Black hole entropy in two dimensions

Robert C. Myers*

Department of Physics, McGill University, Montréal, Québec, Canada H3A 2T8
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Black hole entropy is studied for an exactly solvable model of two-dimensional gravity, using recently developed Noether charge techniques. This latter approach is extended to accommodate the nonlocal form of the semiclassical effective action. In the two-dimensional model, the final black hole entropy can be expressed as a local quantity evaluated on the horizon. This entropy is shown to satisfy an increase theorem on either the global or apparent horizon of a two-dimensional black hole.

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I. INTRODUCTION

The quantum instability of black holes was first demonstrated by Hawking [1]. An external observer detects the emission of thermal radiation from the black hole with a temperature proportional to its surface gravity κ:

$$k_B T = \frac{\hbar \kappa}{2\pi} \ . \tag{1}$$

This result draws interesting connections between quantum field theory, general relativity, and thermodynamics, but also leads to a celebrated conflict between quantum theory and general relativity [2]. If the thermal emissions continue indefinitely, the black hole would ultimately vanish, having radiated away its entire mass. In the process, a pure initial quantum state would appear to evolve into a mixed final state, since the information associated with the black hole's internal conditions is irrevocably lost. Hence unitary time evolution, a basic tenet of quantum theory, would appear to be violated.

Exactly what happens in the final moments of black hole evaporation remains an open question, since it requires an understanding of physics at high curvatures as well as of back reaction effects. However, this is a question which has recently come under intense study in the context of two-dimensional theories of gravity. Callan, Giddings, Harvey, and Strominger (CGHS) [3] began with a theory of two-dimensional dilaton gravity coupled to N massless scalar fields:

$$I_0 = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[e^{-2\phi}[R + 4(\nabla\phi)^2 + 4\lambda^2] \right.$$
$$\left. - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right] . \tag{2}$$

This action is closely related to the effective action describing the radial modes of a four-dimensional extremal black hole in string theory [4]. The equations of motion for this action (2) are exactly soluble. Further since this

theory is in two dimensions, the leading quantum contributions induced by the matter fields can be calculated [5]. One accounts for these effects by including the following nonlocal term to the effective gravity action [6]:

$$I_1 = -\frac{N\hbar}{96\pi} \int d^2x \sqrt{-g(x)}$$
$$\times \int d^2y \sqrt{-g(y)} R(x) G(x,y) R(y), \tag{3}$$

where $G(x,y)$ is the Green's function for the two-dimensional D'Alembertian ∇^2. With a large N expansion in which $N\hbar$ is held fixed, one has a systematic expansion in which the classical and one-loop actions contribute at the same order, and which incorporates the dominant semiclassical effects, including both the Hawking radiation and the back reaction effects on the geometry. Russo, Susskind, and Thorlacius (RST) [7] modified the semiclassical action by adding a local covariant counterterm

$$I_2 = -\frac{N\hbar}{48\pi} \int d^2x \sqrt{-g} \, \phi \, R \tag{4}$$

with which the theory is again exactly soluble. Thus the analysis of the solutions is simplified, and combined with a particular choice of boundary conditions in the strong coupling regime [7–9], one can produce a physical picture for the entire process of the formation and evaporation of a black hole.

The RST model is then a natural framework in which to examine questions about information loss and black hole entropy [9, 10]. Since (classically) a horizon limits one's ability to collect information about the universe, it may seem natural to associate entropy with such a boundary. Bekenstein was the first to suggest that black holes should have an intrinsic entropy proportional to the surface area of the horizon, A[11]. The discovery of Hawking radiation [1] allowed a precise result to be formulated:

$$S = \frac{k_B A}{4G\hbar} \ . \tag{5}$$

This formula applies for any black hole solution of Einstein's equations [12]. If as in an effective quantum cor-

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rected action, the Einstein action is perturbed by higher curvature interactions, the black hole temperature (1) remains unchanged but the entropy formula (5) no longer applies [13]. It is only recently that exact expressions have been derived for black hole entropy in such modified theories [14–18]. In particular, Wald [14] developed a very general technique, which may be applied to any diffeomorphism invariant theory in any number of dimensions. In his Noether charge approach (see below), a first law of black hole mechanics [19] is derived

$$\frac{\kappa}{2\pi}\delta S = \delta M - \Omega^{(\alpha)}\delta J_{(\alpha)} \ , \tag{6}$$

where M, $J_{(\alpha)}$, and $\Omega^{(\alpha)}$ are the mass, the angular momentum [20], and the angular velocity of the black hole, respectively. From this equation one is able to identify the entropy S as given by the integral of a local geometric expression over a cross section of the horizon. In Eq. (6) and for the remainder of the paper, we adopt the standard convention of setting $\hbar = c = k_B = 1$.

If the resulting entropy expressions are to play the true role of an entropy, they should satisfy a second law as well — i.e., S should never decrease as a black hole evolves. In Einstein gravity, such a result is provided by Hawking's area theorem [21], which states that in any classical processes involving black holes, the total surface area of the event horizon will never decrease. Some partial results for black hole entropy in higher curvature theories have also been found [22].

Wald's techniques [14] for determining the black hole entropy were originally developed for application to higher curvature theories in four or higher dimensions. In this paper, these techniques are applied to the two-dimensional dilaton gravity models, described above. Section II describes Wald's method [14] with an application to the classical action (2). This calculation reproduces the black hole entropy already derived by other methods. (Reference [18] has also applied the Noether charge technique to determine the black hole entropy for I_0.) A second law is also proven for this entropy expression. Section III extends the Wald's method to accommodate the nonlocal form of the the semiclassical action in Eq. (3). Even though the action is nonlocal, in conformal gauge the contribution to the entropy is a local expression evaluated on the horizon. The total black hole entropy satisfies an increase theorem for eternal black hole solutions (i.e., black holes in equilibrium with an external heat bath). Section IV extends the latter result to dynamical black holes (i.e., with no external heat bath). In this case, vacuum corrections to the matter stress-energy must be accounted for to evaluate properly the black hole entropy. The corrected expression is also shown to satisfy a second law. Section V presents a discussion of our results, and in particular a comparison with the recent results of Ref. [10]. Throughout the paper, we employ the conventions of [23].

