Semiclassical black hole in thermal equilibrium with a nonconformal scalar field

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(Received 1 July 1994)

Analytic semiclassical corrections to the Schwarzschild metric are found perturbatively, to first order in $\epsilon = \hbar/M^2$, for a quantized scalar field with arbitrary curvature coupling. The approximation scheme developed by Anderson, Hiscock, and Samuel is used to provide approximate algebraic expressions for the components of the vacuum stress-energy tensor. The linearized Einstein equations are solved to find the metric perturbations caused by the quantized field. Microcanonical boundary conditions are imposed on a spherical wall enclosing the black hole. The various physical effects of the back reaction, and their dependence on the value of the curvature coupling, are discussed in detail. The perturbations are found most often to lower the temperature of the black hole. Requiring that the entropy of the system be increased by the quantized field results in upper and lower bounds on the value of the curvature coupling constant, $-3.431 \le \xi \le 7/10$.

PACS number(s): 04.70.Dy, 04.62.+v, 97.60.Lf

I. INTRODUCTION

Since the discovery by Hawking [1] that black holes emit particles in a thermal spectrum, most of the work on quantum effects in black hole spacetimes has been done using a fixed spacetime background. Although these background field calculations are important and give substantial insight into the effects of the spacetime geometry on quantized fields, they do not give the change in the spacetime geometry which is caused by quantum effects. So far only a few direct calculations of the back reaction of quantized fields on the spacetime geometry of a black hole have been done. These include those of Bardeen [2], Hajicek and Israel [3], and York [4] for evaporating black holes and those of York [5], Hochberg and Kephart [6], and Hochberg, Kephart, and York [7] for a Schwarzschild black hole in equilibrium with radiation in a cavity.

In the equilibrium calculations, attention was restricted to conformally invariant quantized fields. However, fields which are not conformally invariant also exist in nature, and in particular, the gravitational field itself is not conformally invariant. For this reason it is of interest to investigate the effects on black holes of quantized fields which are not conformally invariant. In this paper we present the results of such an investigation for the case of a massless scalar field with an arbitrary coupling ξ to the scalar curvature.[1]

The best method currently available to compute the effects which quantized fields have on the spacetime geometry is to use the semiclassical back-reaction equations

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle \ . \qquad (1)$$

To solve these equations for a given class of spacetimes one needs to be able to compute the vacuum expectation value of the stress-energy tensor of the quantized fields $\langle T_{\mu\nu} \rangle$. This has been the primary obstacle which has slowed progress in determining how quantized fields alter the spacetime geometry of a black hole.

Numerical calculations of the stress-energy tensor have been made for the conformally coupled quantized scalar field in Schwarzschild spacetime by Fawcett [8] and Howard and Candelas [9]. Jensen and Ottewill [10] have computed the stress-energy tensor for the electromagnetic field in Schwarzschild spacetime. Anderson, Hiscock and Samuel have developed a method which allows the computation of the stress-energy tensor for massive and massless quantized scalar fields with arbitrary curvature coupling in a general static, spherical spacetime and have applied this method to the Schwarzschild and Reissner-Nordström spacetimes [11,12].

The methods we employ are valid in general and can be performed numerically if desired, but the semiclassical back-reaction equations are much easier to solve

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[1]For $\xi = \frac{1}{6}$ the massless scalar field is conformally invariant.

if an analytic expression for $\langle T_{\mu\nu}\rangle$ exists. Approximate analytic expressions for the stress-energy tensor of quantized conformally coupled fields in static Einstein ($R_{\mu\nu} = \Lambda g_{\mu\nu}$) spacetimes were developed by Page [13], Brown and Ottewill [14], and Brown, Ottewill, and Page [15]. An analytic approximation to the vacuum stress-energy of conformally invariant massless fields in a general static spacetime has been developed by Frolov Zel'nikov [16]. Jensen and Ottewill [10] in the course of their numerical calculation developed an analytic approximation to the stress-energy of the electromagnetic field in Schwarzschild spacetime, which is distinct from, and more accurate than, the Page, Brown, Ottewill, and Frolov-Zel'nikov approximations. For a general static spherically symmetric spacetime, Anderson, Hiscock, and Samuel [11,12] have developed an analytic approximation for a massless scalar field with an arbitrary curvature coupling constant ξ. In each of these approximations the quantized fields have been in a thermal state with an arbitrary temperature; for purposes of back reaction calculations, the obvious temperature to examine is that of the black hole itself, since for any other temperature the stress-energy tensor diverges on the horizon. This corresponds to placing the black hole in the Hartle-Hawking vacuum state [17], in which the black hole is in (possibly unstable) thermal equilibrium with an external heat bath.

