

Temperature, energy, and heat capacity of asymptotically anti-de Sitter black holes

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We investigate the thermodynamical properties of black holes in (3+1)- and (2+1)-dimensional Einstein gravity with a negative cosmological constant. In each case, the thermodynamic internal energy is computed for a finite spatial region that contains the black hole. The temperature at the boundary of this region is defined by differentiating the energy with respect to entropy, and is equal to the product of the surface gravity (divided by 2π) and the Tolman redshift factor for temperature in a stationary gravitational field. We also compute the thermodynamic surface pressure and, in the case of the 2+1 black hole, show that the chemical potential conjugate to angular momentum is equal to the proper angular velocity of the black hole with respect to observers who are at rest in the stationary time slices. In 3+1 dimensions, a calculation of the heat capacity reveals the existence of a thermodynamically stable black hole solution and a negative heat capacity instanton. This result holds in the limit that the spatial boundary tends to infinity only if the cosmological constant is negative; if the cosmological constant vanishes, the stable black hole solution is lost. In 2+1 dimensions, a calculation of the heat capacity reveals the existence of a thermodynamically stable black hole solution, but no negative heat capacity instanton.

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I. INTRODUCTION

A variety of theoretical arguments indicate that black holes have thermodynamical properties. This thermal character is expected to hold for all black holes, yet much of the literature on black hole thermodynamics is restricted to the case of spacetimes that are asymptotically flat in spacelike directions. Since asymptotic flatness is not always an appropriate theoretical idealization, and is never satisfied in reality, it is important to develop a theoretical framework for the description of black hole thermodynamics that is divorced from the assumption of asymptotic flatness of the spacetime. This is one of the primary motivations behind the formalism developed in Refs. [1–8]. This approach to black hole thermodynamics can be applied to gravitational and matter fields within a bounded, finite spatial region, so the asymptotic behavior of the gravitational field becomes irrelevant. In this way, it is possible to treat black hole spacetimes that are asymptotically curved and black holes in spatially closed universes. Even for black hole spacetimes that are asymptotically flat, there are several advantages to be gained by working in a spatially finite region [8]. For example, with the temperature fixed at infinity, the heat capacity for a Schwarzschild black hole is negative [9] and the formal expression for the partition function is not logically consistent [10]. On the other hand, with the temperature fixed at a finite spatial boundary, the heat capacity is positive and there is no inconsistency in the black hole partition function [1].

In this paper we employ spatially finite boundary conditions to investigate the thermodynamical properties of

black holes in (3+1)- and (2+1)-dimensional Einstein gravity with a negative cosmological constant. In these cases, the spacetimes are not asymptotically flat but, rather, they are asymptotically anti-de Sitter. Previous works on the thermodynamics of asymptotically anti-de Sitter black holes lack a firm conceptual foundation, because the temperature is identified with $\kappa_H/(2\pi)$ where κ_H is the surface gravity of the black hole and the thermodynamic internal energy is identified with the conserved mass at infinity [11,12]. However, the surface gravity and the mass at infinity each depend on the normalization of a timelike Killing vector field, and in the absence of an asymptotically flat region there is no physically preferred choice. As we will show, $\kappa_H/(2\pi)$ equals the physical temperature of the black hole as measured by an observer who is situated not at infinity but at a spatial location where the (normalization-dependent) gravitational redshift factor equals unity. For asymptotically anti-de Sitter spacetimes, the physical temperature of a black hole or any hot object will redshift to zero at spatial infinity [13].

We begin in Sec. II with a review of Ref. [7]. This includes a definition of the total energy E of the gravitational field within a region of space with boundary B . In addition, conserved charges are defined whenever the history of the boundary B admits a Killing vector field. The charge associated with a rotational symmetry is the angular momentum J , and the charge associated with a timelike Killing vector field defines a conserved mass M . The energy E and the mass M are not the same—we discuss the distinction between them. We also show that the conserved charges as defined here, in the limit that

the boundary B is pushed to spatial infinity, agree with the Arnowitt-Deser-Misner (ADM) charges [14] that are defined through an analysis of the surface terms in the gravitational Hamiltonian [15]. A spatial stress tensor s^{ab} for the boundary B is also defined.

There is some freedom of choice in the definitions of E , M , and J , which is reflected in the presence of two arbitrary functions of the boundary geometry. This choice determines the “zero-point configuration,” that is, the gravitational canonical data for which E , M , and J vanish. The two arbitrary functions can be defined so that E , M , and J vanish for a stationary slice of anti-de Sitter spacetime, or for a stationary slice of flat spacetime. (For the 2+1 case, the zero point configuration can be chosen as a stationary slice of the zero-mass black hole solution.)

In Sec. III we compute the energy and spatial stress for the region of a static slice of Schwarzschild–anti-de Sitter spacetime within a spherical boundary B . We then identify E as the thermodynamic internal energy and the entropy \mathcal{S} as one-quarter the area of the black hole event horizon. The temperature T at the boundary B is defined by $\partial E/\partial \mathcal{S}$, and is equal to the product of $\kappa_H/(2\pi)$ and the redshift factor [13] for temperature in a stationary gravitational field. In particular, the temperature depends on the location of the boundary B , and correctly redshifts to zero in the limit $B \rightarrow \infty$. Moreover, the temperature is independent of the choice of a zero point configuration for the energy. We also find that the surface pressure \mathcal{P} , as defined by the derivative of E with respect to the area of the boundary B , is given by the trace of the spatial stress tensor s^{ab} .

Our results show that there are no Schwarzschild–anti-de Sitter black hole solutions with temperature at B less than some critical value T_0 , and there are two possible black hole solutions with a given temperature T greater than T_0 . Moreover, for $T > T_0$, the smaller of the two black holes has a negative heat capacity and the larger of the two black holes has a positive heat capacity. These results hold in the limit of a vanishing cosmological constant, and are interpreted as follows [1,11]. For low temperatures, the equilibrium states are described semiclassically by thermal gravitons propagating on flat or anti-de Sitter backgrounds. For high temperatures, the equilibrium states are classically approximated by the larger black hole with positive heat capacity. The Euclidean section of the smaller black hole is an instanton that dominates the semiclassical evaluation of the rate of nucleation of black holes [16] from flat or anti-de Sitter space.

We analyze solutions in the limit in which the boundary goes to infinity and the temperature is adjusted so that the black hole horizon size remains fixed. The results depend crucially on whether the cosmological constant is strictly negative or zero. If the cosmological constant is negative, then our results are qualitatively unchanged in this limit. In particular, solutions include both a large, thermodynamically stable black hole and a small black hole instanton. If the cosmological constant is zero, the large black hole is lost in the limit and only the small black hole instanton solution remains. This shows that

a black hole in infinite space can be thermodynamically stable if the cosmological constant is negative, but not if the cosmological constant is zero [11].

