Static charged black holes in (2 + 1)-dimensional dilaton gravity

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A one-parameter family of static charged black hole solutions in (2 + 1) dimensional general relativity minimally coupled to a dilaton $\phi \propto \ln(r/\beta)$ with a potential term $e^{b\phi}\Lambda$ is obtained. Their causal strutures are investigated, and the thermodynamical temperature and entropy are computed. One particular black hole in the family has the same thermodynamical properties as the Schwarzschild black hole in 3+1 dimensions. Solutions with cosmological horizons are also discussed. Finally, a class of black holes arising from the dilaton with a negative kinetic term (tachyon) is briefly discussed.

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I. INTRODUCTION

$(2+1)$-dimensional gravity continues to be a source of fascination for theorists, primarily because of the potential insight into quantum gravity that it offers. The lower dimensional setting affords a significant amount of technical simplification of the gravitational field equations, bringing into sharper focus conceptual issues that are often obscured in the more complicated $(3+1)$-dimensional case. However, such technical simplicity is not without consequence: for example, in the case of $(2 + 1)$-dimensional general relativity the metric outside of a matter source of finite spatial size is locally flat, and the mass affects the space-time only globally, seemingly implying that $2 + 1$ gravity will not admit black hole solutions.

Fortunately the actual situation is significantly more interesting. Recently, Banados *et al.* showed that by identifying certain points of $(2 + 1)$-dimensional anti–de Sitter space, one obtains a solution with almost all the usual features of a black hole [1]. This Banados-Teitelboim-Zanelli (BTZ) black hole has so far attracted much interest in its classical, thermodynamical, and quantum properties [2]. In particular, it is also a solution to a low energy string theory [3]. Furthermore, by taking the product of the $(1 + 1)$-dimensional string-theoretic black hole of Mandal, Sengupta, and Wadia (MSW) [4] with $\mathbf{S}^1$, another $(2+1)$-dimensional black hole solution to the string theory may be obtained (hereafter referred as $2+1$ MSW black hole)—if a product with $\mathbf{R}$ is taken instead, one gets a black string [5]. One can further conformally transform [5,6] this black hole such that the transformed metric is a solution to an Einstein-Maxwell-dilaton action with special values of the couplings. Since causal structures are preserved under conformal transformations, as long as the transformations are at least finite outside (and on) the event horizon, one may obtain in this manner another black hole solution to the nonvacuum Einstein field equations in 2+1 dimensions.

Motivated by the above, we consider in this paper black hole solutions to the Einstein-Maxwell-dilaton action

$$S = \int d^3x\sqrt{-g}\left(R - \frac{B}{2}(\nabla\phi)^2 - e^{-4a\phi}F_{\mu\nu}F^{\mu\nu} + 2e^{b\phi}\Lambda\right), \quad (1)$$

with arbitrary couplings Λ, a, b, and B, where R is the Ricci scalar, ϕ is the dilaton field, and $F_{\mu\nu}$ is the usual Maxwell field. We find that in addition to the aforementioned black holes in $2 + 1$ dimensions, there exists a one-parameter family of static nonasymptotically anti–de Sitter charged black hole solutions, as well as solutions which admit cosmological horizons. We shall still refer to Λ as the cosmological constant, although in the presence of a nontrivial dilaton, the space does not behave as either de Sitter ($\Lambda < 0$) or anti–de Sitter space ($\Lambda > 0$)—note that the sign of Λ differs from the conventional one. The constants a and b govern the coupling of ϕ to $F_{\mu\nu}$ and Λ, respectively. Without loss of generality, the parameter B in (1) can be set to 8 as usual. However, we shall leave B as an arbitrary parameter so as to more conveniently permit its continuation to negative values. The BTZ and MSW black holes are two of the solutions of the field equations which follow from (1), obtained for particular values of the couplings a and b. Of course Eq. (1) can be viewed as general relativity with an unusual matter Lagrangian. For example, one can easily see that the local energy density in the perfect fluid form is negative when ϕ =const and $\Lambda > 0$.

The corresponding action for the low energy string theory can be obtained by setting $B = 8$, $b = 4$, $a = 1$ and carrying out the conformal transformation [6]

$$g^S_{\mu\nu} = e^{\frac{4\phi}{(n-1)}}g^E_{\mu\nu}, \quad (2)$$

where S and E denote the string and Einstein metrics, respectively, with n the number of spatial dimensions. The corresponding string action is

$$S = \int d^3x\sqrt{-g^S}e^{-2\phi}\{R[g^S] + 4(\nabla\phi)^2 - F^2 + 2\Lambda\}. \quad (3)$$

A number of other special cases of the action (1) have previously been considered by a number of authors. Barrow *et al.* obtained the most general static and circularly symmetric solutions for $\Lambda = 0$, as well as a solution with a nonvanishing Λ, $B = 2$, $F_{\mu\nu} = 0$, $b = -\sqrt{2}$, and $a = 0$ [7]. They did not obtain any black hole solution. Burd and Barrow [8], as well as Muslimov [9], discussed the action (1) in the context of pure scalar field cosmology; several exact solutions were obtained for a $(2+1)$-dimensional spatially flat Friedmann-Robertson-Walker (FRW) model with $B = 2$ and $b < 0$. Shiraishi derived static multicentered solutions for the $\Lambda = 0$ case [10]. We will later show that his solutions actually correspond to those of massless charged particles. Note that Maki and Shiraishi considered a similar type of action in more than two spatial dimensions [6]. They found exact solutions for a configuration of multiple black holes. It is also worthwhile to point out that the action (1) describes a Liouville-type gravity [11]; in the absence of electromagnetism, the action is an extension of the usual Liouville action in curved spacetime to 2+1 dimensions. Exact black hole solutions to the lowest dimensional (i.e., $1+1$) Liouville-type gravity have recently been found [11].