II. ENTROPY AS NOETHER CHARGE AND CLASSICAL ENTROPY

Wald's derivation of black hole entropy relies on constructing a Noether charge associated with the diffeomor-

phism invariance of the action. The present discussion will be a brief introduction to these techniques. In particular for the most part, the discussion will be limited to applications in two dimensions for the theories studied here. The interested reader is referred to Refs. [14, 18, 17] for complete details.

A key concept in Wald's approach is the notion of a Killing horizon. A Killing horizon is a null hypersurface whose null generators are orbits of a Killing vector field. If the horizon generators are geodesically complete to the past (and if the surface gravity is nonvanishing), then the Killing horizon contains a spacelike cross section B, called the *bifurcation surface*, on which the Killing field χ^a vanishes [24]. Such a bifurcation surface is a fixed point of the Killing flow, and lies at the intersection of the two null hypersurfaces that comprise the full Killing horizon. Since the CGHS and RST models of two-dimensional gravity are exactly soluble, it is straightforward to establish that the event horizon of any stationary black hole is a Killing horizon. In these models, the bifurcation surface reduces to the point at the origin of the Kruskal-like coordinates. Wald's construction applies to black holes with bifurcate Killing horizons, but given the final local geometric expression for the black hole entropy, the latter may be evaluated at any point on the horizon of an arbitrary black hole solution.

Another essential element of Wald's approach is the Noether current associated with diffeomorphisms [25]. Let L be a Lagrangian built out of some set of dynamical fields, including the metric, collectively denoted as ψ. Under a general field variation $\delta\psi$, the Lagrangian varies as

$$\delta(\sqrt{-g}L) = \sqrt{-g}E \cdot \delta\psi + \sqrt{-g}\,\nabla_a\theta^a(\delta\psi), \tag{7}$$

where the centered dot denotes a summation over the dynamical fields including contractions of tensor indices, and $E = 0$ are the equations of motion. With symmetry variations, for which $\delta(\sqrt{-g}L) = 0$, θ^a is the Noether current which is conserved when the equations of motion are satisfied — i.e., $\nabla_a\theta^a(\delta\psi) = 0$ when $E = 0$. Rather than vanishing for a diffeomorphisms, $\delta\psi = \mathcal{L}_\xi\psi$, the variation of the Lagrangian is a total derivative, $\delta(\sqrt{-g}L) = \mathcal{L}_\xi(\sqrt{-g}L) = \sqrt{-g}\nabla_a(\xi^a L)$. Thus the conserved Noether current J^a is

$$J^a = \theta^a(\mathcal{L}_\xi\psi) - \xi^a L \ ,$$

where $\nabla_a J^a = 0$ when $E = 0$. Further since J^a is conserved for any diffeomorphism (i.e., for *all* vector fields ξ^a), there exists a globally defined scalar Q, called the Noether potential, satisfying $J^a = \epsilon^{ab}\nabla_b Q$ [26], where ϵ_{ab} is the volume form in two dimensions. Again, this equation for J^a holds up to terms which vanish when the equations of motion are satisfied. Q is a local function of the dynamical fields and a linear function of ξ^a and its derivatives. One can also show then that the Noether charge evaluated for a spacelike interval M reduces to the boundary terms $N = Q_+ - Q_-$, where $Q_\pm$ denotes the Noether potential evaluated at the end points of M.

Given these results, Wald [14] derives a first law of black hole mechanics [19]. One begins by evaluating the Noether charge on a surface in a stationary black

hole background. The diffeomorphism vector is chosen to be the Killing field which generates the horizon, and the surface is a spacelike interval extending from asymptotic infinity to the bifurcation point. Then the dynamical fields are varied infinitesimally to a nearby solution (which need not be stationary), and one finds an identity relating a surface term at infinity to another on the horizon. This identity has the form of the first law (6) (but of course there are no angular momentum terms in two dimensions). The boundary term at the horizon is interpreted as yielding the variation of the entropy, which is then given by $S = 2\pi Q(\tilde{\chi})|_{x_B}$. Here the Noether potential is evaluated at the bifurcation point, and $\tilde{\chi}^a$ is the Killing vector scaled to have unit surface gravity.

By construction Q involves the Killing field $\tilde{\chi}^a$ and its derivatives; however, this dependence can be eliminated as follows [14]. Using Killing vector identities, Q becomes a function of only $\tilde{\chi}^a$ and the first derivative $\nabla_a \tilde{\chi}_b$. At the bifurcation point though, $\tilde{\chi}^a$ vanishes and $\nabla_a \tilde{\chi}_b = \epsilon_{ab}$. Thus eliminating the term linear in $\tilde{\chi}^a$ and replacing $\nabla_a \tilde{\chi}_b$ by ϵ_{ab} yields a completely geometric functional of the metric and the matter fields, which may be denoted $\tilde{Q}$. The expression $2\pi\tilde{Q}$ yields the correct entropy when evaluated at the bifurcation point, or in fact when evaluated at an arbitrary point on the Killing horizon [17]. Thus this latter expression is a natural candidate for the entropy of a general nonstationary black hole. Actually, a number of ambiguities arise in the construction of $\tilde{Q}$, but none of these have any effect when the charge is evaluated on a stationary horizon [17]. These ambiguities may become significant for nonstationary horizons, though. In this case, a choice which yields an entropy that satisfies the second law would be a preferred definition [22].