Solutions to the semiclassical Einstein equations which make use of these results have been obtained only in a very limited number of cases. York used the approximate expression for the vacuum stress-energy of a conformal scalar field developed by Page to calculate the perturbative back reaction of the Schwarzschild metric [15]. He treated the vacuum stress-energy as the source for a small metric perturbation and calculated the resulting changes in the metric components to first order in $(M_P/M)^2$, where M is the mass of the black hole and M_P is the Planck mass. York calculated the back reaction in a perturbative fashion, and only to first order, because the Page approximation only generates an approximate stress-energy tensor for Einstein spacetimes. The full semiclassical solution representing an uncharged, nonrotating black hole in thermal equilibrium with a conformally coupled scalar field will certainly not be an Einstein spacetime. More recently, Hochberg, Kephart, and York [6,7] have extended this work to include the effects of the quantized spinor and vector fields as well as the conformal scalar field, again using the approximations of Page, Brown, Ottewill [13–15] and Jensen and Ottewill [10]. Anderson, Hiscock, and McLaughlin have also calculated the first-order changes in the Schwarzschild metric for an arbitrary collection of spinor, vector, and (possibly nonconformal) scalar fields, using the various approximations, with particular emphasis on the effects of the back reaction on the black hole interior [18].

In this paper we calculate the first-order [in $\hbar$, or equivalently in $(M_P/M)^2$] perturbative corrections to the Schwarzschild metric when a Schwarzschild black hole is placed in thermal equilibrium with a nonconformally coupled quantized scalar field, using the analytic approximation to $\langle T_{\mu\nu}\rangle$ developed by Anderson, Hiscock, and Samuel. Previous work [11] has shown that the accuracy of this analytic approximation does not depend strongly on the value of ξ, the curvature coupling. While this approximation is not restricted to Einstein spacetimes, we limit our calculations to first order since the approximate expression for $\langle T_{\mu\nu}\rangle$ diverges on the black hole event horizon in any spacetime with nonzero $R_{\mu\nu}$. This is a defect of the approximation; full numerical calculations of the stress-energy display no such divergent behavior [11,12].

We find that the perturbed geometry is significantly affected by the value of the curvature coupling constant ξ. This means that the perturbed temperature and other physical quantities are also significantly affected. In fact, by requiring that the quantized fields increase the thermodynamic entropy of the system we can put both an upper and a lower bound on the value of ξ. This is the first time that entropy has ever been used to place a bound on the value of a coupling constant in a theory.

In Sec. II we describe the approximate stress-energy tensor of a quantized scalar field with arbitrary curvature coupling. The metric perturbations created by this stress-energy are calculated to first order in $(M_P/M)^2$ in Sec. III. The geometry of the perturbed black hole is studied in Sec. IV, including the effective of the metric perturbations on the black hole's temperature. The effective potential for particle orbits in the perturbed metric is examined in Sec. V. An analysis of the entropy contributed by the quantized field is used to place physical limits on the curvature coupling constant ξ in Sec. VI. We use the sign conventions of Misner, Thorne, and Wheeler [19] and choose units such that $G = c = k_B = 1$.

II. APPROXIMATE VACUUM STRESS-ENERGY TENSOR

In Refs. [11,12] it was shown that the stress-energy tensor for the Hartle-Hawking vacuum state of a massless scalar field with arbitrary curvature coupling in the Schwarzschild geometry has the form

$$\langle T_{\mu\nu}\rangle = C_{\mu\nu} + (\xi - \tfrac{1}{6})D_{\mu\nu} \;, \tag{2}$$

and analytic approximations to $C_{\mu\nu}$ and $D_{\mu\nu}$ were found. The tensor $C_{\mu\nu}$ represents the approximate vacuum stress-energy for the conformal scalar field and is identical to the expression first calculated by Page [13].

We will begin by considering the components of these tensors in the usual Schwarzschild coordinates, in which the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \;, \tag{3}$$

where $d\Omega^2$ is the metric of the two-sphere.

Using the notation of York [5], the approximate conformal vacuum stress-energy components are

$$C^t{}_t = -3\frac{\epsilon}{\lambda M^2}(f - h), \quad C^r{}_r = \frac{\epsilon}{\lambda M^2}(f + h) \;, \tag{4}$$

$$C^\theta{}_\theta = C^\phi{}_\phi = \frac{\epsilon}{\lambda M^2} f \ , \tag{5}$$

where

$$f(r) = \frac{1-(4-6M/r)^2(2M/r)^6}{(1-2M/r)^2} \ , \tag{6}$$

$$h(r) = 24\left(\frac{2M}{r}\right)^6 \ , \tag{7}$$

and the constants ϵ and λ are defined by

$$\epsilon = \hbar/M^2, \quad \lambda = 90(8^4)\pi^2 \ . \tag{8}$$

The non-conformal portions of the approximate stress-energy are given by [12]

$$D^t{}_t = \epsilon M^2 \frac{r^2+4Mr-20M^2}{128\pi^2 r^6} \ , \tag{9}$$

$$D^r{}_r = -\epsilon M \frac{(2r-3M)(r^2+4Mr+12M^2)}{384\pi^2 r^6} \ , \tag{10}$$

$$D^\theta{}_\theta = D_\phi{}^\phi = \epsilon M \frac{r^3+4Mr^2+12M^2 r-96M^3}{384\pi^2 r^6} \ . \tag{11}$$