The organization of this paper is as follows. In Sec. II we adopt a static and circularly symmetric ansatz and then write down and solve the field equations. In Sec. III, we consider the solutions with $B > 0$ (this corresponds to positive kinetic energy in flat space for the dilaton). The quasilocal mass [12] associated with these solutions is computed and a mass parameter identified. For positive quasilocal mass, we obtain a one-parameter family of static charged black hole solutions. We discuss their causal structures and in Sec. IV compute their relevent thermodynamic properties (i.e., Hawking temperature T_H and entropy S). We will see how a nontrivial dilaton alters the causal structures and thermodynamical properties with respect to the BTZ black hole. One black hole solution has thermodynamical properties which are the same as the $(3+1)$-dimensional Schwarzschild black hole; in both cases, $T_H \propto (\text{mass})^{-1}$. In Sec. V, we consider solutions with cosmological horizons, and in Sec. VI, we briefly investigate the case $B < 0$. In this latter case we find a solution corresponding to a massless charged black hole. We find no black hole solutions of positive quasilocal mass when $\Lambda \leq 0$. We summarize our work in a concluding section.

Our conventions are as by Wald [13]; we set the gravitational coupling constant G, which has a dimension of an inverse energy, equal to unity.

II. EXACT SOLUTIONS

Varying (1) with respect to the metric, Maxwell, and dilaton fields, respectively, yields (after some manipulation)

$$R_{\mu\nu} = \frac{B}{2}\nabla_\mu\phi\nabla_\nu\phi + e^{-4a\phi}(-g_{\mu\nu}F^2 + 2F_\mu{}^\alpha F_{\nu\alpha}) -2g_{\mu\nu}e^{b\phi}\Lambda, \tag{4}$$

$$\nabla^\mu(e^{-4a\phi}F_{\mu\nu}) = 0, \tag{5}$$

$$\frac{B}{2}(\nabla^\mu\nabla_\mu\phi) + 2ae^{-4a\phi}F^2 + be^{b\phi}\Lambda = 0. \tag{6}$$

We wish to find static, circularly symmetric solutions to these equations that have a regular horizon. In $2+1$ dimensions, the most general such metric has two degrees of freedom [14], and can be written in the form

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + H^2(r)d\theta^2. \tag{7}$$

This is different from the usual ansatz for a circularly symmetric metric, but turns out to simplify the calculations. For an electric point charge $F_{\mu\nu}$ has just one independent component, $F_{01} = -F_{10} = e^{4a\phi}f(r)$ (the magnetic field is a scalar in $2+1$ dimensions and will not be considered here). Adopting the metric (7), Eq. (5) implies that

$$f(r) = \frac{q}{H(r)}, \tag{8}$$

where q is an integration constant. Now, Eqs. (4)–(8) together yield

$$U'' + U'\frac{H'}{H} = 4e^{b\phi}\Lambda, \tag{9}$$

$$\frac{H''}{H} = -\frac{B}{2}(\phi')^2, \tag{10}$$

$$U'\frac{H'}{H} + U\frac{H''}{H} = 2\left(e^{b\phi}\Lambda - \frac{e^{4a\phi}q^2}{H^2}\right), \tag{11}$$

$$\frac{B}{2}\left[U\left(\phi'\frac{H'}{H} + \phi''\right) + U'\phi'\right] - \frac{4ae^{4a\phi}q^2}{H^2} + be^{b\phi}\Lambda = 0, \tag{12}$$

where a prime $= \frac{d}{dr}$, denoting the ordinary derivative with respect to r. The first one is the R_{tt} equation. R_{tt} and R_{rr} together yield Eq. (10) and $R_{\theta\theta}$ yields (11). The last equation is the local energy conservation equation, $\nabla^\mu T_{\mu\nu} = 0$. From Eq. (10) it is easy to see that $H^2 \propto r^2 \iff \phi = $const. Thus one generally cannot have $g_{tt} = \frac{-1}{g_{rr}}$ and $g_{\theta\theta} = r^2$ simultaneously in $2+1$ dimensions when one has a nontrivial solution for the dilaton.

For the BTZ charged black hole, one has $U(r) = -m + \Lambda r^2 - 2Q^2\ln(\frac{r}{r_0})$ [15], $H(r) = r$, and $\phi = a = b = 0$. It is easy to check that the field equations are satisfied. For the $2+1$ MSW black hole, one can verify that conformal transformation (2) [with $\phi = -\frac{1}{4}\ln(\frac{r}{\beta})$, $\beta = $const] yields the Einstein metric

$$ds^2 = -(8\Lambda\beta r - 2m\sqrt{r})dt^2 + \frac{dr^2}{(8\Lambda\beta r - 2m\sqrt{r})} + \gamma^2 r\, d\theta^2, \tag{13}$$

where m is the square root of the mass per unit length and γ is an integration constant with dimension $[L]^{\frac{1}{2}}$. Equation (13) is an exact solution to Eqs. (9)–(12) with $q = 0$, $b = 4$, and $B = 8$. If we perform the duality transformation to the corresponding string metric, a charged solution can be obtained [5]. In terms of Einstein metric, we have

$$ds^2 = -(8\Lambda\beta r - 2m\sqrt{r} + 8Q^2)dt^2 + \frac{dr^2}{(8\Lambda\beta r - 2m\sqrt{r} + 8Q^2)} + \gamma^2 r d\theta^2, \tag{14}$$

where Q is the charge. It can be verified that metric (14) is also an exact solution to Eqs. (9)–(12) when $Q^2 = \frac{q^2\beta}{\gamma^2}$, $b = 4a = 4$, and $B = 8$.

We see that when $H^2 = \gamma^2 r$ and $H^2 = r^2$ in the Einstein metric (7), one gets the MSW black hole (13) [or (14)] and the BTZ black hole, respectively. Consider, then, the ansatz

$$H^2 = \gamma^2 r^N. \tag{15}$$

Since r and H^2 have a dimension of $[L]$ and $[L]^2$, respectively, the dimension of γ is $[L]^{\frac{2-N}{2}}$. In addition, we further assume

$$\phi = k \ln\left(\frac{r}{\beta}\right), \tag{16}$$

where k is a real number. Inserting (15) and (16) in (9)–(12), we get the exact solutions

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + \gamma^2 r^N d\theta^2, \tag{17a}$$

with

$$U(r) = \left[Ar^{1-\frac{N}{2}} + \frac{8\Lambda\beta^{2-N}}{(3N-2)N}r^N + \frac{8Q^2}{N(2-N)}\right] \tag{17b}$$

and where

$$k = \pm\sqrt{\frac{N(2-N)}{2B}}, \tag{18}$$

$$4ak = bk = N - 2, \qquad 4a = b. \tag{19}$$

The constant of integration q in Eq. (8) is related to the charge Q via $Q^2 = \frac{q^2\beta^{2-N}}{\gamma^2}$, whereas A is (as we shall subsequently demonstrate) a constant of integration proportional to the quasilocal mass.