Using Wald's technique, results have been established to compute the entropy for a general Lagrangian of the following form: $L = L(\psi_m, \nabla_a \psi_m, g_{ab}, R_{abcd})$, that is involving only second derivatives of the metric g_{ab}, and first derivatives of the matter fields, denoted by ψ_m. The final entropy may then be written [17, 18] (see also [16])

$$S = 2\pi\tilde{Q} = -2\pi Y^{abcd}\epsilon_{ab}\epsilon_{cd}, \qquad (8)$$

where the tensor Y^{abcd} is defined by

$$Y^{abcd} \equiv \frac{\partial L}{\partial R_{abcd}} .$$

This result is sufficient to determine the black hole entropy for the classical CGHS action (2). Only the first term in the action makes a contribution with the outcome that

$$S_0 = 2e^{-2\phi} . \qquad (9)$$

For general higher curvature theories, the definition (8) represents a particular (simple) choice given the ambiguities in Wald's construction. In the present case though where I_0 is only quadratic in derivatives, the formula (9) does not suffer from any such ambiguities. One can derive the same result by integrating the thermodynamic relation $dS = dM/T$ given the temperature as a function of the mass [10]. (In fact, the temperature is a constant

in the present case.) Alternatively, Frolov [27] produced this formula using Euclidean path integral techniques for a class of static black hole solutions. Interpreting this formula in terms of the associated four-dimensional black hole, one finds that Eq. (9) is precisely one-quarter the area of the event horizon, as prescribed by Eq. (5) [9]. This expression was previously derived with the Noether charge technique in Ref. [18]. Since the classical CGHS model is exactly soluble, it is relatively straightforward to establish a second law for this entropy, as we will now describe.

A review of the CGHS model can be found in Ref. [28]. The solutions are most easily described in conformal gauge — i.e., choose the metric to have the form $ds^2 = -e^{2\rho} dx^+ dx^-$ using the freedom of coordinate invariance. Among the resulting equations of motion, one finds [3] $\partial_+ \partial_- (\rho - \phi) = 0$. Hence one has $\rho = \phi + \frac{1}{2}[w_+(x^+) + w_-(x^-)]$. Now a coordinate transformation of the form $x^\pm = h^\pm(\sigma^\pm)$ leaves the line element in the same form $ds^2 = -e^{2\rho'} d\sigma^+ d\sigma^-$ with $\rho' = \rho + \frac{1}{2}\ln(\partial_{\sigma^+} h^+) + \frac{1}{2}\ln(\partial_{\sigma^-} h^-)$. So this residual coordinate freedom within conformal gauge allows one to shift the conformal function ρ to set $w_+ = 0 = w_-$. This choice with $\rho = \phi$ is called Kruskal gauge.

In Kruskal gauge, the general solution is [3]

$$e^{-2\rho} = e^{-2\phi} = -\lambda^2 x^+ x^- - x^+ P_+(x^+) + \Delta_+(x^+)$$
$$-x^- P_-(x^-) + \Delta_-(x^-) + m_0 \qquad (10)$$

where

$$P_\pm = \int_0^{x^\pm} dy^\pm T_{\pm\pm}(y^\pm), \quad \Delta_\pm = \int_0^{x^\pm} dy^\pm y^\pm T_{\pm\pm}(y^\pm),$$
$$(11)$$

and $T_{\pm\pm} = \frac{1}{2}\sum_{i=1}^{N}(\partial_\pm f_i)^2 \geq 0$. Many of the features of these black hole solutions are illustrated by the static vacuum solution

$$ds^2 = -\frac{dx^+ dx^-}{m_0 - \lambda^2 x^+ x^-}$$

which is a black hole with Arnowitt-Deser-Misner (ADM) mass $M = \lambda m_0$ (if $m_0 > 0$). In this case, the global structure is essentially the same as that of a Schwarzschild black hole. There are past and future spacelike curvature singularities at $x^+ x^- = m_0/\lambda^2$, which are hidden behind the future and past event horizons at $x^\pm = 0$. Asymptotically as $x^- \to -\infty$ (or $x^+ \to -\infty$), the metric becomes flat as can be seen from the coordinate transformation $\pm \lambda x^\pm = e^{\pm\lambda\sigma^\pm}$ (or $\mp\lambda x^\pm = e^{\mp\lambda\sigma^\pm}$), which yields $ds^2 \simeq dx^+ dx^-/(\lambda^2 x^+ x^-) = -d\sigma^+ d\sigma^-$. The solution in these asymptotic regions is called the linear dilaton vacuum, since the dilaton grows linearly in a spacelike direction, $\phi \simeq \frac{\lambda}{2}(\sigma^- - \sigma^+)$.

Now returning to the proof of the second law, it will be assumed that T_{++} vanishes in the asymptotic future as $x^+ \to \infty$. In this asymptotic region, observers at points where $\partial_+ e^{-2\phi} < 0$ will inexorably be drawn to the singularity where $\phi \to \infty$. Asymptotically on the global event horizon, one then has $\partial_+ e^{-2\phi} = 0$. By integrating the equations of motion or by differentiating the general

solution (10), one finds

$$\partial_+ e^{-2\phi} = -[\lambda^2 x^- + P_+(x^+)] \ .$$

Hence the global future event horizon can be identified as $x_H^- = -\frac{1}{\lambda^2}P_+(\infty)$. Combining this result with Eqs. (9) and (11) yields

$$\partial_+ S_0 = 2 \int_{x^+}^{\infty} dy^+ T_{++}(y^+), \qquad (12)$$

where $\partial_+ e^{-2\phi}(\infty) = 0$ was also used. Since the integral term is positive definite, the entropy is always increasing along the future event horizon. Note that at an early point on the horizon before any matter has crossed into the black hole, the entropy is increasing because of matter contributions to the future in Eq. (12). This behavior illustrates the teleological nature of the global event horizon — the entropy begins increasing early on in anticipation of the infalling matter.

