phi = \ln(\sinh 2r) + \frac{1}{2}\ln\left(\tanh^2 r - \frac{2}{k}\right). \tag{6.10}$$

The exact background that corresponds to the 3D black string is obtained by a coordinate transformation on Y_1, Y_2. We take $\cos\alpha = \sin\beta$ and define $t = Y_1 - Y_2$ and $X = Y_1 + Y_2$. The metric we obtain is

$$dS^2 = (k-2)dr^2 - (k-2)\frac{\sinh^2 r}{1 + q + 2q(1-2/k)\sinh^2 r}dt^2 + (k-2)\frac{\cosh^2 r}{1 - q + 2q(1-2/k)\cosh^2 r}dX^2,$$
$$\Phi = \frac{1}{2}\ln\left\{\left[1 + q + 2q\left(1 - \frac{2}{k}\right)\sinh^2 r\right] \times \left[1 - q + 2q\left(1 - \frac{2}{k}\right)\cosh^2 r\right]\right\}, \tag{6.11}$$

where $q = \sin 2\alpha$. (These metric and dilaton fields were derived also in [33].) Notice that in all the models and in the ungauged WZW $\sqrt{G}e^{\Phi} = \sinh 2r$.

VII. EXACT $O(n, d-n) \times O(n, d-n)$ TRANSFORMATIONS

In the previous section we were discussing WZW where the group elements were parametrized by

$$g = \exp\left(i\sum_{i=1}^{d}\theta_1^i T^i\right)\tilde{g}(X)\exp\left(i\sum_{i=1}^{d}\theta_2^i T^i\right).$$

The results we got hold also when the group elements are parametrized by

$$g=\exp\left(\sum_{i=1}^{d}\theta_1^iT^i\right)\tilde g(X)\exp\left(\sum_{i=1}^{d}\theta_2^iT^i\right) \qquad (7.1)$$

(all the θ coordinates are noncompact) but requires a minus sign in front of the (ij) components of the exact metric in all the dual models. In a noncompact group we can also parametrize the group by

$$g=\exp\left(i\sum_{i=1}^{n}\theta_1^iT^i+\sum_{i=n+1}^{d}\theta_1^iT^i\right)\tilde g(X)\exp\left(i\sum_{i=1}^{n}\theta_2^iT^i+\sum_{i=n+1}^{d}\theta_2^iT^i\right) \qquad (7.2)$$

[$2n$ compact and $2(d-n)$ noncompact fields] where, as in Sec. III, $\mathrm{tr}T^iT^j=\delta_{ij}$. In this case the ungauged action takes the form

$$S=\frac{k}{2\pi}\int d^2\sigma\left(\sum_{i=1}^{n}(\partial_+\theta_1^i\partial_-\theta_1^i+\partial_+\theta_2^i\partial_-\theta_2^i)-\sum_{i=n+1}^{d}(\partial_+\theta_1^i\partial_-\theta_1^i+\partial_+\theta_2^i\partial_-\theta_2^i)\right.$$
$$\left.+2M_{ij}(X)\partial_-\theta_1^i\partial_+\theta_2^j+2N^1_{\mu i}(X)\partial_+X^\mu\partial_-\theta_1^i+2N^2_{\mu i}(X)\partial_-X^\mu\partial_+\theta_2^i+F_{\mu\nu}(X)\partial_+X^\mu\partial_-X^\nu\right) \qquad (7.3)$$

and the holomorphic currents $J^j=\mathrm{tr}\partial_+gg^{-1}T^j$ and the antiholomorphic currents $\bar J^j=\mathrm{tr}g^{-1}\partial_-gT^j$ with $j>n$ do not have a prefactor i. In the algebraic Hamiltonian approach we shall parametrize these U(1) currents as

$$J^j=i\partial_{\theta_1^j}\ \text{ for } i=1,\ldots,n,$$
$$J^j=\partial_{\theta_1^j}\ \text{ for } i=n+1,\ldots,d, \qquad (7.4)$$

$$\bar J^j=i\partial_{\theta_2^j}\ \text{ for } i=1,\ldots,n,$$
$$\bar J^j=\partial_{\theta_2^j}\ \text{ for } i=n+1,\ldots,d. \qquad (7.5)$$

Then, in the gauged model we have

$$L_0+\bar L_0=\frac{\Delta_G+\bar\Delta_G}{k-\tilde c_G}+\frac{1}{k}\sum_{i=1}^{n}(\partial^2_{\theta_1^i}+\partial^2_{\theta_2^i})-\frac{1}{k}\sum_{i=n+1}^{d}(\partial^2_{\theta_1^i}+\partial^2_{\theta_2^i})$$
$$=\frac{2\Delta_G}{k-\tilde c_G}+\frac{1}{k}\eta(\partial^2_{\theta_1}+\partial^2_{\theta_2}), \qquad (7.6)$$

where η is the $d\times d$ matrix $\eta=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ with n entries 1. (We have used $\Delta_G=\bar\Delta_G$.)