The solutions (17) depend on the dimensionless couplings a and b (or alternatively N) and on the integration constants A and γ. By performing the coordinate transformation $\gamma^2 r^N \to r^2$, (17) becomes

$$ds^2 = -\left(Ar^{\frac{2}{N}-1} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)dt^2 + \frac{4r^{\frac{4}{N}-2}dr^2}{N^2\gamma^{\frac{4}{N}}\left(Ar^{\frac{2}{N}-1} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)} + r^2 d\theta^2, \tag{20a}$$

where from now on r denotes the usual radial coordinate. We have absorbed $\gamma^{1-\frac{2}{N}}$ into the constant A and normalized $\beta^{2-N}\gamma^{-2}$ to 1. Now, $Q^2 = q^2$ and

$$\phi = \frac{2k}{N}\ln\left(\frac{r}{\beta(\gamma)}\right). \tag{20b}$$

Before proceeding further we note the following. First, it is obvious that as $r \to \infty$, $\phi \to \infty$ too. However, the kinetic term, Maxwell term, and the potential term in action (1) all vanish in that limit when $2 \geq N > 0$ (i.e., $B > 0$). On the other hand, when $B < 0$, they all diverge. Second, for nonvanishing Q, when $N \to 2$ (or equivalently, $a \to 0$ and $b \to 0$) the metric diverges and the present solution has no smooth connection to the $N = 2$ case, similar to the situation in Ref. [10]. [This can easily be seen as follows: one can write the charged metric coefficient U as $U_{Q=0} + h(r)$; for $N = 2$, Eqs. (9)–(12) imply $h(r) = -2Q^2\ln(\frac{r}{r_0})$. When $N \neq 2$, the same set of equations forces $h(r)$ to be a constant, inversely proportional to $(2-N)N$. A smooth connection to $N = 2$ case is possible only when $Q = 0$.] Third, when $N = 1$, (20a) reduces to the $2+1$ MSW charged black hole. Fourth, if $N = 2$, $Q = 0$, and $\Lambda = 0$, one obtains the usual vacuum one particle solution in $2+1$ dimensions [16]. Fifth, if both Λ and A vanish, (20a) reduces to the Shiriashi single-particle solution [10] in Schwarzschild form. As we will later show that A is linearly proportional to the quasilocal mass, we see that Shiriashi's solution in fact describes a static massless charged particle. Sixth, when $Q = A = 0$ (the vacuum solution), the metric (20a) does not have an infinitely long throat for small r except $N = 2$ (BTZ case). Also, (20a) fails to fulfill the falloff conditions given in [17] for asymptotically anti–de Sitter spaces. Finally, as mentioned previously, the action (1) is related to the string action (3) by a conformal transformation (2) if $b = 4a = 4$ and $B = 8$. From (18) and (19) we see that this choice of parameters forces $N = 1$. In addition, we have mentioned earlier that for the uncharged $N = 2$ (BTZ) black hole it is a solution to string theory (with an addition of a three form $H_{\rho\mu\nu}$; see Ref. [3] for details). If N differs from 1 or 2, the solution (20) "decouples" from string theory.

III. BLACK HOLE SOLUTIONS FOR $2 > N > \frac{2}{3}$, $\Lambda > 0$, $A < 0$

In this section we seek to determine under what circumstances the set of solutions (20a) has black hole event horizons. The location of the horizon(s) will be given by the real positive roots of $g_{tt} = 0$. However, there exists no general method to obtain roots for a general value of N. In this section, we will restrict ourselves to the case where both B and Λ are positive; the former condition implies $2 > N > 0$ [$N \neq \frac{2}{3}$, see (18) and (20a)]. We first investigate the case $2 > N > \frac{2}{3}$. The other case, $\frac{2}{3} > N > 0$, will be shown in Sec. V, to have cosmological horizons.

We first determine the quasilocal mass associated with the solutions (20a) when $2 > N > \frac{2}{3}$. The spacetime manifold $\mathcal{M}$ for these solutions is topologically the product of a two-dimensional (2D) spacelike hypersurface and a real line interval, $\Sigma \times I$, with the boundary of the former denoted by $\partial\Sigma = B$. The boundary of $\mathcal{M}$, $\partial\mathcal{M}$ consists of initial and final spacelike hypersurfaces t' and t'' (whose induced metrics are denoted by h_{ij}) respectively, and a timelike hypersurface $\mathcal{B} = B \times I$ joining these (whose induced metric is denoted by γ_{ij}). Foliating the boundary element $\mathcal{B}$ into one-dimensional hypersurfaces B with induced one-metrics σ_{ab}, the 2D-metric γ_{ij} can be written as

$$\gamma_{ij}\, dx^i dx^j = -N^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt),$$

where N is the lapse function and V^a is the shift vector. For the solutions (20a), $V^a = 0$. Using this decomposition it may be shown

$$E = \int_B d^2x \sqrt{\sigma}\varepsilon,$$

where σ is the determinant of σ_{ab} and ε is the energy density associated with the two-surface B for the system. E is the total quasilocal energy defined by integration over B [12]. There is also a Killing vector field ξ^i on the boundary $\mathcal{B}$ for the solutions (20a); in this case an associated conserved charge may be defined as

$$Q_\xi = \int_B d^2x \sqrt{\sigma}\varepsilon u^i \xi_i,$$

where u^i is the unit normal to the slices B. Q_ξ is conserved in the sense that Q_ξ is independent of the particular surface B (within $\mathcal{B}$) that is chosen for its evaluation (provided there is no additional matter stress energy in the neighborhood of $\mathcal{B}$). This property is not shared by the energy E. If ξ^i is a timelike Killing vector, Q_ξ is the quasilocal mass M of the system.