Following the suggestion of Refs. [10, 29], it may also be interesting to follow the progression of the entropy along the apparent horizon. In the stationary black holes, which play an important role in the Wald's derivation of the entropy, the apparent and global horizons will coincide. To define an apparent horizon in two-dimensional gravity, one must appeal to the related four-dimensional black hole. There, $e^{-2\phi}$ is proportional to the area of the transverse two-spheres. Trapped points then satisfy $\partial_+ e^{-2\phi} < 0$ and $\partial_- e^{-2\phi} < 0$ (i.e., points for which the area necessarily decreases in the forward light cone). The apparent horizon or the boundary of the region of trapped points is defined by $\partial_+ e^{-2\phi} = 0$. From the general solution (10), one has then $x_A^- = -\frac{1}{\lambda^2}P_+(x^+)$. From the definitions (11), one sees that the apparent horizon can only move out (to more negative x^-) as it evolves forward in x^+. The future directed tangent vector is $t^a \partial_a = \partial_+ + \frac{\partial x_A^-}{\partial x^+}\partial_- = \partial_+ - \frac{1}{\lambda^2}T_{++}\partial_-$. Now the variation of the entropy is given by

$$t^a \partial_a S_0 = 2\partial_+ e^{-2\phi} - \frac{2}{\lambda^2}T_{++}\partial_- e^{-2\phi}$$

$$= \frac{2}{\lambda^2}T_{++}[\lambda^2 x^+ + P_-(x_A^-)] \ .$$

Considering only the evolution of the black hole to the future of the past event horizon, $x_H^+ = -\frac{1}{\lambda^2}P_-(-\infty)$ (where we assumed that any outgoing radiation vanishes as $x^- \to -\infty$). Hence in the second factor above one has

$$\lambda^2 x^+ + P_-(x_A^-) > P_-(x_A^-) - P_-(-\infty)$$

$$= \int_{-\infty}^{x_A^-} dy^- \, T_{--}(y^-) > 0 \ .$$

Hence along the apparent horizon, $t^a \partial_a S > 0$ has been established — i.e., the entropy only increases as the apparent horizon evolves. Note that when $T_{++} = 0$, the apparent horizon remains at a fixed value of x^-, and that the variation of S_0 also vanishes.

III. SEMICLASSICAL CORRECTIONS TO ENTROPY

The black holes considered in the previous section are fixed classical backgrounds. One can compute the Hawking radiation for these backgrounds using the relation with the trace anomaly for massless fields coupled to two-dimensional gravity [5]. The temperature is found to be a constant $\lambda/(2\pi)$, independent of the mass [3]. The back reaction of the geometry due to the Hawking radiation can be incorporated by adding the semiclassical contributions to the action. In the RST model, there are two semiclassical terms given in Eqs. (3) and (4), and both will make new contributions to the black hole entropy. The second of these, I_2, is a local term and falls into the class covered by Eq. (8). One finds then that I_2 contributes

$$S_2 = -\frac{N}{12}\phi$$

to the black hole entropy.

Being nonlocal, I_1 does not lend itself directly to Wald's analysis [14]. However, one can introduce an auxiliary field to rewrite this action in a local form as

$$\tilde{I}_1 = -\frac{N}{96\pi} \int d^2x \, \sqrt{-g} \, [(\nabla\eta)^2 - 2R\eta] \ . \qquad (13)$$

The η equation of motion is $\nabla^2\eta + R = 0$, for which the solution may be written

$$\eta_0(x) = -\int d^2y \, \sqrt{-g(y)} \, G(x,y) \, R(y) \ .$$

Substituting η_0 into the action (13), one recovers the original nonlocal expression I_1 in Eq. (3). The local action may be analyzed as in the previous section, yielding an entropy contribution

$$S_1 = \frac{N}{12}\eta(x_H) = -\frac{N}{12}\int d^2y \, \sqrt{-g(y)} \, G(x_H, y) \, R(y)$$

$$= \frac{N}{6}\rho(x_H), \qquad (14)$$

where the final local result applies in conformal gauge. This approach may appear suspect since it involves adding extra dynamical degrees of freedom to the theory. It will now be shown that with minor modifications the Noether charge approach can be applied to the nonlocal action, and that Eq. (14) is in fact the correct result.

In the analysis of the previous section, the Lagrangian or the action was a functional of only the dynamical fields ψ and their derivatives. Further diffeomorphism invariance then dictates that the variations $\delta\psi = \mathcal{L}_\xi\psi$ induce the variation $\delta L = \mathcal{L}_\xi L$. In the nonlocal action I_1, it is not immediately apparent that the Green's function $G(x,y)$ fits into this framework. In fact though, $G(x,y)$ is implicitly a functional of the metric defined through the equation $\nabla_x^2 G(x,y) = \delta^2(x-y)$, which is most usefully written as

$$(\partial_a \sqrt{-g} g^{ab} \partial_b)_x G(x,y) = \sqrt{-g}\delta^2(x-y) = \tilde{\delta}^2(x-y) \ . \qquad (15)$$

Here $\tilde{\delta}^2(x-y)$ is a density distribution satisfying $\int d^2x\, f(x)\,\tilde{\delta}^2(x-y) = f(y)$, and so it is independent of the metric. Varying the metric in Eq. (15) yields

$$(\partial_a\sqrt{-g}g^{ab}\partial_b)_x \delta G(x,y)$$
$$+(\partial_a\sqrt{-g}[\tfrac{1}{2}g^{cd}\delta g_{cd}\,g^{ab} - \delta g^{ab}]\partial_b)_x G(x,y) = 0$$

where $\delta g^{ab} = g^{ac}g^{bd}\delta g_{cd}$. The variation of the Green's function is then

$$\delta G(x,y) = \int d^2z\, G(x,z)$$
$$\times(\partial_a\sqrt{-g}[\delta g^{ab} - \tfrac{1}{2}g^{cd}\delta g_{cd}\,g^{ab}]\partial_b)_z G(z,y).$$
$$(16)$$

Thus the metric variations produce a well-defined albeit nonlocal variation of the Green's function. This result (16) is of course used to derive the metric equations of motion for the RST model. Next, one can verify that $\delta g_{ab} = \mathcal{L}_\xi g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$ induces the appropriate variation

$$\delta G(x,y) = \mathcal{L}_\xi G(x,y)$$
$$= (\xi^a\partial_a)_x G(x,y) + (\xi^a\partial_a)_y G(x,y) \ . \quad (17)$$