These components have a number of interesting properties. For example, the energy density of the scalar-field $\rho_{sf} = -T^t{}_t$ has the following form at the horizon:

$$\rho_{sf}(2M) = 24\epsilon \frac{15\xi - 4}{\lambda M^2} \ . \tag{12}$$

Note that at the horizon ρ_{sf} is negative for all $\xi < 4/15$ (e.g., for both the conformally coupled and minimally coupled scalar field), but is positive for larger values of the curvature coupling. The energy density is everywhere positive outside the event horizon for $4/15 < \xi < 1.2575$. For larger values of ξ, there is a region outside (but not including) the horizon which has negative energy densities. By taking the derivative of ρ_{sf} with respect to ξ, one finds that the energy density at $r = 2(6^{1/2}-1)M \approx 2.9M$ is independent of the curvature coupling; for all ξ it is given by

$$\rho_{sf} = \epsilon \frac{21-8(6)^{1/2}}{5120\pi^2 M^2[(6)^{1/2}-1]^6} \approx 3.00\times 10^{-6}\frac{\epsilon}{M^2} \ . \tag{13}$$

There are similar radii at which the other components, $T^r{}_r$ and $T^\theta{}_\theta$ are independent of ξ; however, the radius is different for each component.

The trace of the stress-energy tensor $\langle T^\alpha{}_\alpha\rangle$ can also change sign at the horizon depending on the value of the curvature coupling:

$$\langle T^\alpha{}_\alpha\rangle(2M) = 24\epsilon\frac{19-90\xi}{\lambda M^2} \ . \tag{14}$$

As with the density, there is a critical radius at which the trace is independent of the curvature coupling, namely, $r = [(10)^{1/2}-1]2M \approx 4.325M$. The sign of the trace at large r depends on the curvature coupling; the full trace may be written as

$$T_\alpha{}^\alpha = \frac{\epsilon M^2(\xi-\frac{1}{6})}{64\pi^2 r^4}\left(1+\frac{4M}{r}-\frac{36M^2}{r^2}\right) + \frac{\epsilon M^4}{60\pi^2 r^6} \ . \tag{15}$$

While the nonconformal terms dominate the trace at large r, that is only because the conformal classical stress-energy tensor has zero trace. When any single component is examined, the nonconformal contributions to the vacuum stress-energy components drop off as r^{-3} or faster as one moves away from the black hole, while the conformal components approach constant values. Thus, far from the event horizon, the vacuum stress-energy becomes indistinguishable from that of a conformally coupled field.

III. METRIC PERTURBATIONS

It is now possible to use the approximate expressions for the stress-energy tensor components to calculate the first order in $\hbar$ (or, more properly, first order in ϵ) perturbations to the Schwarzschild metric. The resulting combination of perturbed metric and approximate stress-energy tensor makes sense only in a perturbative fashion; for example, if one computes the divergence of the stress-energy tensor in the perturbed metric, one will obtain not zero, but quantities which are of order ϵ^2.

Following York [5] it is convenient to work in ingoing Eddington-Finkelstein coordinates:

$$\nu = t + r + 2M\ln\left(\frac{r}{2M}-1\right) \ , \tag{16}$$

$$\tilde{r} = r \ . \tag{17}$$

The Schwarzschild metric then takes the form

$$ds^2 = -\left(1-\frac{2M}{\tilde{r}}\right)d\nu^2 + 2\,d\nu\,d\tilde{r} + \tilde{r}^2 d\Omega^2 \ . \tag{18}$$

The components of the stress-energy tensor in this coordinate system are

$$T_\nu{}^\nu = T_t{}^t \ . \tag{19}$$

$$T_{\tilde{r}}{}^{\tilde{r}} = T_r{}^r \ , \tag{20}$$

$$T_{\tilde{r}}{}^\nu = \frac{2\epsilon}{\lambda M^2}\left(1-\frac{2M}{\tilde{r}}\right)^{-1}(2f-h) + \left(\xi-\frac{1}{6}\right)D_{\tilde{r}}{}^\nu \ , \tag{21}$$

where

$$D_{\tilde{r}}{}^{\nu} = -\epsilon M \frac{\tilde{r}^2 + 6M\tilde{r} + 24M^2}{192\pi^2\tilde{r}^7} . \quad (22)$$

A general static, spherically metric may be written in Eddington-Finkelstein coordinates as

$$ds^2 = -e^{2\psi(r)}\left(1 - \frac{2m(r)}{r}\right)d\nu^2 + 2e^{\psi(r)}d\nu\, dr + r^2 d\Omega^2 . \quad (23)$$

The linear perturbations (first order in ϵ) to the metric may be described by expanding the metric functions $\psi(r)$ and $m(r)$ as

$$e^{\psi(r)} = 1 + \epsilon\rho(r), \quad m(r) = M[1 + \epsilon\mu(r)] . \quad (24)$$