Comparing the decomposition of γ_{ij} to a static metric in $2+1$ dimensions of the form

$$ds^2 = -W^2(r)dt^2 + \frac{dr^2}{V^2(r)} + r^2 d\theta^2, \tag{21}$$

the quasilocal energy E and mass M are then, respectively, given by [12]

$$E = 2[V_0(r) - V(r)] \tag{22}$$

and

$$M = 2W(r)[V_0(r) - V(r)], \tag{23}$$

provided the matter action contains no derivatives of the metric. Here $V_0(r)$ is an arbitrary function which determines the zero of energy relative to some background spacetime and r is the radius of the spacelike hypersurface boundry; we will choose $V_0(r) = 1/g_{rr}(r; A = 0)$ for g_{rr} given by (20a). For $2 > N > \frac{2}{3}$, Eqs. (20a) and (23) yield

$$M = N\gamma^{\frac{2}{N}} r^{1-\frac{2}{N}} \left(Ar^{\frac{2}{N}-1} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)^{\frac{1}{2}} \left[\left(\frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)^{\frac{1}{2}} - \left(Ar^{\frac{2}{N}-1} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)^{\frac{1}{2}}\right]. \tag{24}$$

Since $2 > (\frac{2}{N} - 1)$ the upper bound of the radius r is infinity regardless of the sign of A. Taking this limit we find $M = -\frac{AN\gamma^{\frac{2}{N}}}{2}$. It is obvious that if $M > 0$ (positive mass), $A < 0$ and vice versa. For convenience, we absorb the constant $\gamma^{\frac{2}{N}}$ into M (so that M has dimension $[L]^{\frac{N-2}{N}}$). Thus one can identify

$$A = -\frac{2M}{N}. \tag{25}$$

Throughout this paper, we will restrict $M \geq 0$ ($A \leq 0$). The quasilocal energy can be calculated similarly. As $r \to \infty$, one can check that $E \to 0$ in (20a), similar to the BTZ black hole [12].

Substituting (25) into the metric (20a), the equation $-g_{tt} = 0$ gives

$$-\frac{2M}{N} r^{\frac{2}{N}-1} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N} = 0. \tag{26}$$

Note that with our choice of N, the middle and last terms are always positive. The first term is always negative. Event horizons exist only in the situations $2 > N > 2/3$ and $\Lambda > 0$.

The location(s) of the event horizon(s) in the coordinates (20a) will in general depend upon M, Λ, Q, and N. Before proceeding to this general case we consider some illuminating examples: there exist five rational numbers within the range $2 > N > \frac{2}{3}$ such closed form solutions to (26) may be obtained. They are $N = \frac{6}{5}, \frac{6}{7}, \frac{4}{3}, \frac{4}{5}$, and 1. The first two give a cubic equation to (26), the next two give a quartic one, and the last a quadratic. This latter case corresponds to the string-theoretic black hole discussed in the preceding section. We consider the other four in turn. For each of these the metric is given by Eq. (20a), with A as in Eq. (25).

Consider first the case $N = \frac{6}{5}$, for which the solutions to (26) are the zeros of the equation

$$y = r^2 - \frac{2M}{5\Lambda} r^{\frac{2}{3}} + \frac{2Q^2}{\Lambda}, \tag{27}$$

where we note that both $\frac{2M}{5\Lambda}$ and $\frac{2Q^2}{\Lambda}$ are positive. A straightforward graphical analysis shows that there are three possible cases for Eq. (27). The first case corresponds to $y_{\min} > 0$ which is equivalent to the condition $\frac{Q^2}{\Lambda} > (\frac{2M}{15\Lambda})^{\frac{3}{2}}$, for which (27) has no real positive root. There is no black hole: the charge Q is too large with respect to a given M and Λ, and the singularity is naked. The second case corresponds to $y_{\min} = 0$, which is equivalent to $\frac{Q^2}{\Lambda} = (\frac{2M}{15\Lambda})^{\frac{3}{2}}$, and has one real positive root which is located at $r_{\min} = (\frac{2M}{15\Lambda})^{\frac{3}{4}}$, corresponding to an extremal black hole. Finally, if $y_{\min} < 0$, then $\frac{Q^2}{\Lambda} < (\frac{2M}{15\Lambda})^{\frac{3}{2}}$ there are two real positive roots and the black hole space-time has both an outer and inner horizon. Generally, Eq. (27) has at most two real positive roots. The above situations are qualitatively the same as the Reissner-Nordström black hole in 3 + 1 dimensions. In general, the (positive) roots of (27) are given by

$$r = \left(-\sqrt{\frac{8M}{15\Lambda}}\cos\theta\right)^{\frac{3}{2}}, \quad \cos(3\theta) = \left(\frac{15\Lambda^{\frac{1}{3}}}{2M}\right)^{\frac{3}{2}} Q^2 \tag{28}$$

which define the location of the horizons. When $Q = 0$ [implying $\cos(3\theta) = 0$] the horizon is at $r_h = (\frac{2M}{5\Lambda})^{\frac{3}{4}}$; for the extremal case, $\cos(3\theta) = 1$ which implies $r_h = (\frac{2M}{15\Lambda})^{\frac{3}{4}}$. In general the location of the outer horizon is determined by the largest positive value of r in (28). For example in the special case that $Q^2\sqrt{\Lambda} = \frac{1}{\sqrt{2}}(\frac{2M}{15})^{\frac{3}{2}}$ the outer horizon is $r_+ = (\frac{15Q^2}{M})^{\frac{3}{2}}$ and the inner one is $r_- = [(\frac{15Q^2}{2M})(\sqrt{3}-1)]^{\frac{3}{2}}$.