Hence the Noether charge analysis can be applied to the nonlocal action treating the Green's function as a functional of the metric. The only change as compared to the discussion in previous section is to refer the construction of the Noether charge to the action, rather than the Lagrangian. For example, Eq. (7) is replaced by [30]

$$\delta\psi \cdot \frac{\delta I_1}{\delta\psi} = E_1 \cdot \delta\psi + \nabla_a\theta_1^a(\delta\psi) \ , \quad (18)$$

where standard functional differentiation is understood on the left-hand side [e.g., for a scalar field, $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^2(x-y)$]. Here, E_1 and θ_1^a are the contributions of I_1 to the total equations of motion, and the total boundary current, respectively. Since the action is manifestly covariant, one knows that

$$\mathcal{L}_\xi\psi \cdot \frac{\delta I_1}{\delta\psi} = \nabla_a\gamma^a(\xi) \ .$$

Note that γ^a does not take the form $\xi^a\omega$ since several integration by parts are required to yield Eq. (17) from the variation in Eq. (16). Of course, γ^a contains nonlocal expressions involving the Green's function. The new contribution to the total Noether current is then $J_1^a = \theta_1^a(\mathcal{L}_\xi\psi) - \gamma^a(\xi)$. It is not hard to show explicitly that when the equations of motion are satisfied, the Noether potential receives a new contribution of the form

$$Q_1(x) = \frac{N}{48\pi}\left[(\nabla^a\xi^b\epsilon_{ab})(x)\int d^2y\sqrt{-g(y)}\,G(x,y)\,R(y)\right.$$
$$\left.+2(\xi^a\epsilon_{ab})(x)\int d^2y\sqrt{-g(y)}\nabla_x^b G(x,y)\,R(y)\right].$$

Now as before, one eliminates the explicit dependence of the Noether potential on the vector field by retaining only the first term, and replacing $\nabla^a\xi^b$ with ϵ^{ab}. Thus one arrives at the contribution of I_1 to the black hole

entropy

$$S_1 = 2\pi\tilde{Q}_1(x_H) = -\frac{N}{12}\int d^2y\sqrt{-g(y)}\,G(x_H,y)\,R(y) \ ,$$
$$(19)$$

where one can evaluate this result at any point x_H on the horizon, while the integration runs over the entire spacetime.

Hence the total black hole entropy for the RST model is

$$S_{\rm RST} = S_0 + S_1 + S_2$$
$$= \frac{N}{6}\left(\frac{12}{N}e^{-2\phi}(x_H) - \frac{\phi(x_H)}{2} - \frac{1}{4}\ln\frac{N}{3} - \frac{1}{4}\right)$$
$$-\frac{N}{12}\int d^2y\sqrt{-g(y)}\,G(x_H,y)\,R(y) \qquad (20)$$

where an extra constant has been added for convenience. In conformal gauge with $ds^2 = -e^{-2\rho}dx^+dx^-$, one has $R = e^{-2\rho}(-2\nabla^2\rho) = 8e^{-2\rho}\partial_+\partial_-\rho$, and the above expressions for the Noether charge and entropy reduce to local terms involving the conformal factor evaluated at the horizon. In particular, the total entropy (20) becomes

$$S_{\rm RST} = \frac{N}{6}\left(\frac{12}{N}e^{-2\phi} + \rho - \frac{\phi}{2} - \frac{1}{4}\ln\frac{N}{3} - \frac{1}{4}\right) \ . \quad (21)$$

In the conformal gauge, the equations and solutions for the RST model are very similar to those of the classical CGHS model [3,8–10]. The equations are most easily analyzed in terms of [31]

$$\chi = \frac{12}{N}e^{-2\phi} + \rho - \frac{\phi}{2} - \frac{1}{4}\ln\frac{N}{3},$$
$$\Omega = \frac{12}{N}e^{-2\phi} + \frac{\phi}{2} + \frac{1}{4}\ln\frac{N}{48}, \qquad (22)$$

where the constants are chosen following [10]. One finds that the combination $\chi - \Omega = \rho - \phi + \frac{1}{2}\ln\frac{N}{12}$ is a free field [7], i.e.,

$$\partial_+\partial_-(\chi - \Omega) = 0 \ . \qquad (23)$$

As in the CGHS model, one uses the residual freedom in coordinate transformations to fix to Kruskal gauge where $\chi = \Omega$ (or $\rho = \phi + \frac{1}{2}\ln\frac{N}{12}$).

A new aspect of the semiclassical RST equations is that they are ill defined for a critical value of the dilaton. This critical point is also revealed by the fact that $\Omega \geq \Omega_{\rm cr} = \frac{1}{4}$ for any real value of ϕ. To complete the model, the behavior of the fields must be resolved at this critical point. Russo, Susskind, and Thorlacius [8, 9] suggested that one impose

$$\partial_+\Omega|_{\rm cr} = 0 = \partial_-\Omega|_{\rm cr} \qquad (24)$$

where the $\Omega = \Omega_{\rm cr}$ surface is timelike. This constraint ensures that the curvature remains finite as the boundary is approached [32].

In Kruskal gauge, apart from Eq. (23), the remaining gravity equations are

$$\partial_+\partial_-\Omega = -\lambda^2,$$
$$\partial_\pm^2\chi = \partial_\pm^2\Omega = -\tilde{T}_{\pm\pm}, \qquad (25)$$

where the matter stress-energy tensor has been scaled to $\widetilde{T}_{\pm\pm} = \frac{6}{N}\sum_{1=1}^{N}(\partial_{\pm}f_i)^2 \geq 0$. The general solution of these equations is

$$\chi = \Omega = -\lambda^2 x^+ x^- - x^+\widetilde{P}_+(x^+) + \widetilde{\Delta}_+(x^+)$$
$$-x^-\widetilde{P}_-(x^-) + \widetilde{\Delta}_-(x^-) + m_0 \qquad (26)$$

where

$$\widetilde{P}_{\pm} = \int_0^{x^{\pm}} dy^{\pm}\,\widetilde{T}_{\pm\pm}(y^{\pm}),$$

$$\widetilde{\Delta}_{\pm} = \int_0^{x^{\pm}} dy^{\pm}\,y^{\pm}\,\widetilde{T}_{\pm\pm}(y^{\pm}) \ .$$

These solutions are eternal black holes in equilibrium with a heat bath at infinity with a temperature $\frac{\lambda}{2\pi}$. Note that a black hole can remain in equilibrium with a single heat bath even while matter is falling in since the Hawking temperature is independent of the mass of the black hole.