In Sec. III we also compute the conserved mass M for the Schwarzschild–anti-de Sitter black hole. (The angular momentum vanishes.) With an appropriate choice of zero point configuration, and in the limit $B \rightarrow \infty$, M is equal to the black hole mass parameter and $\partial M/\partial \mathcal{S}$ is equal to $\kappa_H/(2\pi)$. As mentioned above, $\kappa_H/(2\pi)$ is not the physical temperature at infinity. However, in the limit that the boundary tends to infinity and the black hole horizon size remains fixed, $\partial M/\partial(\kappa_H/2\pi)$ yields the correct expression for the heat capacity.

In Sec. IV we analyze the thermodynamical properties of the stationary black hole solution to (2+1)-dimensional Einstein gravity with a negative cosmological constant [12,17]. In this case, there are no black hole solutions in the limit that the cosmological constant vanishes. First we compute the energy, spatial stress, mass, and angular momentum for a stationary slice of the 2+1 black hole spacetime within an axially symmetric boundary. The entropy \mathcal{S} is twice the “area” (circumference) of the event horizon, and we again identify E as the thermodynamic internal energy. Just as for the Schwarzschild–anti-de Sitter black hole in 3+1 dimensions, the temperature at the boundary B is the product of $\kappa_H/(2\pi)$ with the redshift factor, and the surface pressure is given by the trace of the spatial stress. The chemical potential conjugate to angular momentum is defined by $\partial E/\partial J$, and is shown to be equal to the proper angular velocity of the black hole with respect to the Eulerian observers who are at rest in the stationary time slices.

Our results show that there is a unique black hole solution for each temperature T at the boundary B . Assuming the temperature is positive, then the heat capacity is positive and the black hole is thermodynamically stable. Unlike the case in 3+1 dimensions, there is no negative heat capacity instanton, and therefore no obvious mechanism to allow for the nucleation of black holes from anti-de Sitter space. These conclusions are qualitatively unchanged in the limit in which $B \rightarrow \infty$ and $T \rightarrow 0$ in such a way that the black hole size remains fixed.

For both the (3+1)-dimensional Schwarzschild–anti-de Sitter black hole and the (2+1)-dimensional black hole, we derive the temperature by identifying E with the internal energy and *assuming* the appropriate expression for entropy as a function of horizon size. In order to actually derive the entropy, it is necessary to perform a quantum (or at least semiclassical) calculation. One approach, which is based on path integral techniques [9,18], is to identify the entropy in the classical approximation with the “microcanonical action” for the Euclideanized (or complexified) black hole spacetime [8]. Even without the assumption of the appropriate form for the entropy, the calculations in this paper can be interpreted in terms of black hole mechanics.

II. REVIEW

We consider a spacetime manifold \mathcal{M} of dimension D which is topologically the product of a spacelike hypersurface and a real line interval, $\Sigma \times I$. The boundary of

Σ is denoted $\partial\Sigma = B$. The spacetime metric is $g_{\mu\nu}$ with the associated curvature tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ and derivative operator ∇_μ . The boundary of \mathcal{M} , $\partial\mathcal{M}$, consists of initial and final spacelike hypersurfaces t' and t'' , respectively, and a timelike hypersurface $\mathcal{T} = B \times I$ joining these. The

induced metric on the spacelike hypersurfaces t' and t'' is denoted by h_{ij} , and the induced metric on \mathcal{T} is denoted by¹ γ_{ij} .

The gravitational action appropriate for fixation of the metric on the boundary $\partial\mathcal{M}$ is [19]

$$S^1 = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{-g} (R - 2\Lambda) + \frac{1}{\kappa} \int_{t'}^{t''} d^{D-1} x \sqrt{h} K - \frac{1}{\kappa} \int_{\mathcal{T}} d^{D-1} x \sqrt{-\gamma} \Theta. \quad (2.1)$$

Here, κ is a coupling constant and Λ is an optional cosmological constant. For simplicity we have omitted matter contributions to the action. The symbol $\int_{t'}^{t''} d^{D-1} x$ denotes an integral over the boundary element t'' minus an integral over the boundary element t' . The function K is the trace of the extrinsic curvature K_{ij} for the bound-

ary elements t' and t'' , defined with respect to the future pointing unit normal. Likewise, Θ is the trace of the extrinsic curvature Θ_{ij} of the boundary element \mathcal{T} , defined with respect to the outward pointing unit normal.

Under variations of the metric, the action (2.1) varies according to

$$\delta S^1 = (\text{terms that vanish when the equations of motion hold}) + \int_{t'}^{t''} d^{D-1} x P^{ij} \delta h_{ij} + \int_{\mathcal{T}} d^{D-1} x \pi^{ij} \delta \gamma_{ij}. \quad (2.2)$$

The coefficient of δh_{ij} in the boundary terms at t' and t'' is, by definition, the gravitational momentum

$$P^{ij} = \frac{1}{2\kappa} \sqrt{h} (K h^{ij} - K^{ij}). \quad (2.3)$$

Likewise, the coefficient of $\delta \gamma_{ij}$ in the boundary term at \mathcal{T} is

$$\pi^{ij} = -\frac{1}{2\kappa} \sqrt{-\gamma} (\Theta \gamma^{ij} - \Theta^{ij}). \quad (2.4)$$

In addition to the terms displayed in Eq. (2.2), δS^1 includes an integral over the corner $t'' \cap \mathcal{T}$ (and an integral over $t' \cap \mathcal{T}$) whose integrand is proportional to the variation of the ‘‘angle’’ $\mathbf{u} \cdot \mathbf{n}$ between the unit normal \mathbf{u} to t'' (or t') and the unit normal \mathbf{n} to \mathcal{T} [20,21]. We will not need these terms in the analysis that follows.

The action S^1 yields the classical equations of motion when the induced metric on $\partial\mathcal{M}$ is held fixed in the variational principle. In general, the functional $S = S^1 - S^0$, where S^0 is a functional of the metric on $\partial\mathcal{M}$, also yields the classical equations of motion when the metric is fixed on $\partial\mathcal{M}$, since in that case δS^0 vanishes. For simplicity, we define S^0 to be a functional of γ_{ij} only. The variation δS differs from δS^1 of Eq. (2.2) only in that π^{ij} is replaced by $\pi^{ij} - (\delta S^0 / \delta \gamma_{ij})$.