We consider next the causal structure of the $N = \frac{6}{5}$ metric. Since Eq. (27) is cubic in $r^{\frac{2}{3}}$ and has at most two real positive roots, we proceed by first obtaining the causal structure in the uncharged case, and then deducing the structure for the $Q \neq 0$ space-time (a method similar to Ref. [18]). The conformal radial coordinate $r_* \equiv \int \sqrt{-\frac{g_{rr}}{g_{tt}}}dr$ for $N = \frac{6}{5}$ and $Q = 0$ is given by

$$r_* = \frac{3}{10\Lambda\alpha\gamma^{\frac{5}{3}}}\left[\ln\left(\frac{r^{\frac{1}{3}}-\alpha}{r^{\frac{1}{3}}+\alpha}\right) + 2\arctan\left(\frac{r^{\frac{1}{3}}}{\alpha}\right)\right], \tag{29}$$

where $\alpha = (\frac{2M}{5\Lambda})^{\frac{1}{4}} = r_h^{\frac{1}{3}}$. The advanced and retarded null coordinates are defined as $u = t - r_*$, $v = t + r_*$, as usual. Defining the Kruskal coordinates as

$$\mathcal{U} = -\frac{3}{5\Lambda\alpha\gamma^{\frac{5}{3}}}e^{-\frac{5\Lambda\alpha\gamma^{\frac{5}{3}}}{3}u}, \qquad \mathcal{V} = \frac{3}{5\Lambda\alpha\gamma^{\frac{5}{3}}}e^{\frac{5\Lambda\alpha\gamma^{\frac{5}{3}}}{3}v}, \tag{30}$$

the metric can be written as

$$ds^2 = -\left(\frac{5Mr^2}{3}\right)\left(\frac{1}{\alpha^4} - \frac{1}{r^{\frac{4}{3}}}\right)\left(\frac{r^{\frac{1}{3}}+\alpha}{r^{\frac{1}{3}}-\alpha}\right) \times e^{-2\arctan\left(\frac{r^{\frac{1}{3}}}{\alpha}\right)} d\mathcal{U}\, d\mathcal{V} \tag{31}$$

and

$$\mathcal{U}\mathcal{V} = -\left(\frac{3}{5\Lambda\alpha\gamma^{\frac{5}{3}}}\right)^2\left(\frac{r^{\frac{1}{3}}-\alpha}{r^{\frac{1}{3}}+\alpha}\right)e^{2\arctan(\frac{r^{\frac{1}{3}}}{\alpha})}. \tag{32}$$

The Penrose diagram is given in Fig. 1. As $r \to \infty$, $\mathcal{U}\mathcal{V} \to -(\frac{3}{5\Lambda\alpha\gamma^{\frac{5}{3}}})^2 e^{\pi}$, which corresponds to a vertical (i.e., a timelike) line in the Penrose diagram. The horizon is at $r = r_h = \alpha^3$, which implies $\mathcal{U}\mathcal{V} = 0$. On passing from $r > r_h$ to $r < r_h$, the metric (31) changes sign and one has to transform $\mathcal{U} \to \tilde{\mathcal{U}} = -\mathcal{U}$ so that the time direction is still vertical. Now, $\tilde{\mathcal{U}}\mathcal{V} = -(\frac{3}{5\Lambda\alpha\gamma^{\frac{5}{3}}})^2$ when $r = 0$, indicating that the singularity is timelike as well. It is lengthy but straightforward to check that the Ricci and Kretschmann scalars diverge at $r = 0$. Thus the space-time has a timelike scalar curvature singularity. (A similar causal structure has been found for a class of (1 + 1)-dimensional black holes in Ref. [19].) A test particle may travel along a future directed timelike curve from region I, passing through r_h to region II. Without hitting the singularity, it can reemerge from region II and enter another region I. In fact, one has an infinitum of regions I and II. When charge is added, the manifold splits into three different regions: region I ($r > r_+$), region II ($r_+ > r > r_-$), and region III ($r_- > r$). In this case, whenever one crosses the horizons r_+ or r_-, the space and time coordinates interchange roles and the singularity is spacelike. We can deduce that the charged black hole has a causal structure that looks like Fig. 2, and for the extremal case, it becomes Fig. 3. These causal structures are similar to some of those found in Ref. [20],

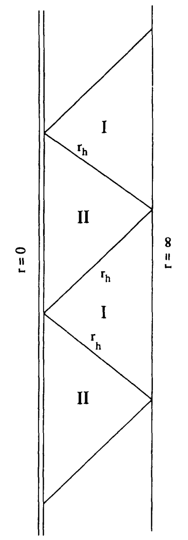

FIG. 1. Penrose diagram of the $N = 6/5$ and $4/3$ uncharged black holes. The double line indicates the curvature singularity.

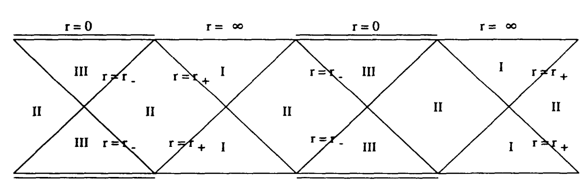

FIG. 2. Penrose diagram for the $N = 6/5$ and $4/3$ charged black holes (generic case).

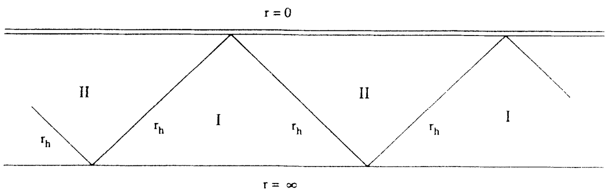

FIG. 3. Penrose diagram of the 6/5 and 4/3 charged black holes (extreme case).

where the authors investigated dimensionally continued Lovelock black holes with a cosmological constant $\Lambda > 0$ in both odd and even space-time dimensions. It is interesting to note that in their static charged black hole solutions in both odd and even dimensions, they always have a term Λr^2, similar to ours [see Eq. (20a)]. It is tempting to consider the same kind of dimensional continuation with the presence of a dilaton; however, we will not discuss it here.

We next turn to our attention to the case $N = \frac{6}{7}$. The analogue of (27) is now

$$y = r^2 - \frac{M}{7\Lambda} r^{\frac{4}{3}} + \frac{Q^2}{2\Lambda}. \tag{33}$$

As with the previous case, a graphical analysis indicates that there are three kinds of curves, corresponding to $\frac{Q^2}{\Lambda} > (\frac{2M}{21\Lambda})^3$ (a naked singularity), $\frac{Q^2}{\Lambda} = (\frac{2M}{21\Lambda})^3$ (an extremal black hole), and $\frac{Q^2}{\Lambda} < (\frac{2M}{21\Lambda})^3$ (a double-horizon black hole) respectively. The roots are implicitly given by

$$r = \left[\frac{M}{21\Lambda}(1 - 2\cos\theta)\right]^{\frac{3}{2}},$$

$$\cos(3\theta) = -1 + \left(\frac{21}{M}\right)^3 \left(\frac{Q\Lambda}{2}\right)^2. \tag{34}$$

It is easy to see that $Q = 0$ gives $r_h = (\frac{M}{7\Lambda})^{\frac{3}{2}}$. For the extremal case, one has $\cos(3\theta) = 1$, which yields a positive $r_h = (\frac{2M}{21\Lambda})^{\frac{3}{2}}$. As an intermediate example, if one sets $Q\Lambda = 2(\frac{M}{21})^{3/2}$, then the outer and inner horizons are $r_+ = (\frac{21Q^2}{4M})^{\frac{3}{4}}(\sqrt{3}+1)^{\frac{3}{2}}$ and $r_- = (\frac{21Q^2}{4M})^{\frac{3}{4}}$.