Comparing Eqs. (21) and (22), one sees that the entropy may simply be written

$$S_{\text{RST}} = \frac{N}{6}(\chi - \Omega_{\text{cr}}) = \frac{N}{6}(\Omega - \Omega_{\text{cr}}) \qquad (27)$$

in the Kruskal gauge. A second law is established without any further effort when one realizes that in the present RST model, Ω replaces $e^{-2\phi}$ in both the equations of motion and entropy of the classical CGHS model. The derivation of the second law for the CGHS model can be applied to the present case by simply replacing $e^{-2\phi}$ by Ω. So in any of the eternal black holes, the entropy (27) can only increase on the future global event horizon, as well as on the apparent horizon. Note that the apparent horizon can be defined by $\partial_+\Omega = 0$ (see below) [9, 10].

The evolution of the classical black hole entropy in these solutions might also be considered. One has $\partial_a\Omega = \Omega'\partial_a\phi = -\frac{1}{4}\Omega'e^{2\phi}\partial_a S_0$ where $\Omega' = \frac{\partial\Omega}{\partial\phi} = \frac{1}{2} - \frac{24}{N}e^{-2\phi}$. Now in the physical region of interest (i.e., $\phi_{\text{cr}} \geq \phi > -\infty$) $\Omega' < 0$, and so the prefactor $(-\frac{1}{4}\Omega'e^{2\phi})$ is nonnegative, vanishing only at $\phi = \phi_{\text{cr}}$. Hence given that Ω never decreases, it must then also be true that the classical entropy S_0 increases on both global and apparent event horizons.

IV. EVAPORATING BLACK HOLES

Since the RST model provides a full semiclassical picture of black hole physics, one can also describe evaporating black holes (i.e., black holes without an external heat bath). In the Kruskal gauge, these are given by

$$\chi = \Omega = -\lambda^2 x^+ x^- - x^+\widetilde{P}_+(x^+) + \widetilde{\Delta}_+(x^+)$$
$$+m_0 - \frac{1}{4}\ln[-4\lambda^2 x^+ x^-] \qquad (28)$$

in a region where there is only infalling matter. Now the (global or apparent) horizon will originate at an early time on a timelike portion of the $\Omega = \Omega_{\text{cr}}$ boundary. Since Ω_{cr} is the minimum value for Ω, the quantity in

(27) must begin by increasing as the horizon pulls away from the $\Omega = \Omega_{\text{cr}}$ boundary. The final moment at which the black hole has completely evaporated is distinguished as the point where the horizon returns to the $\Omega = \Omega_{\text{cr}}$ boundary, which turns there from being spacelike to timelike. Hence, late in the evolution of the black hole, the function in (27) must be decreasing as Ω returns to Ω_{cr}. While it may seem disappointing that S_{RST} in (27) is sometimes decreasing, in fact it is not the entropy for these evaporating black holes. The reason is that Eq. (27) was derived using the equations of motion (25), whereas the evaporating solutions satisfy different equations of motion, due to a difference in the vacuum state of the quantum fields.

To produce evaporating black hole solutions, the constraints (25) are replaced by

$$\partial_{\pm}^2\Omega = -\widetilde{T}_{\pm\pm} - t_{\pm}, \qquad (29)$$

where $t_{\pm}$ are quantum corrections to the vacuum energy. For the solutions (28), $t_{\pm} = -\frac{1}{4x^{\pm 2}}$. The origin of this term in the semiclassical equations can be understood arising from the anomalous transformation properties of the normal ordered stress-energy tensor [10, 33].

One can also understand these contributions as arising from properly defining the scalar Green's function for calculations in a particular vacuum. Recall that in conformal gauge, the D'Alembertian $\nabla^2 = -4e^{-\rho}\partial_+\partial_-$ has a family of zero modes of the form $w_+(x^+)$ and $w_-(x^-)$. These will then give rise to ambiguities in the definition of the Green's function, which must be resolved by choosing appropriate boundary conditions. The relevance of this ambiguity here is that the above calculations used

$$\int d^2y\sqrt{-g(y)}\,G(x,y)R(y) = \rho(x), \qquad (30)$$

where it was assumed that $\rho = \rho_K$, the conformal factor for the Kruskal gauge metric. In fact if the vacuum was defined with repect to time for another choice of coordinates $\sigma^{\pm}$, one should have $\rho = \rho_0$, the conformal factor for the corresponding vacuum metric. Recall that for $x^{\pm} = h^{\pm}(\sigma^{\pm})$, one has $\rho_0 = \rho_K + \omega_+(x^+) + \omega_-(x^-)$ where $\omega_{\pm} = \frac{1}{2}\ln(\partial_{\sigma^{\pm}}h^{\pm})$. Hence the difference between the conformal factors is precisely in the zero mode sector. The end result is that one should set $\rho = \rho_0$ in the final entropy (21).