Equating terms in the Einstein equations which are first order in ϵ then gives the following equations for the linear metric perturbations:

$$\frac{\partial\mu}{\partial r} = \frac{12\pi r^2}{\lambda M^3}(f - h) - \left(\xi - \frac{1}{6}\right)M\frac{r^2 + 4Mr - 20M^2}{32\pi r^4} , \quad (25)$$

$$\frac{\partial\rho}{\partial r} = \frac{8\pi r}{\lambda M^2}\left(1 - \frac{2M}{r}\right)^{-1}(2f - h) - \left(\xi - \frac{1}{6}\right)M\frac{(r - 2M)^2(r^2 + 6Mr + 24M^2)}{48\pi r^6} . \quad (26)$$

There is of course a third Einstein equation which is related to these two through the Bianchi identities. The general solutions to Eqs. (25) and (26) are given by

$$K\mu(r) = \frac{1}{3}\left(\frac{r}{2M}\right)^3 + \left(\frac{r}{2M}\right)^2 + 3\left(\frac{r}{2M}\right) - \frac{13}{3} - \left[22 - 120\left(\xi - \frac{1}{6}\right)\right]\left(1 - \frac{2M}{r}\right) + \left[30 - 240\left(\xi - \frac{1}{6}\right)\right]\left(1 - \frac{2M}{r}\right)^2 - \left[11 - 100\left(\xi - \frac{1}{6}\right)\right]\left(1 - \frac{2M}{r}\right)^3 + 4\ln\left(\frac{r}{2M}\right) + C_0 , \quad (27)$$

$$K\rho(r) = \frac{1}{3}\left(\frac{r}{2M}\right)^2 + 2\left(\frac{r}{2M}\right) - \frac{7}{3} + \left[\frac{92}{3} - 400\left(\xi - \frac{1}{6}\right)\right]\left(1 - \frac{2M}{r}\right) - \left[19 - 300\left(\xi - \frac{1}{6}\right)\right]\left(1 - \frac{2M}{4}\right)^2 + \left[\frac{14}{3} - 80\left(\xi - \frac{1}{6}\right)\right]\left(1 - \frac{2M}{r}\right)^3 + 4\ln\left(\frac{r}{2M}\right) + k_0 , \quad (28)$$

where the constants of integration C_0 and k_0 have been chosen such that $K\mu(2M) = C_0$, $K\rho(2M) = k_0$ and $K = 3840\pi$.

The departures from the Schwarzschild metric now become

$$\Delta g_{\nu\nu} = -\left(1 - \frac{2M}{r}\right)2\epsilon\rho(r) + \frac{2M\epsilon\mu(r)}{r} , \quad (29)$$

$$\Delta g_{\nu r} = \epsilon\rho(r) , \quad (30)$$

which are manifestly regular at $r = 2M$.

The transformation back to Schwarzschild-type coordinates can be achieved by letting

$$\frac{\partial\nu}{\partial t} = 1, \quad \frac{\partial\nu}{\partial r} = \epsilon^{-\psi}\left(1 - \frac{2m(r)}{r}\right)^{-1} , \quad (31)$$

The metric then has the form

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right)(1 + 2\epsilon\rho)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 , \quad (32)$$

where now ρ is given by Eq. (28) and $m(r)$ is to be defined using the expression for μ given in Eq. (27).

IV. PROPERTIES OF THE PERTURBED METRIC

In a static geometry the event horizon is identical to the apparent horizon and is given, to first order, by

$$r_H = 2m = 2M[1 + \epsilon\mu(2M)] \equiv 2M_{\rm BH} . \quad (33)$$

The event horizon now defines a mass for the black hole, which has gravitational radius $r_+ = 2M_{\rm BH}$, in the perturbed geometry. We henceforth replace $M_{\rm BH}$ by M, with M representing the "dressed mass" defined via Eq. (33). This redefinition serves to absorb the (physically unmeasurable) integration constant C_0 into M, so that the unknown quantities in the perturbed metric are now reduced to the single remaining integration constant, k_0. The functional form of $\mu(r)$ from this point on is taken to be given by Eq. (27) with C_0 set equal to zero.

The remaining integration constant, k_0 may be fixed by imposing microcanonical boundary conditions. We surround the perturbed black hole with an ideal perfectly reflecting spherical wall at radius r_0. Following York [5] we assume a massless cavity wall and ignore boundary effects due to the quantized fields. We assume the radius of the wall is chosen to be small enough that the metric perturbations are still small, yet large enough that the stress-energy of the scalar field at the wall is strongly dominated by the classical radiation terms. This second condition is necessary if one is to ignore the semiclassical (Boulware vacuum) corrections to the Schwarzschild metric [20] outside the wall. The metric exterior to the wall is then assumed to be Schwarzschild in form, with mass $M^* \equiv m(r_0)$. We can fix the normalization of the interior time coordinate by choosing g_{tt} to be continuous at the wall. This choice forces $\rho(r_0) = 0$, by Eq. (32), and thus fixes the integration constant k_0 in terms of r_0

$$k_0 = -K\rho_0(r_0) \ , \tag{34}$$

where $\rho_0(r) \equiv \rho(r) - k_0 K^{-1}$ is the function ρ with the integration constant k_0 set equal to zero.

