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Model of a gravitational wave in Schwarzschild space-time

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A model of a spherical impulsive gravitational wave propagating through the Schwarzschild field is constructed. The vacuum part of the space-time to the future of the history of the wave is calculated approximately as a Bondi-Sachs space-time, providing a model of the reaction of the Schwarzschild field to the passage of the wave through it and implying the existence of backscattered radiation due to the presence of the isolated matter distribution.

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I. INTRODUCTION

The study of gravitational radiation with wave fronts homeomorphic to a two-sphere (and thus having an isolated matter distribution as source) began with the pioneering papers of Robinson and Trautman [1,2], Bondi, van der Burg, and Metzner [3], Sachs [4], and Newman and Penrose [5]. A special case in which the spherical wave is an impulsive wave [and thus the field (Riemann tensor) has a δ-function singularity on the history of the wave] is described by Penrose's [6] exact solution of Einstein's vacuum field equations, when the wave is propagating through flat space-time. As a tentative model of an explosion involving emission of a gravitational wave from an isolated source we consider in this paper a spherical impulsive wave propagating through the Schwarzschild space-time. The Schwarzschild space-time must react to the wave because if one were to assume that the Schwarzschild space-time were left behind after the disturbance, then a calculation of the surface stress-energy tensor on the history of the disturbance, using the technique described in [7], reveals that the surface stress-energy tensor does not vanish. This means that a lightlike shell of matter and not a gravitational wave is propagating through space-time. The surface stress-energy tensor also does not vanish if the field behind the wave is described by a Robinson-Trautman [1,2] solution of Einstein's vacuum field equations. Thus the field behind the wave is unlikely to be a known *exact*, nonstationary space-time having an isolated matter distribution as a source.

If one considers the Kruskal extension of the Schwarzschild manifold then it is relatively straightforward to construct a model of an impulsive wave whose history is the past (or future) horizon of the white (or black) hole with the appropriate vacuum regions of the Kruskal space-time on either side of the chosen horizon. This emerges from the study of perturbations of white holes by Eardley [8] and later by Redmount [9]. The construction is straightforward because the horizons in question are *hyperplanes* (generated by shear-free, *expansion-free*, null geodesics) and therefore from the point of view of using them as histories of impulsive gravitational waves have more in common geometrically with plane impulsive waves than with spherical impulsive waves.

In this paper the reaction of the Schwarzschild field to a spherical gravitational impulsive wave propagating through it is studied. It will be convenient to write the Schwarzschild line element as (cf. [1,2])

$$ds^2 = -r^2 p_0^{-2}(dx^2 + dy^2) + 2\,du\,dr + \left(1 - \frac{2m}{r}\right) du^2, \tag{1.1}$$

where m is the (constant) Schwarzschild mass and

$$p_0 = 1 + \tfrac{1}{4}(x^2 + y^2). \tag{1.2}$$

Then the hypersurfaces u =const are future-directed null hypersurfaces generated by the null geodesics tangent to the vector field $\partial/\partial r$. These geodesics are shear-free and have expansion r^{-1}, where r is an affine parameter. We shall use one of them, $u = 0$, as the history of an impulsive wave. Thus in this sense the wave we consider has a spherical front. However the field of the wave will have a singularity along a generator of $u = 0$ (see Sec. III below) as well as having the δ-function singularity there and the expected singularity at $r = 0$. It is in this general sense that we will refer throughout this paper to a "spherical" wave (cf. [1]). The region of space-time to the past of the history of the wave will be denoted by $u < 0$ and its vacuum subregion will be taken to be the Schwarzschild space-time with line element (1.1). The vacuum part of the region of space-time to the future of the history of the wave will then correspond to $u > 0$. This latter region of space-time describes the gravitational field behind the wave and embodies the reaction of the Schwarzschild gravitational field to the passage of the wave through it. It is the description of this space-time region which is the main focus of attention in the present paper. We find that we are unable to describe it exactly. Neverthe-

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less the approximate model obtained in Sec. III, which is an example of a Bondi-Sachs [3,4] space-time, enables us to infer the existence of backscattered gravitational radiation due to the presence of the isolated matter distribution left behind after the wave. To this end a brief summary of the well-known Bondi-Sachs theory is given in a convenient formulation for our purpose in Sec. II. The model of the impulsive wave is then constructed in Sec. III, with some helpful calculations summarized in the Appendix, and the paper ends with a discussion of the model in Sec. IV.