One proceeds by defining χ as in Eq. (22) with $\rho = \rho_K$, and choosing the Kruskal gauge $\chi = \Omega$, as before. Then when Eq. (30) yields $\rho = \rho_0 = \rho_K + \omega_+(x^+) + \omega_-(x^-)$, the constraints (25) are modified to those in Eq. (29) with

$$t_{\pm} = (\partial_{\pm}\omega_{\pm})^2 + \partial_{\pm}^2\omega_{\pm} \ . \qquad (31)$$

The entropy contribution from the nonlocal action (19) is also modified to

$$S_1 = \frac{N}{6}\rho_0(x_H^+, x_H^-)$$

$$= -\frac{N}{6}[\rho_K(x_H^+, x_H^-) + \omega_+(x_H^+) + \omega_-(x_H^-)] \ .$$

Thus the total entropy becomes

$$S_{\text{RST}} = \frac{N}{6} \left(\frac{12}{N} e^{-2\phi} + \rho_K + \omega_+ + \omega_- \right.$$
$$\left. - \frac{\phi}{2} - \frac{1}{4}\ln\frac{N}{3} - \frac{1}{4} \right)$$
$$= \frac{N}{6} \left[\Omega - \Omega_{\text{cr}} + \omega_+ + \omega_- \right]. \tag{32}$$

In the case of evaporating black holes described by Eq. (28), the correct vacuum metric is that obtained for the linear dilaton background coordinates, $\pm\lambda x^\pm = e^{\pm\lambda\sigma^\pm}$. Hence

$$\omega_+ = \frac{1}{2}\ln(\lambda x^+), \quad \omega_- = \frac{1}{2}\ln(-\lambda x^-) \ , \tag{33}$$

yielding $t_\pm = -\frac{1}{4x^{\pm 2}}$ and

$$S_{\text{RST}} = \frac{N}{6} \left[\Omega - \Omega_{\text{cr}} + \frac{1}{2}\ln\left(-\lambda^2 x^+ x^-\right) \right]. \tag{34}$$

It is easily shown that the entropy (32) satisfies a second law, on a global event horizon. The constraint equation (29),

$$\partial_+^2 \Omega = -T_{++} - (\partial_+\omega_+)^2 - \partial_+^2\omega_+,$$

yields

$$\partial_+^2 S_{\text{RST}} = -\frac{N}{6} \left[\widetilde{T}_{++} + (\partial_+\omega_+)^2 \right], \tag{35}$$

where factor in brackets is positive definite. Integrating as before yields

$$\partial_+ S_{\text{RST}} = \partial_+ S_{\text{RST}}(x_F^+)$$
$$+ \frac{N}{6} \int_{x^+}^{x_F^+} dy^+ \left[\widetilde{T}_{++} + (\partial_+\omega_+)^2 \right] (y^+)$$

where x_F^+ is the value of x^+ at the end point of the black hole evaporation. Now at $x^+ = x_F^+$ which lies on the $\Omega = \Omega_{\text{cr}}$ boundary, one has $\partial_+\Omega = 0$ by the RST boundary condition (24), and so the sign of $\partial_+ S_{\text{RST}}(x_F^+)$ is determined entirely by $\partial_+\omega_+$ at that point. Assuming that $\partial_+\omega_+(x_F^+) \geq 0$, one has that S_{RST} can only increase along the global horizon. This condition certainly holds for the evaporating black holes where Eq. (33) applies.

The analysis of the evolution of the entropy on an apparent horizon is more complicated because in general it is difficult to determine the location of apparent horizon. Here the discussion will focus on the solutions given in Eq. (28) for which the entropy is given in Eq. (34). The apparent horizon is defined by $\partial_+\Omega = 0$ which yields

$$x_A^- = -\frac{1}{4\lambda^2 x^+} - \frac{1}{\lambda^2}\widetilde{P}_+.$$

The tangent to the horizon is $t^a\partial_a = \partial_+ + \frac{\partial x_A^-}{\partial x^+}\partial_- = \partial_+ + \left(\frac{1}{(2\lambda x^+)^2} - \frac{1}{\lambda^2}\widetilde{T}_{++} \right)\partial_-$. Then

$$t^a\partial_a S_{\text{RST}} = \frac{N}{6} \left[x^+ \widetilde{T}_{++} \left(1 - \frac{1}{4\lambda^2 x^+ x^-} \right) \right.$$
$$\left. + \frac{1}{4x^+} \left(1 + \frac{1}{4\lambda^2 x^+ x^-} \right) \right].$$

All the terms above are positive definite, except possibly the very last factor. Now the vacuum $\Omega = \Omega_{\text{cr}}$

boundary curve is given by $4\lambda^2 x^+ x^- = -1$, and infalling matter which always carries positive energy will always move boundary inside this curve. Thus one must have $1 + \frac{1}{4\lambda^2 x^+ x^-} > 0$, and hence $t^a\partial_a S_{\text{RST}} > 0$. Therefore the entropy always increases on the apparent horizon for these solutions (28), as well.

V. DISCUSSION

In this paper, the black hole entropy for the semiclassical action for RST model was derived using the techniques developed by Wald. Despite the nonlocal form of the semiclassical action, the Noether charge technique can be extended to derive the entropy. However, the result is itself nonlocal, of course. These semiclassical contributions account for the entropy in the Hawking radiation generated by black hole. I expect that this calculation producing black hole entropy contributions for nonlocal terms in the effective action will extend to higher dimensional theories as well. To satisfy this conjecture in general, one must extend the analysis of Ref. [26] to guarantee that the Noether current can always be written in terms of an exact differential form even for the nonlocal terms. Such entropy contributions will be important for theories including massless fields (e.g., photons, neutrinos, gravitons!), where the semiclassical effective action must have nonlocal terms to describe Hawking radiation. A feature of the calculation which may be particular to two dimensions is that the entropy reduces to a manifestly local expression evaluated at the horizon with an appropriate choice of gauge.

It is not suprising that a second law holds for the entropy in the classical CGHS model. In terms of the four-dimensional black hole, this entropy (9) corresponds to the horizon area. Hawking's area theorem [21] holds in the four-dimensional theory, and so ensures that the entropy will never decrease on the global event horizon, under the assumption that cosmic censorship holds. In the two-dimensional model, no cosmic censorship assumption is needed since the general solution (26) is known, or rather from the general solution one knows that cosmic censorship is valid for this theory. One should note that the two-dimensional solutions only correspond to a subset of the possible solutions in the four-dimensional theory.