As noted by York [5], the tangential stress in the wall is related to the vacuum stress-energy by the Einstein equations:

$$S_\phi{}^\phi = -\frac{1}{2} r \left(1 - \frac{2M}{r}\right)^{-1/2} T_r{}^r \ . \tag{35}$$

In the case of conformal coupling ($\xi = \frac{1}{6}$), the radial stress $T_r{}^r$ is positive everywhere outside the horizon, and hence the cavity wall is in tension, as $S_\phi{}^\phi < 0$ [5]. However, when other values of the curvature coupling are allowed, the situation is not so simple. The radial stress of the scalar field is positive everywhere outside the horizon only if $\xi < 0.2275$; for greater values, there is a region outside the horizon with negative radial stress. The inner boundary of this region lies on the horizon for all $\xi \geq 4/15$; the outer boundary steadily grows in radius as ξ is increased. This behavior allows one to consider cavity walls which would have a positive pressure, enclosing a volume with a radial tension; or, alternatively, one could choose the wall radius r_0 to be at a location where $T_r{}^r$ vanishes, in which case the wall has no stress-energy whatsoever (its mass was assumed to be zero).

The surface gravity of the perturbed black hole may be calculated from

$$\kappa^2 = -\tfrac{1}{2}\chi_{\alpha;\beta}\chi^{\alpha;\beta} \ , \tag{36}$$

where χ^α is the timelike Killing vector field, and the right-hand side of Eq. (36) is to be evaluated on the horizon.

For the metric at hand, the value of the surface gravity on the event horizon is

$$\kappa = \frac{1}{4M}\left[1 + \epsilon\frac{k_0 + 12 - 120(\xi - \frac{1}{6})}{3840\pi}\right] \tag{37}$$

to first order in ϵ. The black hole temperature measured at asymptotically flat spatial infinity far outside the box radius r_0, T_*, is $\hbar\kappa(2\pi)^{-1}$. If the perturbed metric of Eq. (32) is Euclideanized, the resulting space will be regular (free of conical singularity) only if the Euclidean time as we have normalized it here is identified with period $2\pi/\kappa$, with κ taking the value defined in Eq. (37). The Euclidean definition of temperature at infinity, viz., $T_E = 1/\text{period}$, and the Lorentzian definition of this quantity thus agree as they must for an equilibrium configuration.

After the imposition of microcanonical boundary conditions, k_0 is a known function of the radius of the cavity wall, r_0. It is then possible to determine, as a function of the wall radius, whether the black hole's temperature T_* (defined through the surface gravity or Euclidean period) is increased or lessened by semiclassical effects. The order ϵ change in the surface gravity, defined as $\delta\kappa = \kappa - 1/(4M)$ may be positive or negative depending on the choice of ξ and r_0. The regions for which $\delta\kappa$ is positive and negative in the (ξ, r_0) plane are shown in Fig. 1. If the curvature coupling is constrained by $0.4691 > \xi > 0.2679$, then the surface gravity (and temperature T_*) are reduced by semiclassical effects for all values of r_0. For smaller values of ξ, there is a region near the horizon for which the semiclassical correction is positive, and then beyond a certain radius it is always negative. Specifically, for the conformally coupled scalar field, we find that $\delta\kappa$ is negative for all $r_0 > 3.0972M$. In the case of minimal coupling ($\xi = 0$), $\delta\kappa$ is negative for all $r_0 > 3.2507M$. For values of ξ larger than 0.4691, there is a range of radii (increasing in width as ξ increases) for which $\delta\kappa$ is positive. There is one value of r_0 at which the semiclassical contribution to the surface gravity is independent of ξ; this occurs at $r_0 \approx 3.41727M$; there, the

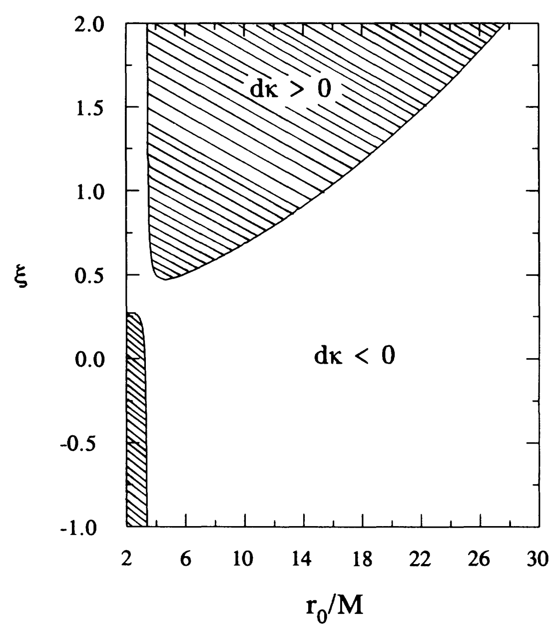

FIG. 1. Regions of positive and negative $\delta\kappa$, the semiclassical correction to the surface gravity (and hence temperature) of the black hole, are plotted as functions of the curvature coupling, ξ, and the radius of the spherical wall enclosing the black hole, r_0. Microcanonical boundary conditions are assumed to hold at the wall.

surface gravity is given by

$$\kappa = \frac{1}{4M}(1 - 1.64 \times 10^{-4}\epsilon) \ . \tag{38}$$

The local equilibrium temperature of the system is not given by T_*, the temperature of the black hole, but by

$$T_{\rm loc}(r) = |g_{tt}(r)|^{-1/2} T_* \ . \tag{39}$$

A simple calculation, similar to that presented for the conformal case in Ref. [5], shows that $T_{\rm loc}$ is independent of the integration constant k_0. Analogously, one can conclude that T_* is the temperature an observer would measure at asymptotically large distances from the black hole (assuming one had physical contact with the enclosed system, as through a small tube extending from r_0 to infinity).