II. THE BONDI-SACHS SPACE-TIMES

A convenient form of the line element of the Bondi-Sachs space-times is given by (see [10])

$$ds^2 = -(\theta^1)^2 - (\theta^2)^2 + 2\theta^3\theta^4, \tag{2.1}$$

with

$$\theta^1 = rp^{-1}(e^{\alpha}\cosh\beta\, dx + e^{-\alpha}\sinh\beta\, dy + a\, du), \tag{2.2a}$$
$$\theta^2 = rp^{-1}(e^{\alpha}\sinh\beta\, dx + e^{-\alpha}\cosh\beta\, dy + b\, du), \tag{2.2b}$$
$$\theta^3 = dr + \tfrac{1}{2}c\,du, \tag{2.2c}$$
$$\theta^4 = du. \tag{2.2d}$$

The six functions $\alpha, \beta, a, b, p, c$ depend on all coordinates x, y, r, u, and u =const are null hypersurfaces generated by the geodesic integral curves of the vector field $\partial/\partial r$ with r an affine parameter along the geodesics. The following assumptions are made regarding the r dependence of the six functions above:

$$\alpha = \frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \cdots, \tag{2.3a}$$
$$\beta = \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \cdots, \tag{2.3b}$$
$$a = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots, \tag{2.3c}$$
$$b = b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \cdots, \tag{2.3d}$$
$$c = 1 + \frac{c_1}{r} + \cdots, \tag{2.3e}$$
$$p = p_0\left(1 + \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{q_3}{r^3} + \cdots\right), \tag{2.3f}$$

with p_0 given by (1.2). To know the metric tensor components up to and including the r^{-1} terms all eighteen functions of x, y, u appearing as coefficients of the various powers of r in (2.3) are required. The vacuum field equations and an outgoing radiation condition then yield (see [10])

$$\alpha_2 = \beta_2 = 0, \tag{2.4}$$

$$a_0 = a_1 = 0 \text{ and } b_0 = b_1 = 0, \tag{2.5}$$

$$a_2 = p_0^4\left\{\frac{\partial}{\partial x}(p_0^{-2}\alpha_1) + \frac{\partial}{\partial y}(p_0^{-2}\beta_1)\right\}, \tag{2.6}$$

$$b_2 = p_0^4\left\{\frac{\partial}{\partial x}(p_0^{-2}\beta_1) - \frac{\partial}{\partial y}(p_0^{-2}\alpha_1)\right\}, \tag{2.7}$$

$$q_1 = 0,\ q_2 = \tfrac{1}{2}(\alpha_1^2 + \beta_1^2),\ q_3 = 0. \tag{2.8}$$

There remain five field equations. The first is an equation we regard as a propagation equation for $c_1(u, x, y)$ off u =const.

$$\dot{M} + |\dot{\gamma}|^2 = 0, \tag{2.9}$$

where the overdot denotes differentiation with respect to u,

$$M = m - \dot{q}_2 - \frac{1}{2}\left\{\frac{\partial}{\partial x}(p_0^{-2}a_2) + \frac{\partial}{\partial y}(p_0^{-2}b_2)\right\}, \tag{2.10}$$

with

$$m = -\tfrac{1}{2}c_1,\quad \gamma = \alpha_1 + i\beta_1. \tag{2.11}$$

In the present formulation the remarkably simple equation (2.9), when averaged over a two-sphere, leads to the well-known Bondi-Sachs mass-loss formula. The propagation equations for a_3, b_3 off u =const are