In the semiclassical RST model, the fact that a second law holds confirms the validity of the interpretation of the Noether potential (20) as an entropy. It may seem unusual that the classical entropy also increases in the semiclassical theory, at least for the eternal black holes. This effect is due to the thermal equilibrium between the black hole and the heat bath. Even though the entropy (27) accounts for the entropy in the radiation, no new entropy is being generated because of the equilibrium condition. The latter is clear from the fact that on the apparent horizon, the entropy only changes when matter crosses the horizon into the black hole. One may expect that in this case the entropy in the radiation is infinite, and so that the semiclassical entropy (27) should diverge, which is clearly not the case. This apparent discrepancy occurs because this divergence would simply be a constant common to all (eternal) black holes, and hence would not affect the variations in the first law (6). In the

Noether charge method, which integrates these variations to determine S, this divergence would be an integration constant, which is naturally omitted. This fortunate circumstance relies on the fact that the black hole temperature is independent of the mass in these two-dimensional models.

In the case of evaporating black holes, the production of entropy in the Hawking radiation is crucial to ensure that a second law holds for the total entropy, even when the classical entropy decreases. Care must be taken to account for the proper quantum vacuum to evaluate the entropy (20) correctly. It is interesting that the vacuum stress-energy t_+ has two contributions (31), one positive definite and the other of indefinite sign. In establishing the second law, it is precisely the latter that is absorbed in the entropy leaving a positive definite "effective" stress-energy in Eq. (35). It is important to emphasize that the positivity of the stress-energy is always crucial in establishing the second law in all of the cases considered. This positivity provides some insight as to why one can expect a second law to apply for the semiclassical results. In any theory if the stress-energy satisfies a null energy condition, the second law follows immediately from the first law, at least for the special case of quasistationary processes [22]. Of course, the conclusions here apply beyond quasistationary situations.

One might refer to any of the above entropy increase theorems as an intrinsic second law, in that they refer to the increase of the black hole entropy, alone. Such a result is distinct from a generalized second law, which would require that the sum of the black hole entropy and that of the external matter interacting with the black hole always increases [11]. One might suppose that an intrinsic version of the second law will be a prerequisite for the generalized second law to hold. There exist arguments in favor of the generalized second law [34], but the results are less conclusive since they only apply to quasistationary processes.

Reference [10] proves the generalized second law in the RST model for a very broad class of processes. This analysis relies on finding a microphysical interpretation for the semiclassical corrections to the black hole entropy. In their derivation, S_1 arises as entanglement entropy from short-range correlations between fluctuations near the horizon. This point of view suggests that similar terms could always be formulated as a local expression, despite apparent nonlocal appearances, even in higher-dimensional theories.

However, the analysis of Ref. [10] also seems to point out a shortcoming in the present approach. These authors also find a further contribution which is required to account properly for long-range correlations. Up to an additive constant, this new term takes the form $\Delta S_L = \frac{N}{6}\ln\left[\ln(-4\lambda^2 x^+ x^-)\right]$. This term is essential to establish that the entropy increases for the generalized second law in situations where a black hole accretes a near critical flux of matter. One speculation on how ΔS_L may arise in the present analysis is that it may be found in a yet more careful examination of Eq. (30). The treatment of boundary terms has been lax for the integration by parts performed in this integral with conformal gauge.

One expects that these terms will be canceled by an integral of the boundary curvature, which should be included on the left-hand side of Eq. (30). (Ultimately, the latter is inherited from the conformal anomaly, which includes a surface term for manifolds with boundary.) The $\Omega = \Omega_{cr}$ boundary though plays a special role in these dilaton gravity models [8, 9, 32]. It may be that a consistent theory requires a modified boundary action for this surface, and that as a result the integration by parts in Eq. (30) produces a residual boundary term, which represents the extra long range entropy contribution. It is an apparent drawback of Wald's technique that no contribution such as ΔS_L arises directly. One may note that when ΔS_L is added to the present black hole entropy, it still satisfies an intrinsic second law.

Reference [10] also argues that the generalized second law can always be violated in special situations if one attempts to apply it to the global event horizon. Hence they conclude that one should formulate the second law on the apparent horizon instead. These violations only occur on short time scales. On longer time scales (e.g., the entropy differences between approximately stationary phases in the evolution of a black hole), the global horizon should serve as equally well as the apparent horizon in a second law, since the two surfaces should be almost the same. For the present two-dimensional models, the intrinsic version of the second law applies to either type of horizon. In higher dimensions, the second law is usually discussed in the context of the global horizon, although Ref. [29] has considered the laws of black hole mechanics for apparent horizons.

A related question is how to account for the semiclassical entropy after the black hole ceases to exist. At the final point in the existence of the black hole, the black hole entropy can be attributed entirely to the semiclassical contributions, which indicates it is entirely associated with the Hawking radiation. Now certainly this radiation does not disappear even after the black hole is completely evaporated. Thus one may consider whether or not there is a sensible way to consider the evolution of the entropy after the black hole vanishes. A natural candidate is to continue evaluating the total entropy (32) along the null ray which extends the global horizon to future null infinity. One finds quite generally that the entropy continues to increase along this surface. The dominant contribution though rapidly becomes the classical "area" term as the surface expands. Another natural surface to consider would be the $\Omega = \Omega_{cr}$ boundary. It would be interesting to consider the evolution of the entropy expression (32) along this surface, where the entire contribution would be in the vacuum correction terms. These speculations might also lead one to consider the behavior of the entropy expression (32) along an arbitrary (outgoing) light-like surface. Again under fairly general conditions, the entropy is found to increase. This increase may be expected since in free space such a surface is naturally expanding, which would increase both the classical "area" term as well as the semiclassical entanglement entropy. What would make this result far more interesting is if a version of the first law could also be devised on such an arbitrary surface.

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