The unstable circular photon orbit at $r_p = 3M$ (at order ϵ^0) is of interest for a variety of reasons. Locating this orbit at order ϵ we find

$$\begin{aligned} r_p &= 3M\left[1 + \epsilon\frac{685 - 10320(\xi - \frac{1}{6}) + 1296\ln(\frac{3}{2})}{1244160\pi}\right] \\ &\approx 3M(1 + \epsilon[3.1 \times 10^{-4} - 2.6 \times 10^{-3}(\xi - \tfrac{1}{6})]). \end{aligned} \tag{40}$$

The semiclassical correction to the radius of the photon orbit is positive (moving the orbit out beyond $r_p = 3M$) for all $\xi < \xi_1 \approx 0.284$. This includes both the conformal and minimally coupled cases. Values of ξ larger than ξ_1 give semiclassical corrections which move the photon orbit inward.

Quadratic curvature invariants play an important role in quantum field theory in curved spacetime. Of the four invariants, $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$, $R_{\alpha\beta}R^{\alpha\beta}$, R^2, and $\Box R$, the middle two are of order ϵ^2, while $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ has an order ϵ correction, and the leading term in $\Box R$ is of order ϵ. We find that, to order ϵ,

$$\begin{aligned} C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6} + \frac{\epsilon}{480\pi r^9}\Bigg\{ & [2368 + 1920(\xi - \tfrac{1}{6})]M^5 - [240 - 960(\xi - \tfrac{1}{6})]M^4 r \\ & -[160 + 480(\xi - \tfrac{1}{6})]M^3 r^2 - [104 + 720(\xi - \tfrac{1}{6})]M^2 r^3 \\ & +12Mr^4 + r^5 + 48M^2 r^3 \ln\left(\frac{r}{2M}\right)\Bigg\} \end{aligned} \tag{41}$$

and

$$\Box R = \frac{\epsilon M^2}{10\pi r^9}[96M^3 - 40M^2 r - (\xi - \tfrac{1}{6})(3240M^3 - 1600M^2 r + 60Mr^2 + 15r^3)] \ . \tag{42}$$

As with previously examined quantities, there are specific radii at which the values of these curvature invariants are independent of ξ. It is interesting that for the square of the Weyl curvature $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$, this occurs precisely at the event horizon, $r = 2M$. At the horizon,

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{3}{4M^4} + \frac{\epsilon}{384\pi M^4} \ , \tag{43}$$

independent of the curvature coupling.

V. EFFECTIVE POTENTIAL FOR PARTICLE TRAJECTORIES

To investigate further the physical consequences resulting from the changes in the spacetime geometry when back reaction effects are taken into account, one can examine the effects on the trajectories of particles close to the black hole. An efficient way to do this is to make use of the effective potential formalism. We consider an equatorial slice $\theta = \pi/2$ of the spacetime geometry which can be done without loss of generality. Then following Hochberg, Kephart, and York [21] we note that the four-momentum of a test particle in the spacetime is given by

$$p^\mu = (\dot{t}, \dot{r}, 0, \dot{\phi}) \ , \tag{44}$$

where the overdot means differentiation with respect to an affine parameter. The effective potential is obtained from the square of the four-momentum. The result is

$$E^2 = -g_{tt}g_{rr}\dot{r}^2 + V^2 \ , \tag{45}$$

where

$$V^2 = -g_{tt}\left(\frac{L^2}{r^2} + m^2\right) \ . \tag{46}$$

Here E and L are the particle's conserved energy and angular momentum defined by

$$E = -\chi^\alpha p_\alpha = -p_t \ , \tag{47}$$

$$L = \zeta^\alpha p_\alpha = p_\phi \ , \tag{48}$$

where χ and ζ are, respectively, the timelike and axial

Killing vector fields, and m is the rest mass of the test particle.