Following the coordinate transformations outlined above, we can obtain the causal structures. We only show the uncharged case (Fig. 4). It is similar to the one for an extremal Reissner-Nordström (RN) black hole in $3+1$ dimensions. There is a timelike singularity at which the Ricci and Kretschmann scalars diverge. The causal structure for generic charged black holes is obtained from the one for generic RN black holes by a $\frac{\pi}{2}$ rotation. For the extremal one, it is simply obtained through a $\frac{\pi}{2}$ rotation of Fig. 4. Note that those causal structures are similar to the ones in Ref. [20] for asymptotically flat charged Lovelock black holes in even dimensions.

Substituting $N = \frac{4}{3}, \frac{4}{5}$ into (26) yields two quartic equations. The analysis of the roots is similar to the previous two cases, and we find that there are either two

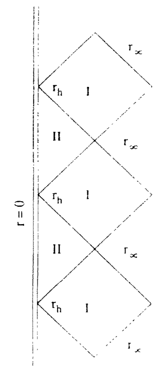

FIG. 4. Penrose diagram of the $N = 6/7$ and 4/5 uncharged black holes.

positive roots, one (the extremal case) or none. The $N = \frac{4}{3}$ black hole, which has the same causal structures as the $N = \frac{6}{5}$ one, has the extremal condition $\frac{Q^2}{\Lambda} = (\frac{M}{8\Lambda})^{\frac{4}{3}}$; the horizon is at $r_h = (\frac{M}{8\Lambda})^{\frac{2}{3}}$. For the $N = \frac{4}{5}$ one, which has the same causal structures as the $N = \frac{6}{7}$ black hole, the extremal condition is $\frac{Q^2}{\Lambda} = (\frac{3M}{40\Lambda})^4$, and the horizon is at $r_h = (\frac{3M}{40\Lambda})^2$, respectively.

For general N such that $2 > N > \frac{2}{3}$, a similar graphical analysis shows that we either get a black hole (generic or extremal) or a naked singularity. We rewrite (26) as

$$x^{2\lambda} - \lambda x^2 + \delta(\lambda - 1) = 0, \tag{35a}$$

where $\lambda = \frac{2-N}{2N}$, $x = \frac{r}{r_m}$, $\delta = \frac{Q^2}{\Lambda r_m^2}$ and

$$r_m = \left[\frac{(3N-2)(2-N)M}{8N\Lambda}\right]^{\frac{N}{3N-2}} \tag{35b}$$

The extremal condition is when $\delta = 1$, for which $r_h = r_m$ (i.e., $x = 1$). For a given M and Λ two horizons exist if $\delta < 1$, they coalesce if $\delta = 1$ and if $\delta > 1$ there is only a naked curvature singularity.

So far we have been addressing $\Lambda > 0$. When Λ flips sign or vanishes, one can see from metric (20a) that there is no event horizon for a positive mass. We also comment that higher order polynomials (e.g. cubic or quartic) in the variable in equation $g_{tt} = 0$ for a generic static black hole metric are not uncommon. For example, in the $(3+1)$-dimensional charged Schwarzschild–de Sitter metric, one has a quartic equation (see, e.g., [21]) whereas in $(1+1)$-dimensional string theory, higher loop corrections may lead to higher order polynomials in $g_{tt} = 0$ for a charged black hole [18] and multiple real and positive roots may be expected. [In (14) $g_{tt} = 0$ is a quadratic equation in r since it corresponds to the lowest loop-order string theoretic action (3).]

IV. THERMODYNAMICS

An important thermodynamical quantity in a static black hole is the Hawking temperature T_H. Given a static and circularly symmetric black hole metric, T_H is given by

$$T_H = \frac{|g'_{tt}|}{4\pi}\sqrt{-g^{tt}g^{rr}}|_{r=r_+} \tag{36a}$$

along with a blueshift factor which may be computed as in Ref. [12]. Using Eqs. (20a), (25), and (35) the Hawking temperature becomes

$$T_H = \frac{\lambda M \gamma^{\frac{2}{N}}}{2\pi r_+}\left[\left(\frac{r_+^2}{r_m^2}\right)^{1-\lambda} - 1\right] \tag{36b}$$

with r_+ the location of the (outer) event horizon. It is easy to check that for the uncharged BTZ black hole, $T_H = \frac{1}{2\pi}\sqrt{M\Lambda}$ as expected; for the uncharged MSW black hole, $T_H = \frac{\gamma^2\Lambda}{\pi}$, independent of mass.

Given the outer horizon r_+ as a function of M, Q, and Λ, one can use Eq. (36) to get the Hawking temperature in terms of these quantitites. For general N (including the four cases discussed above) r_+ is defined in terms of these quantities via (35): for a given λ and δ, one can numerically solve (35), obtaining $r_+ = x_0(\lambda,\delta)r_m$, where x_0 is the largest positive root of (35), yielding

$$T_H = \frac{\lambda M \gamma^{\frac{2}{N}}}{2\pi x_0(\lambda,\delta) r_m}\left[(x_0^2(\lambda,\delta))^{1-\lambda} - 1\right]. \tag{36c}$$

For extremal black holes, $x_0 = 1$ and $T_H = 0$. Thus they are stable end points of Hawking evaporation.