Now foliate the boundary element \mathcal{T} into $(D-2)$ -dimensional hypersurfaces B with induced $(D-2)$ -metrics σ_{ab} . The $(D-1)$ -metric γ_{ij} can be written according to the familiar ADM decomposition [14] as

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab} (dx^a + V^a dt)(dx^b + V^b dt), \quad (2.5)$$

where N is the lapse function and V^a is the shift vector. The corresponding variation of γ_{ij} is [7]

$$\delta \gamma_{ij} = (-2u_i u_j / N) \delta N + (-2\sigma_{a(i} u_{j)}) / N \delta V^a + (\sigma_{(i}^a \sigma_{j)}) \delta \sigma_{ab}, \quad (2.6)$$

where u_i is the unit normal to the slices B and $\sigma_a^i = \delta_a^i$ projects covariant tensors from \mathcal{T} to the slices B . With this result, the contribution to the variation of S from the boundary element \mathcal{T} becomes

$$\begin{aligned} \delta S|_{\mathcal{T}} &= \int_{\mathcal{T}} d^{D-1} x [\pi^{ij} - (\delta S^0 / \delta \gamma_{ij})] \delta \gamma_{ij} \\ &= \int_{\mathcal{T}} d^{D-1} x \sqrt{\sigma} [-\varepsilon \delta N + j_a \delta V^a + (N/2) s^{ab} \delta \sigma_{ab}], \end{aligned} \quad (2.7)$$

where the coefficients of the varied fields are defined by

$$\varepsilon = \frac{2}{N\sqrt{\sigma}} u_i \pi^{ij} u_j + \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta N}, \quad (2.8)$$

$$j_a = -\frac{2}{N\sqrt{\sigma}} \sigma_{ai} \pi^{ij} u_j - \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta V^a}, \quad (2.9)$$

$$s^{ab} = \frac{2}{N\sqrt{\sigma}} \sigma_i^a \pi^{ij} \sigma_j^b - \frac{2}{N\sqrt{\sigma}} \frac{\delta S^0}{\delta \sigma_{ab}}. \quad (2.10)$$

The leading terms in Eqs. (2.8)–(2.10) can be rewritten in terms of the extrinsic curvature k_{ab} that is defined by parallel transporting the unit normal \mathbf{n} to \mathcal{T} across a $(D-2)$ -dimensional slice B . Thus, k_{ab} is the extrinsic curvature of B considered as the boundary $B = \partial\Sigma$ of a spacelike hypersurface Σ whose unit normal \mathbf{u} is orthogonal to \mathbf{n} . Also let P^{ij} denote the gravitational momen-

¹We use latin letters i, j, k, \dots , as indices both for tensors on \mathcal{T} and for tensors on a generic hypersurface Σ . The two uses of such indices can be distinguished by the context in which they occur.

tum for the hypersurfaces Σ that are “orthogonal” to \mathcal{T} , and let $a_\mu = u^\nu \nabla_\nu u_\mu$ denote the acceleration of the unit normal u_μ for this family of hypersurfaces. The resulting expressions are [7]

$$\varepsilon = \frac{1}{\kappa} k - \varepsilon_0, \quad (2.11)$$

$$j_i = \frac{-2}{\sqrt{h}} \sigma_{ij} P^{jk} n_k - (j_0)_i, \quad (2.12)$$

$$s^{ab} = \frac{1}{\kappa} [k^{ab} + (n_\mu a^\mu - k) \sigma^{ab}] - (s_0)^{ab}. \quad (2.13)$$

In these equations we have expressed the terms proportional to the functional derivatives of S^0 as ε_0 , $(j_0)_i$, and $(s_0)^{ab}$. Also note that the indices in Eq. (2.12) refer to the hypersurface Σ . Thus, $j_i = j_a \sigma_a^i$ where $\sigma_a^i = \delta_a^i$ projects tensors from Σ to B , and $\sigma_{ij} = \sigma_i^a \sigma_{aj}$.

We will assume that S^0 is a linear functional of the lapse N and shift V^a , so that ε_0 and $(j_0)_i$ are functionals of the two-metric σ_{ab} only [7]. This condition implies that S^0 is functionally homogeneous of degree 1 in the lapse and shift:

$$S^0 = \int_{\mathcal{T}} d^{D-1}x \left(N \frac{\delta S^0}{\delta N} + V^a \frac{\delta S^0}{\delta V^a} \right). \quad (2.14)$$

By varying this expression with respect to N , V^a , and σ_{ab} , we find

$$\begin{aligned} & \int_{\mathcal{T}} d^{D-1}x \frac{N\sqrt{\sigma}}{2} (s_0)^{ab} \delta \sigma_{ab} \\ &= \int_{\mathcal{T}} d^{D-1}x [-N\delta(\sqrt{\sigma}\varepsilon_0) + V^a \delta(\sqrt{\sigma}(j_0)_a)]. \end{aligned} \quad (2.15)$$

This relationship is useful for the determination of $(s_0)^{ab}$ when ε_0 and $(j_0)_a$ are directly given as functions of σ_{ab} and its derivatives. For later use, we note that the variations in σ_{ab} can be split into variations in the determinant and variations that preserve the determinant

$$\delta \sigma_{ab} = \frac{2}{d} \left(\frac{\sigma_{ab}}{\sqrt{\sigma}} \right) \delta \sqrt{\sigma} + (\sqrt{\sigma})^{2/d} \delta \left(\frac{\sigma_{ab}}{(\sqrt{\sigma})^{2/d}} \right), \quad (2.16)$$

where $d = D - 2$ is the dimension of B .

From its definition through Eq. (2.7), $-\sqrt{\sigma}\varepsilon$ is equal to the time rate of change of the action, where changes in time are controlled by the lapse function N on \mathcal{T} . Thus, ε is identified as an energy surface density for the system and the total quasilocol energy is defined by integration over a $(D - 2)$ -surface B [7]:

$$E = \int_B d^{D-2}x \sqrt{\sigma} \varepsilon. \quad (2.17)$$

We also refer to j_i as the momentum surface density and s^{ab} as the spatial stress [7].

When there is a Killing vector field ξ on the boundary \mathcal{T} , an associated conserved charge is defined by [7]

$$Q_\xi = \int_B d^{D-2}x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i. \quad (2.18)$$

If there is no matter stress-energy in the neighborhood of \mathcal{T} , Q_ξ is conserved in the sense that Q_ξ is independent of the particular surface B (within \mathcal{T}) that is chosen for its evaluation [7]. This property is not shared by the energy E . If the system contains a rotational symmetry given by a Killing vector field ζ on \mathcal{T} , the conserved charge is the angular momentum $J = Q_\zeta$. If the $(D - 2)$ -surface B is chosen to contain the orbits of ζ , then the angular momentum can be expressed as

$$J = \int_B d^{D-2}x \sqrt{\sigma} j_i \zeta^i. \quad (2.19)$$

This is the integral over B of the ϕ component of the momentum surface density j_i , where $\zeta^i = (\partial/\partial\phi)^i$.