Substituting in the perturbed metric [Eqs. (23) and (24)] and keeping terms up to order ϵ, one finds

$$\begin{aligned} V^2 &= V_0{}^2(1+\epsilon V_1)\ , \\ V_0{}^2 &= \left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+m^2\right)\ , \\ V_1 &= 2\rho(r) - \frac{2M\mu(r)}{r-2M}\ . \end{aligned} \tag{49}$$

The relative change in the effective potential, V_1, depends on the box size r_0, the radial coordinate r, and the curvature coupling constant ξ. It is a linear function of ξ and can thus be written as

$$V_1 = V_{1C} + (\xi - \tfrac{1}{6})V_{1D}\ . \tag{50}$$

A graphical analysis of V_{1C} shows that, as found by Hochberg, Kephart, and York [21], $V_{1C} < 0$ for large values of r_0. In fact, this appears to be the case for all $r \leq r_0$ if $r_0 > 2.8M$. A careful analytical analysis of V_{1D} shows that it is positive for all $r \leq r_0$ if $r_0 > 2(1+6^{1/2})M$. Since boundary effects will be important near the horizon unless r_0 is much larger than these values we will only consider the case $r_0 \gg 2M$. Then since V_{1C} and V_{1D} have opposite signs but V_{1D} is multiplied by $(\xi - \frac{1}{6})$ we see that for $\xi < \frac{1}{6}$, $V_1 < 0$ for large r_0. For $\xi > \frac{1}{6}$ the sign of V_1 depends on the relative strengths of V_{1C} and V_{1D} as well as the size of ξ. For large enough values of r_0, and values of ξ of order unity, a simple analysis shows that the V_{1C} term dominates and again $V_1 < 0$. The physical implications of a decreased effective potential as pointed out in Ref. [21] include a larger effective cross section for capture of photons and massive particles by the black hole.

VI. ENTROPY

The thermodynamical entropy S of the black hole dressed with the quantized scalar field can be computed using the corrected metric given in this paper. To do so we will use the method given in Ref. [7] in which the entropy is computed using the relationship

$$dS = \frac{dE}{T_{\text{loc}}} = \frac{1}{T_{\text{loc}}}\left(\frac{\partial E}{\partial M}\right)_r dM\ . \tag{51}$$

Here E is the quasilocal energy of the system. For a classical field minimally coupled to gravity it has been rigorously shown that $E = r - r(g^{rr})^{1/2}$ [22]. Although a rigorous derivation does not yet exist for the quasilocal energy for the fields we are considering ($\xi \neq 0$), we use this expression in our computation of the entropy. Our reason for doing so is that this is the simplest possible expression[2] for the quasilocal energy which both is a continuous function of the radial coordinate r and gives the correct ADM mass in the limit $r \to \infty$.

Substituting Eq. (39) into (51) and using (32) allows one to write the equation for the entropy in terms of μ and ρ_0. Integrating Eq. (51) and changing the integration variable from M to $w = 2M/r$ results in the following expression for the total entropy $S(r)$, for $r < r_0$:

$$S = \frac{4\pi M^2}{\hbar} + \Delta S\ , \tag{52}$$

$$\Delta S = 8\pi \int_1^w [\tilde{w}^{-1}(\rho_0 - \mu) + \frac{\partial \mu}{\partial \tilde{w}} - nK^{-1}\tilde{w}^{-1}]d\tilde{w}\ , \tag{53}$$

where

$$n = \left.\frac{\partial(K\mu)}{\partial w}\right|_{w=1}\ . \tag{54}$$

This expression is valid not only for the massless scalar field but for any massless free field [7].

The form of the components of the stress-energy tensor for all massless fields is such that it is possible to write Eq. (53) as an integral over a linear combination of components of the stress-energy tensor [23]. This is done by first noting that

$$\begin{aligned} \mu &= -\frac{4\pi}{\epsilon M}\int_{2M}^r d\tilde{r}\,\tilde{r}^2\langle T^t{}_t\rangle\ , \\ \rho_0 &= \frac{4\pi}{\epsilon}\int_{2M}^r d\tilde{r}\frac{\tilde{r}^2}{\tilde{r}-2M}(\langle T^r{}_r\rangle - \langle T^t{}_t\rangle)\ . \end{aligned} \tag{55}$$

Changing the integration variables in these integrals to w and substituting into Eq. (53) results in a double integral. This double integral can be reduced after some algebra and use of the Bianchi identity to a single radial integral. The result which is valid for $r < r_0$ is

$$\Delta S(r) = \frac{8\pi M}{\hbar}\int_{2M}^r 4\pi\tilde{r}^2[T_r{}^r - T_t{}^t - T_\mu{}^\mu \ln(r/\tilde{r})]d\tilde{r}\ . \tag{56}$$

We have fixed the integration constant so that $\Delta S = 0$ at the event horizon of the black hole. This would seem to be the obvious boundary condition to use. Because we are integrating over M, in principle one could also have an arbitrary function of $r/\hbar^{1/2}$ and ξ. However, as pointed out in Ref. [7], such a function does not occur in the solutions to the back-reaction equations and therefore it seems unlikely that it would occur as an integration "constant" for the entropy.

When evaluated for conformal scalar fields ($\xi = 1/6$), massless fermions, and Abelian vector gauge fields, it turns out that ΔS is non-negative and monotonically increasing with radius, which is the expected physical behavior for entropy [7].