For $\delta = 0$ (i.e., $Q = 0$), (36c) becomes

$$T_H = \frac{\gamma^{\frac{2}{N}}\Lambda}{\pi N}\left(\frac{(3N-2)M}{4\Lambda}\right)^{\frac{2(N-1)}{3N-2}} . \tag{37}$$

Note that (1) describes Einstein-Hilbert action coupled to matter whose kinetic energy is quadratic; thus for those "dirty black holes" the entropy is still given by $S = 4\pi r_h$ [22]:

$$S = 4\pi r_h = 4\pi\left(\frac{(3N-2)M}{4\Lambda}\right)^{\frac{N}{3N-2}} . \tag{38}$$

Using this relation between S and r_h, it is easy to check that Eq. (37) can also be derived from the quasilocal energy E given by Eq. (22), except that the blueshift factor $\frac{1}{g_{tt}(r)}$ will be introduced in Eq. (37). From Eq. (38), one can also see that S can also be obtained through the equation $\frac{\partial S}{\partial M} = \frac{1}{T}$ of thermodynamics of event horizons in $2+1$ dimensions [23] and Eq. (37).

When $Q \neq 0$, the relation $S = 4\pi r_+$ still holds since the addition of a point electric charge in the matter action in (1) will not change the fact that the entropy is proportional to the area of the horizon [22]. Given such a relationship between S and r_+, following the argument in Ref. [23], one can loosely see that in $2+1$ dimensions, the area of an event horizon (generated by null geodesics) never decreases in time if $R_{\nu\mu}k^\nu k^\mu \geq 0$, where k^μ is the tangential vector field of the null geodesics in a congruence. Similar to the Reissner-Nordström and Schwarschild–de Sitter types space-times in Ref. [23], this condition is always satisfied in action (1) with an arbitrary dilaton and a point electric charge, as long as $B > 0$. Thus the entropy cannot decrease with time in all the black holes discussed above.

For $N = \frac{6}{5}$, $T_H \propto \gamma^{\frac{5}{3}}\Lambda^{\frac{3}{4}}M^{\frac{1}{4}}$ and $S \propto \Lambda^{-\frac{3}{4}}M^{\frac{3}{4}}$, whereas for $N = \frac{4}{3}$, $T_H \propto \gamma^{\frac{3}{2}}\Lambda^{\frac{2}{3}}M^{\frac{1}{3}}$ and $S \propto \Lambda^{-\frac{2}{3}}M^{\frac{2}{3}}$. We see that for these black holes their last breath is cold: i.e., $M = 0 \Rightarrow T_H = 0$. On the other hand, for $N = \frac{6}{7}$, $T_H \propto \gamma^{\frac{7}{3}}\Lambda^{\frac{3}{2}}M^{-\frac{1}{2}}$; therefore $S \propto \Lambda^{-\frac{3}{2}}M^{\frac{3}{2}}$, and the last breath is hot ($M = 0 \Rightarrow T_H \to \infty$). Finally (and perhaps most interestingly), for $N = \frac{4}{5}$, $T_H \propto \gamma^{\frac{5}{2}}\Lambda^2 M^{-1}$ and $S \propto \Lambda^{-2}M^2$. Thus this $N = \frac{4}{5}$ black hole has the same thermodynamic properties (apart from the blueshift factor) as the $(3+1)$-dimensional Schwarzschild one. We believe that this is the first example of this kind among lower dimensional black holes.

We close this section with some further comments on the temperature. In Ref. [20] it was noted that in $\mathcal{D} = 2n$ ($n \geq 2$) even space-time dimensions, Lovelock black holes radiate away with increasing temperature, while in $\mathcal{D} = 2n-1$ ($n \geq 2$) odd dimensions, they cool off as they radiate. In particular, the uncharged BTZ black hole (a special case in Ref. [20]) has vanishing temperature as it radiates away all its mass. In our dilatonic case with $\mathcal{D} = 3$ we have both situations, the former for $N > 1$ and the latter for $1 > N > 2/3$. So the additon of a dilaton indeed changes some of the generic thermodynamic properties of $(2+1)$-dimensional black holes.

V. COSMOLOGICAL HORIZONS FOR $2 > N > 0$, $\Lambda < 0$

Another class of space-time horizons of physical interest are the cosmological horizons. It is a well-known fact that for the de Sitter case, a cosmological horizon is present. However, as we mentioned earlier, when there is a nontrivial dilaton coupling, the case $\Lambda < 0$ no longer behaves as a de Sitter space. Therefore, it is interesting to see under what circumstances cosmological horizons may arise.

Similar to Secs. III and IV, we still demand that $B > 0$, or equivalently $2 > N > 0$ and the mass is positive. The existence of cosmological horizons is indicated by the fact that as the radial coordinate goes large enough (but still finite), the metric signature flips sign. The limit $r \to \infty$ cannot be taken to define quasilocal mass. However, in the case of small Λ and M [12], we can take the limits $r^{-\frac{2}{N}+1} \gg A$, $\Lambda r^2 \ll 1$ in Eq. (24) to identify the mass parameter in our solution. It is easy to see that the mass reduces to Eq. (25) and so we again identify $A = -\frac{2M}{N}$.

We first consider the "massless" case (i.e., $A = 0$). Now the metric is

$$ds^2 = -\left(\frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)dt^2 + \frac{4r^{\frac{4}{N}-2}dr^2}{N^2\gamma^{\frac{4}{N}}\left(\frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}\right)} + r^2 d\theta^2. \tag{39}$$

This metric will have a horizon whenever $\frac{\Lambda}{(3N-2)} <$

0, with the (cosomological) horizon located at $r_c = Q\sqrt{\frac{(2-3N)}{(2-N)\Lambda}}$. When $\Lambda \to 0$, $r_c \to \infty$ and we recover the Shiriashi solution [10].

We next consider the $\Lambda = 0$, $A \neq 0$ case. Now the metric is

$$ds^2 = -\left(-\frac{2M}{N}r^{\frac{2}{N}-1} + \frac{8Q^2}{(2-N)N}\right)dt^2 + \frac{4r^{\frac{2}{N}-1}dr^2}{N^2\gamma^{\frac{4}{N}}(-\frac{2M}{N}r^{\frac{2}{N}-1} + \frac{8Q^2}{(2-N)N})} + r^2 d\theta^2. \quad (40)$$

For positive M, there is always a "cosmological" horizon located at $r_c = (\frac{4Q^2}{(2-N)M})^{\frac{N}{2-N}}$.