If the Killing vector field ξ is timelike, then the negative of the corresponding charge (2.18) defines a conserved mass for the system, $M = -Q_\xi$. If the Killing vector field is also surface forming, then the mass can be evaluated on a surface B whose unit normal is proportional to ξ . In this case the conserved mass is given by

$$M = \int_B d^{D-2}x \sqrt{\sigma} N \varepsilon, \quad (2.20)$$

where N is the lapse function defined by $\xi = Nu$. If, in addition, ξ restricted to B has unit norm, $N = 1$, then the conserved mass M coincides with the energy (2.17) of the hypersurface Σ whose boundary is B . However, if ξ does not have unit norm at B , then the mass M will differ from the energy E . Moreover, the energy E evaluated on other slices of \mathcal{T} will not, in general, equal the conserved mass M . These distinctions between mass and energy are especially important for spacetimes that are asymptotically anti-de Sitter, since in that case the magnitude of the timelike Killing vector field diverges as it approaches infinity. Thus, the timelike Killing vector does not approach the unit normal to the (asymptotically) stationary time slices at spatial infinity, and the mass M and energy E do not coincide.

For asymptotically flat or asymptotically anti-de Sitter spacetimes, the ADM charges at infinity, as defined by an analysis of the surface terms in the Hamiltonian [14,15,22,23], coincide with (the negative of) the conserved charges (2.18). In order to verify this connection, note that the Hamiltonian derived from the action (2.1) is [7]

$$\begin{aligned} H &= \int_{\Sigma} d^{D-1}x (N\mathcal{H} + V^i \mathcal{H}_i) \\ &+ \int_B d^{D-2}x \sqrt{\sigma} (N\varepsilon - V^i j_i). \end{aligned} \quad (2.21)$$

When the constraints $\mathcal{H} = 0 = \mathcal{H}_i$ hold, the first integral in the Hamiltonian vanishes. Thus the energy (2.17) is the value of the Hamiltonian with $N = 1$ and $V^i = 0$ on the boundary B ; that is, the value of the Hamiltonian that generates a unit time translation on the boundary in the direction orthogonal to the hypersurface Σ . Now push the boundary B to infinity along an asymptotically stationary time slice Σ . The ADM charges are

defined by the value of the Hamiltonian that generates an evolution that asymptotically coincides with an asymptotic Killing vector ξ . Therefore each ADM charge is given by the boundary integral in the Hamiltonian (2.21), where N and V^i are chosen such that (asymptotically) $N\mathbf{u} + V^i(\partial/\partial x^i) = \xi$. On the other hand, setting ξ equal to $N\mathbf{u} + V^i(\partial/\partial x^i)$ in Eq. (2.18), we see that the conserved charge associated with the asymptotic Killing vector ξ is

$$Q_\xi = - \int_B d^{D-2}x \sqrt{\sigma} (N\varepsilon - V^i j_i). \quad (2.22)$$

This is the negative of the boundary term in the Hamiltonian H . Therefore, if B is taken to infinity, Q_ξ is the negative of the ADM charge associated with ξ . Specifically, the ADM mass and angular momentum agree with the mass M and angular momentum J of Eqs. (2.20) and (2.19), in the limit that B is taken to infinity.

It should be recognized that both the conserved charges (2.18) and the ADM charges depend on the normalization of the (asymptotic) Killing vector field. Thus, the charge associated with the Killing vector $c\xi$, where c is a constant, is equal to the product of c and the charge associated with ξ . A second and more subtle ambiguity in the charges arises because of the presence of the terms ε_0 and $(j_0)_i$ in the energy surface density and momentum surface density. These terms depend on the boundary metric σ_{ab} . In the standard ADM analysis at infinity, these terms are effectively chosen such that the mass and angular momentum vanish for a flat time slice of flat Minkowski spacetime if $\Lambda = 0$, or for a static time slice of anti-de Sitter spacetime if $\Lambda < 0$. (Alternatively, in 2+1 dimensions with $\Lambda < 0$, mass and angular momentum are conveniently chosen to vanish for a static time slice of the zero mass black hole solution.) In the asymptotically flat case, this choice of a zero point configuration for the mass and angular momentum is typically built into the analysis through the use of coordinate derivatives acting on the metric tensor components, where the coordinate system is asymptotically Cartesian.

III. SCHWARZSCHILD–ANTI–DE SITTER SPACETIMES

In this section we will consider the Schwarzschild–anti-de Sitter black hole solutions to (3+1)-dimensional Einstein gravity with a negative cosmological constant Λ . We will adopt units for which $\kappa = 8\pi$ (thus $G = 1$), and also set $\Lambda = -3/l^2$. The metric written in static spherical coordinates is

$$ds^2 = -N^2(r)dt^2 + f^{-2}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.1)$$

where

$$N^2(r) = f^2(r) = (1 - 2m/r + r^2/l^2). \quad (3.2)$$

Let Σ be the interior of a $t = \text{const}$ slice with two-boundary B specified by $r = R = \text{const}$. The term ε_0

in the energy surface density is a function of R , and the term $(j_0)_a$ in the momentum surface density is chosen to be zero. If variations in the metric σ_{ab} are restricted to variations in the radius R , then Eqs. (2.15) and (2.16) imply

$$\int_{\mathcal{T}} d^3x \frac{N}{2} \sigma_{ab}(s_0)^{ab} \delta\sqrt{\sigma} = - \int_{\mathcal{T}} d^3x N \frac{\partial(\sqrt{\sigma}\varepsilon_0)}{\partial\sqrt{\sigma}} \delta\sqrt{\sigma}. \quad (3.3)$$

It follows that

$$\sigma_{ab}(s_0)^{ab} = -2 \frac{\partial(R^2\varepsilon_0)}{\partial(R^2)}, \quad (3.4)$$

since N is constant on \mathcal{T} .

A straightforward calculation of the trace of the extrinsic curvature k_{ab} for the spherical boundary $r = R$ in Schwarzschild–anti–de Sitter spacetime yields

$$k = -\frac{2f(R)}{R} = -\frac{2}{R} \sqrt{1 - 2m/R + R^2/l^2}. \quad (3.5)$$

The acceleration of the unit normal u_μ satisfies

$$n_\mu a^\mu = \frac{f(R)N'(R)}{N(R)} = \frac{m/R + R^2/l^2}{R\sqrt{1 - 2m/R + R^2/l^2}}, \quad (3.6)$$

where the prime indicates a radial derivative. From these results, the energy surface density (2.11) is given by

$$\varepsilon = -\frac{1}{4\pi R} \sqrt{1 - 2m/R + R^2/l^2} - \varepsilon_0(R), \quad (3.7)$$

and the total energy (2.17) is

$$E = -R\sqrt{1 - 2m/R + R^2/l^2} - 4\pi R^2\varepsilon_0(R). \quad (3.8)$$

Also, the trace of the spatial stress (2.13) is given by

$$\sigma_{ab}s^{ab} = \frac{1}{4\pi R} \left(\frac{1 - m/R + 2R^2/l^2}{\sqrt{1 - 2m/R + R^2/l^2}} \right) + \frac{1}{R} \frac{\partial(R^2\varepsilon_0)}{\partial(R)}, \quad (3.9)$$

where Eq. (3.4) has been used. For the boundary $r = R$ of the $t = \text{const}$ slices of Schwarzschild–anti–de Sitter, the momentum surface density (2.12) vanishes.