Let us consider now the rotation of the θ coordinates. The translation of the coordinates θ_1^i,θ_2^i with $i=1,\ldots,n$ is generated by iT^i but the translation of the coordinates θ_1^i,θ_2^i with $i=n+1,\ldots,d$ is generated by T^i. So let us define the generators of the $\mathrm{U}(1)^d$ gauge (before the rotation of the θ coordinates) by $\mathsf{T}^j=T^j$ for $j=1,\ldots,n$ and $\mathsf{T}^j=-iT^j$ for $j=n+1,\ldots,d$. Then $\mathrm{tr}\mathsf{T}^i\mathsf{T}^j=\eta^{ij}$. Now we want to repeat the procedure we were using in order to obtain the dual models. Consider a rotation

$$\theta_1\to\theta_1'=O_1\theta_1,\ \ \theta_2\to\theta_2'=O_2\theta_2. \qquad (7.7)$$

The generators of the currents $J,\bar J$ in the rotated system are linear combinations of the generator T^i. A rotation $\theta_1'^i=O^i_{1j}\theta_1^j$ and $\theta_2'^i=O^i_{2j}\theta_2^j$ means that in the rotated action the corresponding U(1) currents are generated by $\mathsf{T}_L'^i=O^i_{1j}\mathsf{T}^j$ and $\mathsf{T}_R'^i=O^i_{2j}\mathsf{T}^j$. Therefore in order to preserve the condition for anomaly cancellation (4.1) the matrices O_1 and O_2 must be $O(n,d-n)$ matrices (namely $O_1\eta O_1^T=O_2\eta O_2^T=\eta$). Hence, when $d-n$ of the U(1) isometries have opposite sign the result (4.3) for the inverse metric of the dual models changes as follows: The unit matrix multiplying $\tilde c_G/k$ (the $1/k$ correction to the inverse metric) is replaced by η and the matrices O_1,O_2 that generate the duality are taken to be $\mathrm{O}(n,d-n)$ matrices rather than $\mathrm{O}(d)$. In the case of dual models to the ungauged WZW in Sec. V, we can introduce the $2d$ extra U(1) currents with different signature. Suppose we take $d-m$ pairs of φ coordinates with negative signature and the ungauged WZW has $d-n$ pairs of θ coordinates which are noncompact, then the duality is obtained by $\mathrm{O}(m+n,2d-m-n)$ matrices.

VIII. SUMMARY AND DISCUSSION

In this work we have generalized the one-loop $\mathrm{O}(d,d)$ transformations to the exact to all orders case in WZW and WZW coset models. A general $\mathrm{O}(d,d)$ transformation can be decomposed to constant coordinate transformations, a shift of the antisymmetric tensor by a constant antisymmetric matrix and $\mathrm{O}(d)\times\mathrm{O}(d)$ transformations. The first two are exact and in this paper we have derived the exact $\mathrm{O}(d)\times\mathrm{O}(d)$ transformations. We have found that for coset models $G/U(1)^d$ these transformations operate as follows: writing the inverse metric as the semiclassical inverse metric plus the $1/k$ corrections, only the semiclassical part transforms to a dual semiclassical inverse metric. Therefore, although with the algebraic Hamiltonian approach we could only derive the exact metric and dilaton field, knowing the antisymmetric tensor to one-loop order, we can find all the exact dual metrics and dilaton fields. In the ungauged WZW model with $2d$ Abelian isometries we obtained the dual metrics by considering the equivalent model $G\times U(1)^{2d}/U(1)^{2d}$. Thus although the ungauged WZW is exact, its dual models receive $1/k$ corrections with respect to the one-loop order transformation. Our analysis shows that $\sqrt{G}e^{\Phi}$ is invariant under the exact $O(d)\times O(d)$ transformations. In particular, in the exact $G/U(1)^d$ coset models (as well as in the dual models) $\sqrt{G}e^{\Phi}$ is independent of k.