Finally, for the neutral case, $Q = 0$, and $\frac{2}{3} > N > 0$, it is easy to show that the cosmological horizon is located at $r_c = (\frac{4\Lambda}{(3N-2)M})^{\frac{N}{2-3N}}$. As a matter of fact, we can generally consider all Λ, M, and Q are nonvanishing. For example, if $\frac{2}{3} > N > 0$ and $\Lambda < 0$, a simple graphical analysis shows that only one real positive root exists for $-g_{tt} = 0$ which corrresponds to r_c. We will not discuss general cases in detail. If the assumption $B > 0$ is relaxed, then further cosmological or event horizons may exist. In next section we briefly discuss the existence of black holes in such situations.

VI. BLACK HOLES FOR $N > 2$, $N < 0$, $A < 0$

So far we have been assuming $B > 0$ in our discussion on cosmological and event horizons. Physically this means that the kinetic energy of the dilaton (in flat space) is positive. In this section, we briefly point out that black holes can also arise if the dilaton acts as a tachyon field (i.e., $B < 0$, a negative kinetic energy). Although we can still identify $A = -\frac{2M}{N}$ as the quasilocal mass, these black holes have a number of rather unattractive and unphysical properties. Specifically, when the kinetic term is negative, the terms in the action (1) involving the kinetic, Maxwell and dilaton potential all diverge as $r \to \infty$.

It is trivial to see that when $B < 0$, one must have $N < 0$ or $N > 2$ [see Eq. (18)]. We will illustrate two cases as examples.

Suppose $N = -2$. Now $g_{tt} = 0$ yields

$$r^4 - \frac{2Q^2}{\Lambda}r^2 + \frac{2M}{\Lambda} = 0. \quad (41)$$

Obviously, the roots are given by ${r_\pm}^2 = \frac{Q^2}{\Lambda} \pm \frac{1}{2}\sqrt{\frac{4Q^4}{\Lambda^2} - \frac{8M}{\Lambda}}$ and the extremal condition is given by $Q^4 = 2M\Lambda$. One can further calculate the temperature from Eq. (36). T_H is zero in the extremal case. When $Q^4 < 2M\Lambda$, (i.e., for sufficiently large mass), the horizon disappears. In the pure charge case ($M = 0$), the event horizon is located at $r_h = Q\sqrt{\frac{2}{\Lambda}}$. The temperature of such a black hole has the property $T_H \propto Q^3\Lambda^{-\frac{1}{2}}$. As long as Q is a constant, T_H is nonvanishing, and it keeps radiating. The entropy S is still related to r_h through Eq. (38). Since $B < 0$, the area of the event horizon does not necessarily increase with time. Note that for typical black holes of charge Q and mass M, a naked singularity appears if the charge is too large with respect to the mass, whereas in this case the situation is reversed. It seems that the mass is playing the role of charge and vice versa.

In addition to this, one special black hole solution in the limit $N \to \infty$ can be derived as follows. When one takes the limit $N \to \infty$ (or $b \to \sqrt{-2B}$), the dilaton becomes $\phi = \frac{2}{b}\ln(\frac{r}{\beta})$. Recall that M in Eq. (25) is in fact $\frac{M}{\gamma^{\frac{2}{N}}}$. We demand that $\gamma^{\frac{2}{N}}N$ is finite in that limit and absorbed in M, and the charge Q is large comparable to N. Equation (20a) now becomes

$$ds^2 = -\left(-\frac{2M}{r} + \frac{8\Lambda r^2}{3} - 8Q^2\right)dt^2 + \frac{4r^{-2}dr^2}{(\frac{-2M}{r} + \frac{8\Lambda r^2}{3} - 8Q^2)} + r^2 d\theta^2, \quad (42)$$

where $-N(N-2)$ is absorbed into $8Q^2$ [in fact, it can be checked that (42) can be alternatively derived if a linear dilaton is assumed, instead of a logarithm, then a coordinate transformation is performed to get (42)]. In this limit, an event horizon exists as long as $\Lambda > 0$. For $Q = 0$, it is easily see that $T_H \to \Lambda^{\frac{1}{3}}M^{\frac{2}{3}}$. Thus $T_H = 0$ when $M = 0$.

VII. CONCLUSIONS

We have found a one-parameter ($2 \geq N > \frac{2}{3}$) family of static charged black hole solutions for Einstein gravity minimally coupled to a dilaton $\phi \propto \ln(\frac{r}{\beta})$ with an potential term $e^{b\phi}\Lambda$ for the (2+1)-dimensional action (1). Their causal structures, Hawking temperature, and entropy were investigated. One particular black hole ($N = \frac{4}{5}$) has the same thermodynamic behavior as the Schwarschild one. Solutions of cosmological horizons are also found. In the presence of a tachyon field, black holes are also obtained, which consist of a massless charged black hole as a particular case. There are no black hole solutions for positive quasilocal mass if $\Lambda \leq 0$.

One (perhaps unattractive) feature of the solutions (20a) is that they do not asymptotically approach anti–de Sitter space. This is because ϕ does not approach a constant at spacelike infinity. It would be interesting to investigate whether black holes exist for an asymptotically constant dilaton field. In this context we note that the only asymptotically flat black hole solutions to the Einstein-Hilbert action minimally coupled to a dilaton with a vanishing potential term, is the Schwarschild solution ($\phi = 0$) [24]. In $2+1$ dimensions, this "Schwarschild solution" is not a black hole at all but is instead locally flat space-time [16]. One should therefore not expect to obtain any asymptotically flat black hole solution in action (1). The simplest black hole in (1) is the BTZ one and it is nonasymptotically flat.

We close by commenting on further possible extensions of our work. Addition of angular momentum to our solutions is an obvious generalization. Dimensional reduction of the solutions obtained in this paper to get a new class of black holes in $1+1$ dimensions would be another interesting avenue of research. In the $\phi = 0$ case, this was done by Achucarro [25]. On the other hand, one may dimensionally continue our solutions in the context of Lovelock gravity in Ref. [20]. Regardless, it is always tempting to see how a nontrivial dilaton field alters the causal structure and thermodynamic properties of any possible black hole solution.

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