If we choose

$$\varepsilon_0(R) = -\frac{1}{4\pi R} \sqrt{1 + R^2/l^2}, \quad (3.10)$$

then the energy surface density ε and the energy E vanish for anti-de Sitter spacetime ($m = 0, l$ finite). Also note that with this choice, as $R \rightarrow \infty$ the energy vanishes: $E \sim ml/R \rightarrow 0$. Another natural choice is $\varepsilon_0 = -1/(4\pi R)$. In that case ε and E vanish for flat spacetime ($m = 0, l \rightarrow \infty$), and $E \sim -R^2/l \rightarrow \infty$ as $R \rightarrow \infty$. The simplest and most convenient choice for ε_0 is $\varepsilon_0 = 0$. We will leave $\varepsilon_0(R)$ unspecified in the analysis that follows.

The entropy \mathcal{S} of any stationary black hole in (3+1)-dimensional Einstein gravity is one-quarter the area of

its event horizon. (This includes black holes that are distorted by stationary matter fields relative to the standard Kerr or Kerr–anti–de Sitter family.) This conclusion was first reached by Bekenstein [24] apart from an overall numerical factor, and is derived as a general result in Ref. [8]. Moreover, it is now well recognized [25] that black hole entropy depends only on the geometry of the horizon (in the classical approximation). This suggests that the expression for black hole entropy is independent of the asymptotic behavior of the gravitational field, or the presence of external matter fields. Thus, for the Schwarzschild–anti–de Sitter black hole (3.1), the entropy is $\mathcal{S} = \pi(r_H)^2$, where r_H satisfies

$$N^2(r_H) = 1 - 2m/r_H + (r_H)^2/l^2 = 0. \quad (3.11)$$

For a given m , there is a unique real solution for the event horizon r_H . The expressions (3.5)–(3.9) are real and physically meaningful only for $r_H < R$.

Now identify the energy (3.8) as the thermodynamic internal energy for the Schwarzschild–anti–de Sitter black hole spacetime within the boundary R ; view E as a function of the entropy $\mathcal{S} = \pi(r_H)^2$ and the boundary area $4\pi R^2$. The corresponding temperature is

$$T \equiv \left(\frac{\partial E}{\partial \mathcal{S}} \right) = \frac{1}{\sqrt{1 - 2m/R + R^2/l^2}} \left(\frac{1 + 3(r_H)^2/l^2}{4\pi r_H} \right). \quad (3.12)$$

The second factor in this expression is just $1/(2\pi)$ times the surface gravity κ_H of the black hole, where

$$\kappa_H^2 = -\frac{1}{2}(\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu) = (\partial_i N)h^{ij}(\partial_j N) \quad (\text{evaluated on the horizon}); \quad (3.13)$$

with χ being a Killing vector normal to the horizon. The first factor in Eq. (3.12) is the inverse of the lapse function $1/N = \sqrt{-g^{tt}}$ evaluated at R , and is the Tolman redshift factor for temperature in a stationary gravitational field [13]. Therefore, the temperature at R is the product of $\kappa_H/(2\pi)$ and the redshift factor:

$$T(R) = \frac{1}{2\pi} \frac{\kappa_H}{N(R)}. \quad (3.14)$$

For a given size black hole, the temperature T redshifts to zero as $R \rightarrow \infty$ (assuming the cosmological constant is nonzero). Note that although the surface gravity (3.13) depends on the scale of the coordinate t that labels the stationary time slices, the temperature (3.14) does not. Also observe that the temperature is independent of the choice of function ε_0 .

From the energy E we can also define a thermodynamic surface pressure by

$$\begin{aligned} \mathcal{P} &\equiv - \left(\frac{\partial E}{\partial(4\pi R^2)} \right) \\ &= \frac{1}{8\pi R} \left(\frac{1 - m/R + 2R^2/l^2}{\sqrt{1 - 2m/R + R^2/l^2}} \right) + \frac{\partial(R^2\varepsilon_0)}{\partial(R^2)}. \end{aligned} \quad (3.15)$$

This is precisely one half of the trace of the spatial stress tensor (3.9):

$$\mathcal{P} = \frac{1}{2}\sigma_{ab}s^{ab}. \quad (3.16)$$

The surface pressure does depend on the function ε_0 . The definitions for temperature and surface pressure are captured in the first law of thermodynamics, namely, $dE = T d\mathcal{S} - \mathcal{P}d(4\pi R^2)$.

The heat capacity at constant surface area $4\pi R^2$ is defined by

$$C_R \equiv \left(\frac{\partial E}{\partial T} \right), \quad (3.17)$$

where the energy E is expressed as a function of T and R . The energy (3.8) and temperature (3.12) can be expressed as functions of r_H and R by eliminating m through Eq. (3.11). Then the heat capacity can be written as

$$C_R = \left(\frac{\partial E}{\partial r_H} \right) \left(\frac{\partial T}{\partial r_H} \right)^{-1}. \quad (3.18)$$

It is straightforward to show that E is a monotonically increasing function of r_H for $0 < r_H < R$, so that $\partial E/\partial r_H$ is strictly positive. On the other hand, the temperature T is a positive function of r_H with $T \rightarrow \infty$ both as $r_H \rightarrow 0$ and as $r_H \rightarrow R$. It can be shown that in the range $0 < r_H < R$, T has a single extremum² which is a minimum T_0 . Therefore, Eq. (3.12) has no solution for r_H when $T < T_0$, and has two solutions for r_H when $T > T_0$. Physically, this means that there are no Schwarzschild–anti–de Sitter black hole solutions with temperature at $r = R$ less than T_0 . If the temperature at $r = R$ is fixed to a value less than T_0 then the system will be dominated by thermal radiation in an anti–de Sitter background. If the temperature at $r = R$ is fixed to a value greater than T_0 , there are two black hole solutions, a small black hole with $\partial T/\partial r_H < 0$ and a large black hole with $\partial T/\partial r_H > 0$. Since the sign of the heat capacity (3.18) coincides with the sign of $\partial T/\partial r_H$, only the larger of the two black holes is thermodynamically stable.