Finally, we want to comment about the generality of our results to a general exact CFT. It was shown by Tseytlin [34] that there exist a $(D+2)$-dimensional background with a target space metric having a covariantly constant null Killing vector and a flat "transverse" part that is exact to all orders. The corresponding σ model is invariant under $D+1$ Abelian isometries. In a recent work [35] it was shown that a background of this type is transformed by the one-loop order $O(D,D)$ transformations (only in the transverse directions) to the same class of exact solutions. This specific example requires a special analysis, but we speculate that this class of exact solutions can be obtained by gauging a larger background followed by redefining the fields so that the $1/k$ correction is absorbed and disappears. Then the result in [35] can be considered as a particular example to our results.

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[1] K. Narain, M. Sarmadi, and E. Witten, Nucl. Phys. **B279**, 369 (1987).

[2] A. Giveon, E. Rabinovici, and G. Veneziano, Nucl. Phys. **B322**, 167 (1989); A. Shapere and F. Wilzcek, *ibid.* **B320**, 669 (1989); A. Giveon, N. Malkin, and E. Rabinovici, Phys. Lett. B **220**, 551 (1989).

[3] K. Kikkawa and M. Yamasaki, Phys. Lett. **149B**, 357 (1984); N. Sakai and I. Senda, Prog. Theor. Phys. Suppl. **75**, 692 (1986).

[4] T. Buscher, Phys. Lett. B **201**, 466 (1988).

[5] K. Meissner and G. Veneziano, Phys. Lett. B **267**, 33 (1991).

[6] A. Sen, Phys. Lett. B **271**, 295 (1991).

[7] S. F. Hassan and A. Sen, Nucl. Phys. **B375**, 103 (1992).

[8] A. Giveon and M. Rocek, Nucl. Phys. **B380**, 128 (1992).

[9] X. C. de la Ossa and F. Quevedo, Nucl. Phys. **B403**, 377 (1993).

[10] E. Alvarez, L. Alvarez-Gaume, J. L. F. Barbon, and Y. Lozano, Nucl. Phys. **B415**, 71 (1994).

[11] R. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989).

[12] A. Giveon, Mod. Phys. Lett. A **6**, 2843 (1991).

[13] A. Giveon and E. Kiritsis, Nucl. Phys. **B411**, 487 (1994).

[14] M. Gasperini, J. Maharana, and G. Veneziano, Phys. Lett. B **296**, 51 (1992).

[15] E. Kiritsis, Nucl. Phys. **B405**, 109 (1993).

[16] S. F. Hassan and A. Sen, Nucl. Phys. **B405**, 143 (1993).

[17] M. Henningson and C. Nappi, Phys. Rev. D **48**, 861 (1993).

[18] D. Gershon, Nucl. Phys. **B421**, 80 (1994).

[19] A. A. Tseytlin, Phys. Lett. B **317**, 559 (1993).

[20] For a review see I. Bars, "Curved Spaces Geometry for Strings and Affine Non-Compact Algebras," USC Report No. USC-93/HEP-B3 (unpublished).

[21] R. Dijkgraaf, E. Verlinde, and H. Verlinde, Nucl. Phys. **B371**, 269 (1992).

[22] I. Bars and K. Sfetsos, Phys. Rev. D **46**, 4510 (1992).

[23] A. A. Tseytlin, Phys. Lett. B **268**, 175 (1991).

[24] I. Jack, D. R. T. Jones, and P. Panvel, Nucl. Phys. **B393**, 95 (1993)

[25] K. Sfetsos and A. A. Tseytlin, Nucl. Phys. **B415**, 116 (1994).

[26] E. Witten, Commun. Math. Phys. **144**, 189 (1992).

[27] E. Kiritsis, Mod. Phys. Lett. A **6**, 2871 (1991).

[28] A. Kumar, Phys. Lett. B **293**, 49 (1992).

[29] D. Gershon, Phys. Rev. D **49**, 99 (1994).

[30] G. T. Horowitz and D. L. Welch, Phys. Rev. Lett. **71**, 328 (1993).

[31] A. Sen, Phys. Lett. B **274**, 33 (1993).

[32] J. H. Horne and G. T. Horowitz, Nucl. Phys. **368**, 444 (1992); N. Ishibashi, M. Li, and A. Steif, Phys. Rev. Lett. **67**, 3336 (1991).

[33] K. Sfetsos, Nucl. Phys. **B389**, 424 (1993); I. Bars and K. Sfetsos, Phys. Rev. D **48**, 844 (1993).

[34] A. A. Tseytlin, Phys. Lett. B **288**, 279 (1992); Phys. Rev. D **47**, 3421 (1993).

[35] C. Klimcik and A. A. Tseytlin, Phys. Lett. B **323**, 305 (1994).