Let us now evaluate $\Delta S(\xi, w)$ for the massless scalar field with arbitrary curvature

[2] Any local expression for the quasilocal energy in which either the fields or higher derivatives of the metric explicitly appear will be discontinuous at the cavity wall.

$$\Delta S(\xi, w) = \frac{8\pi}{K}\left(\frac{1}{2}\right)[\tfrac{8}{9}w^{-3} + \tfrac{8}{3}w^{-2} + 8w^{-1} - \tfrac{16}{9} + \tfrac{32}{3}\ln(w) - \tfrac{40}{3}w - 8w^2 + \tfrac{104}{9}w^3] + \frac{320\pi}{K}\left(\xi - \frac{1}{6}\right)\left(\frac{1}{2}\right)(-3 - 2\ln(w) + 2w + 3w^2 - 2w^3). \quad (57)$$

Unlike the previous cases which have been studied [7], the entropy defined in Eq. (57), because it has a term linear in ξ, for $\xi \neq \frac{1}{6}$, cannot be non-negative and monotonically increasing as a function of r, for fixed M, for all values of ξ. It follows from the general result, Eq. (53), that for all massless fields, $[\partial(\Delta S)/\partial r]_M = 0$ at the horizon. We must insist that this be a local minimum of ΔS to prevent the existence of a spherical layer of negative entropy near the horizon. Examining the second derivative ΔS with respect to r at the horizon, this implies that $\xi \leq \frac{7}{10}$. Indeed, for physically acceptable behavior of the entropy, we cannot accept a shell of negative entropy anywhere, that is we cannot permit $[\partial(\Delta S)/\partial r]_M = 0$ for any value of r if further derivatives indicate that ΔS is locally decreasing as r increases. An examination of Eq. (57) with this criterion in mind shows that the largest "permissible" value of ξ is found at the horizon. Investigation of the behavior of Eq. (57) shows that the smallest such value is $\xi \cong -3.431$, which occurs at $r \cong 5.126M$. We conclude that the physically acceptable behavior of the entropy of a massless scalar field around a black hole requires, in the present approximation to the back reaction,

$$-3.431 \leq \xi \leq \tfrac{7}{10}\,. \quad (58)$$

Note that this range includes both minimal ($\xi = 0$) and conformal ($\xi = \frac{1}{6}$) couplings. We believe this is the first instance in which it has been shown that, as a matter of principle, the demand of physically well-behaved entropy can be used to limit the range of a fundamental coupling constant.

These limits have been derived using an analytical approximation to the stress-energy tensor for a massless free scalar field. They will be refined somewhat by using the full numerical values for the stress-energy tensor. Further, massive fields and/or fields with a self-interaction of the form $\lambda\phi^4$ will make contributions to the entropy which may be significantly different from those of the massless free scalar field. This may lead to different bounds of ξ for those fields. These issues are being investigated by the authors in collaboration with T. W. Kephart. The results will appear elsewhere.

ACKNOWLEDGMENTS

One of us (J.W.Y.) wishes to thank T. W. Kephart for helpful discussions. This research was supported in part by National Science Foundation Grants Nos. PHY-9207903 (W.A.H. and J.W.) and PHY-8908741 (J.W.Y.).

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] J. M. Bardeen, Phys. Rev. Lett. **46**, 382 (1980).
[3] P. Hajicek and W. Israel, Phys. Lett. **80A**, 9 (1980).
[4] J. W. York, Jr., in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. DeWitt*, edited by S. M. Christensen (Hilger, Bristol, 1984), pp. 135–147.
[5] J. W. York, Jr., Phys. Rev. D **31**, 775 (1985).
[6] D. Hochberg and T. W. Kephart, Phys. Rev. D **47**, 1465 (1993).
[7] D. Hochberg, T. W. Kephart, and J. W. York, Jr., Phys. Rev. D **48**, 479 (1993).
[8] M. S. Fawcett, Commun. Math Phys. **89**, 103 (1983).
[9] K. W. Howard and P. Candelas, Phys. Rev. Lett. **53**, 403 (1984); K. W. Howard, Phys. Rev. D **30**, 2532 (1984).
[10] B. P. Jensen and A. Ottewill, Phys. Rev. D **39**, 1130 (1989).
[11] P. R. Anderson, W. A. Hiscock, and D. A. Samuel, Phys. Rev. Lett. **70**, 1739 (1993).
[12] P. R. Anderson, W. A. Hiscock, and D. A. Samuel, Montana State University report, 1994 (unpublished).
[13] D. N. Page, Phys. Rev. D **25**, 1499 (1982).
[14] M. R. Brown and A. C. Ottewill, Phys. Rev. D **31**, 2514 (1985).
[15] M. R. Brown, A. C. Ottewill, and D. N. Page, Phys. Rev. D **33**, 2840 (1986).
[16] V. P. Frolov and A. I. Zel'nikov, Phys. Rev. D **35**, 3031 (1987).
[17] For discussion of the various black hole vacuum states, see N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[18] P. R. Anderson, W. A. Hiscock, and J. G. McLaughlin, Montana State University report, 1994 (unpublished).
[19] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[20] W. A. Hiscock, Phys. Rev. D **37**, 2142 (1988).
[21] D. Hochberg, T. W. Kephart, and J. W. York, Phys. Rev. D **49**, 5257 (1994).
[22] J. D. Brown and J. W. York, Jr., Phys. Rev. D **47**, 1407 (1993).
[23] O. B. Zaslavskii, Phys. Lett. A **181**, 105 (1993).