The preceding analysis is qualitatively unchanged in the limit of a vanishing cosmological constant, $\Lambda = 0$ ($l \rightarrow \infty$). Previous work [1,16] on the $\Lambda = 0$ case shows that the Euclidean section of the small black hole with $C_R < 0$ is an instanton that dominates the semiclassical evaluation of the rate of nucleation of black holes in a cavity of size R and temperature T . The large black hole with $C_R > 0$ is the end result of the nucleation process.

Consider the heat capacity with the limit $R \rightarrow \infty$ taken in such a way that the black hole size r_H remains

² $\partial T/\partial r_H$ is equal to a positive function of r_H times a fifth order polynomial in r_H . This polynomial has precisely one positive root, because the constant term is negative and the signs of the coefficients of all higher powers of r_H are positive. The single root lies in the range $0 < r_H < R$ since the polynomial is positive at $r_H = R$.

fixed. If the cosmological constant is negative (l is finite), then the temperature T will go to zero in this limit; if the cosmological constant is zero ($l \rightarrow \infty$), then the temperature will go to $1/(4\pi r_H)$. In general, the result for the heat capacity is

$$\lim_{R \rightarrow \infty} C_R = -2\pi r_H^2 \left(\frac{1 + 3r_H^2/l^2}{1 - 3r_H^2/l^2} \right). \quad (3.19)$$

Now, for large R , the minimum T_0 of the function $T(r_H)$ occurs at $r_H = l/\sqrt{3}$. If l is finite, then Eq. (3.19) confirms that for the large black hole ($r_H > l/\sqrt{3}$) the heat capacity is positive, while for the small black hole ($r_H < l/\sqrt{3}$) the heat capacity is negative. However, if the cosmological constant vanishes ($l \rightarrow \infty$), then the minimum T_0 becomes infinite. In this case the horizon size for the large black hole must become infinite as $R \rightarrow \infty$. Therefore it is not possible to take the limit $R \rightarrow \infty$ while keeping the large black hole size fixed. When the cosmological constant vanishes, only the small black hole instanton can survive the $R \rightarrow \infty$ limit. Correspondingly, for $l \rightarrow \infty$ the heat capacity (3.19) is strictly negative.

The above results indicate that a black hole in infinite space ($R \rightarrow \infty$) can be in thermal equilibrium if the cosmological constant is negative, but not if the cosmological constant is zero. The same conclusion has been reached by Hawking and Page [11].

Finally, consider the conserved charges associated with the Schwarzschild–anti-de Sitter black hole. Since the momentum surface density vanishes, there is no angular momentum (2.19) as expected. The conserved mass (2.20) differs from the energy E by a factor of the lapse function at $r = R$:

$$\begin{aligned} M &= N(R)E \\ &= -R(1 - 2m/R + R^2/l^2) \\ &\quad - 4\pi R^2 \sqrt{1 - 2m/R + R^2/l^2} \varepsilon_0(R). \end{aligned} \quad (3.20)$$

We can view M as a function of the entropy $\mathcal{S} = \pi(r_H)^2$ and the boundary size R by expressing m in terms of r_H through Eq. (3.11). Then the mass varies with entropy according to the relationship

$$\frac{\partial M}{\partial \mathcal{S}} = N \frac{\partial E}{\partial \mathcal{S}} + E \frac{\partial N}{\partial \mathcal{S}} = \frac{\kappa_H}{2\pi} + \frac{E}{2\pi r_H} \frac{\partial N}{\partial r_H}, \quad (3.21)$$

where the result (3.12) and (3.14) has been used. Note that $\partial N/\partial r_H = -2\pi T r_H/R$. If $E(\partial N/\partial r_H)$ vanishes as $R \rightarrow \infty$, then in this limit $\partial M/\partial \mathcal{S}$ is the product of $1/(2\pi)$ and the surface gravity κ_H . This is indeed the case if we choose ε_0 as in Eq. (3.10), since then $E \rightarrow 0$ as $R \rightarrow \infty$. Moreover, with this choice for ε_0 , the conserved mass reduces to $M = m$ in the $R \rightarrow \infty$ limit. However, it is not correct to interpret $M = m$ as the thermodynamic internal energy and $\kappa_H/(2\pi)$ as the temperature at infinity—the physical temperature (3.14) redshifts to zero at infinity. In fact, $\kappa_H/(2\pi)$ is the temperature at the spatial location where the lapse function $N(R)$ (the inverse of the redshift factor) equals unity. Recall that the conserved mass (2.20) and the surface gravity (3.13)

depend on the choice of scale for the time coordinate t . Likewise, the location at which $N(R) = 1$ depends on the choice of time coordinate.

The heat capacity (3.17) can be expressed as

$$\begin{aligned} C_R &= \left(\frac{\partial E}{\partial T} \right) \\ &= \left(\frac{\partial(NE)}{\partial r_H} - E \frac{\partial N}{\partial r_H} \right) \left(\frac{\partial(NT)}{\partial r_H} - T \frac{\partial N}{\partial r_H} \right)^{-1}. \end{aligned} \quad (3.22)$$

If $E(\partial N/\partial r_H)$ and $T(\partial N/\partial r_H)$ vanish as $R \rightarrow \infty$, then in this limit the heat capacity becomes

$$\begin{aligned} \lim_{R \rightarrow \infty} C_R &= \lim_{R \rightarrow \infty} \left(\frac{\partial M}{\partial r_H} \right) \left(\frac{\partial(\kappa_H/2\pi)}{\partial r_H} \right)^{-1} \\ &= \left(\frac{\partial m}{\partial(\kappa_H/2\pi)} \right). \end{aligned} \quad (3.23)$$

Again, the term $E(\partial N/\partial r_H)$ can be dropped if ε_0 is chosen as in Eq. (3.10). Whether or not the term $T(\partial N/\partial r_H) = -2\pi T^2 r_H/R$ can be dropped as $R \rightarrow \infty$ depends on how the limit is taken. If r_H is held fixed in the limit $R \rightarrow \infty$, then $T(\partial N/\partial r_H)$ indeed vanishes (for both l finite and $l \rightarrow \infty$). Thus, with this definition for the limit, the expression $\partial m/\partial(\kappa_H/2\pi)$ correctly yields the heat capacity (3.19). On the other hand, if the temperature T is held fixed as $R \rightarrow \infty$, then $T(\partial N/\partial r_H)$ vanishes only for the small black hole. For the large black hole, $T(\partial N/\partial r_H)$ does not vanish because $r_H \rightarrow \infty$ as $R \rightarrow \infty$.

IV. (2+1)-DIMENSIONAL BLACK HOLE

We now consider (2+1)-dimensional Einstein gravity with a negative cosmological constant Λ . We will adopt units in which $\kappa = \pi$ and set $\Lambda = -1/l^2$. The axially symmetric black hole solution obtained by Bañados *et al.* [12,17] written in stationary coordinates is

$$ds^2 = -N^2(r)dt^2 + f^{-2}(r)dr^2 + r^2[V^\phi(r)dt + d\phi]^2, \quad (4.1)$$

where

$$N^2(r) = f^2(r) = -m + \left(\frac{r}{l}\right)^2 + \left(\frac{j}{2r}\right)^2 \quad (4.2a)$$

and

$$V^\phi(r) = -\frac{j}{2r^2}. \quad (4.2b)$$

An analysis of the Hamiltonian for (2+1) gravity shows that m and j are the ADM mass and angular momentum at infinity [17,23]. The mass parameter m also can be expressed in terms of the initial energy density of a disk of collapsing dust [26] in anti-de Sitter space or alterna-

tively in terms of Casimir invariants in a gauge-theoretic formulation (2+1)-dimensional general relativity [27]. As with the Kerr solution, the lapse function $N(r)$ for the (2+1)-black hole vanishes for two values of r , namely, r_+ and r_- , where

$$(r_{\pm})^2 = \frac{ml^2}{2} \pm \frac{l}{2} \sqrt{m^2 l^2 - j^2}. \quad (4.3)$$

The larger of these, r_+ , is specified as the black hole horizon. Such a horizon exists only for $m > 0$ and $|j| \leq ml$. (When $|j| = ml$, $r_+ = r_-$.)

Let Σ be the interior of a $t = \text{const}$ slice with boundary B specified by $r = R = \text{const}$. The term ε_0 in the energy surface density is a function of R , and the term $(j_0)_i$ in the momentum surface density is chosen to be zero. Equations (2.15) and (2.16) imply

$$\sigma_{ab}(s_0)^{ab} = -\frac{\partial(R\varepsilon_0)}{\partial R}, \quad (4.4)$$

since variations in the metric σ_{ab} consist of variations in the surface "area" $2\pi R$.

Straightforward calculations yield

$$k = -\frac{f(R)}{R} = -\frac{1}{R} \sqrt{-m + R^2/l^2 + j^2/(4R^2)} \quad (4.5)$$

for the trace of the extrinsic curvature of B and, as in Eq. (3.6),

$$n_{\mu} a^{\mu} = \frac{f(R)N'(R)}{N(R)} = \frac{R^2/l^2 - j^2/(4R^2)}{R\sqrt{-m + R^2/l^2 + j^2/(4R^2)}} \quad (4.6)$$

for the acceleration at B of the unit normal u_{μ} to the stationary time slices. From these results the energy surface density (2.11) is given by

$$\varepsilon = -\frac{1}{\pi R} \sqrt{-m + R^2/l^2 + j^2/(4R^2)} - \varepsilon_0(R), \quad (4.7)$$

and the total energy (2.17) is

$$E = -2\sqrt{-m + R^2/l^2 + j^2/(4R^2)} - 2\pi R \varepsilon_0(R). \quad (4.8)$$

The trace of the spatial stress (2.13) is

$$\sigma_{ab}s^{ab} = \frac{1}{\pi R} \left(\frac{R^2/l^2 - j^2/(4R^2)}{\sqrt{-m + R^2/l^2 + j^2/(4R^2)}} \right) + \frac{\partial(R\varepsilon_0)}{\partial R}, \quad (4.9)$$

where Eq. (4.4) has been used.

For the (2+1) black hole, the only nonzero component of the gravitational momentum (2.3) is $P^{r\phi} = -r f(r)(V^{\phi})'/[4\pi N(r)] = -j/(4\pi r^2)$. It follows that the momentum surface density (2.12) is

$$j_{\phi} = \frac{j}{2\pi R}, \quad (4.10)$$

and the total angular momentum (2.19) associated with the Killing vector field $\partial/\partial\phi$ is equal to the parameter j :

$$J = j. \quad (4.11)$$

This result is independent of the boundary size R . This is because the difference in J between two surfaces B_1 and B_2 of some slice Σ is given by the matter momentum density in the $\partial/\partial\phi$ direction, integrated over the region by Σ bounded by B_1 and B_2 [7]. Since the matter momentum density vanishes for the (2+1) black hole, the angular momentum J is the same for any surface B within the stationary slice Σ .

The energy E and angular momentum J will vanish for the zero mass black hole [the metric (4.1) with $m = 0, j = 0$] if we choose

$$\varepsilon_0(R) = -\frac{1}{\pi l}. \quad (4.12)$$

Another natural choice is $\varepsilon_0 = -\sqrt{1 + R^2/l^2}/(\pi R)$. In this case, E and J vanish for anti-de Sitter spacetime. With either of these choices, we find $E \sim ml/R$ for $R \gg l$ so the energy vanishes as $R \rightarrow \infty$.

The results of Ref. [12] show that the entropy \mathcal{S} of the (2+1) black hole (4.1) is $4\pi r_+$, twice the "area" of its event horizon. General arguments like those used in 3+1 dimensions show that $4\pi r_+$ is the entropy for any stationary black hole in (2+1)-dimensional Einstein gravity. We will accept this result and identify the energy E as the thermodynamic internal energy for the black hole spacetime within the spatial region bounded by $r = R$. Then the corresponding temperature is given by

$$\begin{aligned} T &\equiv \left(\frac{\partial E}{\partial \mathcal{S}} \right) \\ &= \frac{1}{\sqrt{-m + R^2/l^2 + (J/2R)^2}} \left(\frac{(r_+/l)^2 - (J/2r_+)^2}{2\pi r_+} \right). \end{aligned} \quad (4.13)$$

Here, E is treated as a function of entropy $\mathcal{S} = 4\pi r_+$, angular momentum J , and boundary "area" $2\pi R$ by solving Eq. (4.3) for m as function of r_+ and setting $j = J$. The first factor in this expression for T is the Tolman redshift factor $1/N(R)$ for temperature in a stationary gravitational field. The second factor is the product of $1/(2\pi)$ and the surface gravity (3.13). Thus, just as in 3+1 dimensions, the temperature at R is given by

$$T(R) = \frac{1}{2\pi} \frac{\kappa_H}{N(R)}. \quad (4.14)$$

For a given size black hole, the temperature redshifts to zero as $R \rightarrow \infty$.

The thermodynamic surface pressure defined by the energy E is equal to the trace of the spatial stress (4.9):

$$\mathcal{P} \equiv -\left(\frac{\partial E}{\partial(2\pi R)} \right) = \sigma_{ab}s^{ab}. \quad (4.15)$$

This result and the result (3.16) for (3+1)-dimensional black holes differ by a factor of one half. The difference stems from the fact that the surface pressure is defined by variations in E with respect to variations in the boundary metric σ_{ab} that preserve the conformally invariant part of

the metric $\sigma_{ab}/(\sqrt{\sigma})^{2/d}$. The factor of $2/d$ in Eq. (2.16) shows that in 2+1 dimensions ($d = 1$) the variations in $\sqrt{\sigma}$ have an extra factor of 2 relative to the variations in $\sqrt{\sigma}$ for 3+1 dimensions ($d = 2$).

Next, we compute the thermodynamic chemical potential conjugate to angular momentum J . It is defined by

$$\begin{aligned} \omega &\equiv \left(\frac{\partial E}{\partial J} \right) = \frac{J/(2r_+^2) - J/(2R^2)}{\sqrt{-m + R^2/l^2 + (J/2R)^2}} \\ &= \frac{V^\phi(R) - V^\phi(r_+)}{N(R)}. \end{aligned} \quad (4.16)$$

Observe that the angular velocity of the black hole horizon with respect to the spatial coordinate system is $-V^\phi(r_+)$. This can be verified by showing that the Killing vector field $\chi^\mu = (\partial/\partial t)^\mu - V^\phi(r_+)(\partial/\partial\phi)^\mu$ is null on the horizon [28]. By definition, the shift vector $V^\phi(R)$ is the angular velocity of the spatial coordinate system relative to the Eulerian observers at R whose four velocities are orthogonal to the stationary time slices $t = \text{const}$. Therefore, $V^\phi(R) - V^\phi(r_+)$ is the angular velocity of the black hole with respect to the Eulerian observers at R . This is an improper angular velocity, in the sense that it is taken with respect to coordinate time t . But coordinate time t is related to the proper time of the Eulerian observers at $r = R$ by a factor of the lapse function $N(R)$. Therefore we see that the chemical potential ω of Eq. (4.16) is the proper angular velocity of the black hole with respect to the Eulerian observers at the boundary B of the system. This is the expected result [4,8].

The definitions for temperature, surface pressure, and chemical potential are captured in the first law of thermodynamics for the (2+1) black hole, namely, $dE = T dS + \omega dJ - \mathcal{P}d(2\pi R)$.

The heat capacity at constant surface ‘‘area’’ $2\pi R$ and constant angular momentum J is

$$C_{R,J} \equiv \left(\frac{\partial E}{\partial T} \right) = \left(\frac{\partial T}{\partial r_+} \right) \left(\frac{\partial T}{\partial r_+} \right)^{-1}, \quad (4.17)$$

where the energy (4.8) and temperature (4.13) are expressed as functions of r_+ , R , and J . The square root factor that appears in both E and T is the lapse function evaluated at R , which can be expressed in terms of r_+ as

$$N(R) = [-(J/2r_+)^2 - (r_+/l)^2 + (R/l)^2 + (J/2R)^2]^{1/2}. \quad (4.18)$$

It is straightforward to show that the derivative $\partial T/\partial r_+$ is positive, so that the temperature is a monotonically increasing function of r_+ . This means that there is a unique black hole with a given temperature $T(R)$ and a given angular momentum J . Moreover, note that for $R > r_+$, the inequalities $r_+ > r_+^2/R \geq ml^2/(2R) \geq |J|l/(2R)$ follow from the explicit form (4.3) of r_+ and the condition $|J| \leq ml$. Thus, we find that r_+ is limited to the range

$$R > r_+ > |J|l/(2R). \quad (4.19)$$

The lapse function (4.18) is real for r_+ in this range, and vanishes at the end points $r_+ = R$ and $r_+ = |J|l/(2R)$. It

follows that the temperature T increases monotonically from $-\infty$ at $r_+ = |J|l/(2R)$ to $+\infty$ at $r_+ = R$. The temperature vanishes for $r_+ = \sqrt{|J|l/2}$, so that $T > 0$ for $r_+ > \sqrt{|J|l/2}$. Now, a simple calculation shows that the derivative $\partial E/\partial r_+$ is positive for $r_+ > \sqrt{|J|l/2}$ and negative for $r_+ < \sqrt{|J|l/2}$. Therefore the sign of the heat capacity (4.17) is the same as the sign of the temperature. In conclusion, there is a unique black hole with positive temperature T at $r = R$ and angular momentum J , and this black hole is thermodynamically stable ($C_{R,J} > 0$). Unlike the case in 3+1 dimensions, for $T > 0$ there is no negative heat capacity instanton.

From Eq. (2.18) we find that the conserved mass associated with the Killing vector field $\xi = (\partial/\partial t)$ is equal to

$$M = -Q_\xi = N(R)E + \frac{J^2}{2R^2}. \quad (4.20)$$

Note that Eq. (2.20) cannot be used for this calculation, since ξ is not orthogonal to the hypersurface Σ . Consider M to be a function of the entropy $\mathcal{S} = 4\pi r_+$, angular momentum J , and boundary size R . If we choose the function ε_0 as in Eq. (4.12), then for a given size black hole E vanishes as $R \rightarrow \infty$. We then find, as in 3+1 dimensions, that $M \rightarrow m$ as $R \rightarrow \infty$ and

$$\lim_{R \rightarrow \infty} \left(\frac{\partial M}{\partial \mathcal{S}} \right) = \frac{\kappa_H}{2\pi}. \quad (4.21)$$

The heat capacity can be expressed as in Eq. (3.22) (with r_H replaced by r_+) and the term $E(\partial N/\partial r_+)$ can be dropped assuming ε_0 is chosen appropriately. The term $T(\partial N/\partial r_+) = -2\pi T^2$ can be dropped if $T = 0$. Thus we obtain

$$\begin{aligned} \lim_{R \rightarrow \infty} C_{R,J} &= \frac{\partial m}{\partial(\kappa_H/2\pi)} \\ &= 4\pi r_+ \left(\frac{(r_+/l)^2 - (J/2r_+)^2}{(r_+/l)^2 + 3(J/2r_+)^2} \right), \end{aligned} \quad (4.22)$$

where $T \rightarrow 0$ as $R \rightarrow \infty$ in such a way that the black hole size r_+ remains fixed. If r_+ satisfies $r_+ > \sqrt{|J|l/2}$, then Eq. (4.22) shows that the heat capacity is positive in the $T \rightarrow 0, R \rightarrow \infty$ limit. This agrees with the general result $C_{R,J} > 0$ for $T > 0$.

Note added. Spatially finite boundary conditions have been used previously by Zaslavskii [29] to study the thermodynamical properties of the (2+1) black hole.

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