$$\begin{aligned}\frac{3}{2}p_0^{-1}\dot{A}_3 - p_0\frac{\partial m}{\partial x} = {} & p_0^3\left\{\frac{\partial}{\partial x}(p_0^{-2}\alpha_1) + \frac{\partial}{\partial y}(p_0^{-2}\beta_1)\right\} - b_2\frac{\partial^2 p_0}{\partial x\partial y} - a_2\frac{\partial^2 p_0}{\partial x^2} \\ & - p_0^2\dot{\beta}_1\left\{\frac{\partial}{\partial x}(p_0^{-1}\beta_1) - \frac{\partial}{\partial y}(p_0^{-1}\alpha_1)\right\} + \frac{\partial p_0}{\partial y}\frac{\partial a_2}{\partial y} + 2\beta_1\dot{\alpha}_1\frac{\partial p_0}{\partial y} \\ & - p_0^2\dot{\alpha}_1\left\{\frac{\partial}{\partial x}(p_0^{-1}\alpha_1) + \frac{\partial}{\partial y}(p_0^{-1}\beta_1)\right\} - \frac{\partial p_0}{\partial x}\left(\dot{q}_2 + \frac{\partial b_2}{\partial y} - b_2\frac{\partial}{\partial y}(\ln p_0) - a_2\frac{\partial}{\partial x}(\ln p_0)\right) \\ & - p_0\frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial a_2}{\partial y} - \frac{1}{2}\frac{\partial b_2}{\partial x} + 2\dot{\alpha}_1\beta_1\right) - p_0^{-1}a_2\dot{\alpha}_1 - p_0^{-1}b_2\dot{\beta}_1,\end{aligned} \tag{2.12}$$

$$\frac{3}{2}p_0^{-1}\dot{B}_3 - p_0\frac{\partial m}{\partial y} = p_0^3\left\{\frac{\partial}{\partial x}(p_0^{-2}\beta_1) - \frac{\partial}{\partial y}(p_0^{-2}\alpha_1)\right\} - a_2\frac{\partial^2 p_0}{\partial x\partial y} - b_2\frac{\partial^2 p_0}{\partial y^2}$$
$$-p_0^2\dot{\beta}_1\left\{\frac{\partial}{\partial x}(p_0^{-1}\alpha_1) + \frac{\partial}{\partial y}(p_0^{-1}\beta_1)\right\} + \frac{\partial p_0}{\partial x}\frac{\partial b_2}{\partial x} - 2\beta_1\dot{\alpha}_1\frac{\partial p_0}{\partial x}$$
$$+p_0^2\dot{\alpha}_1\left\{\frac{\partial}{\partial x}(p_0^{-1}\beta_1) - \frac{\partial}{\partial y}(p_0^{-1}\alpha_1\right\} - \frac{\partial p_0}{\partial y}\left(\dot{q}_2 + \frac{\partial a_2}{\partial x} - a_2\frac{\partial}{\partial x}(\ln p_0) - b_2\frac{\partial}{\partial y}(\ln p_0)\right)$$
$$-p_0\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial b_2}{\partial x} - \frac{1}{2}\frac{\partial a_2}{\partial y} - 2\dot{\alpha}_1\beta_1\right) + p_0^{-1}b_2\dot{\alpha}_1 - p_0^{-1}a_2\dot{\beta}_1. \quad (2.13)$$

In the latter two equations,

$$A_3 = a_3 - \frac{1}{3}(b_2\beta_1 + a_2\alpha_1) - \frac{4}{3}q_2 p_0\frac{\partial p_0}{\partial x} - \frac{2}{3}p_0^3\frac{\partial}{\partial y}(p_0^{-1}\alpha_1\beta_1), \quad (2.14a)$$

$$B_3 = b_3 - \frac{1}{3}(a_2\beta_1 - b_2\alpha_1) - \frac{4}{3}q_2 p_0\frac{\partial p_0}{\partial y} + \frac{2}{3}p_0^3\frac{\partial}{\partial x}(p_0^{-1}\alpha_1\beta_1). \quad (2.14b)$$

We also have the propagation equations for α_3, β_3 off u =const:

$$8\dot{\alpha}_3 - 4m\alpha_1 + 12\alpha_1\dot{\beta}_1\beta_1 + 12\beta_1^2\dot{\alpha}_1 - 8\alpha_1^2\dot{\alpha}_1 - \alpha_1\frac{\partial a_2}{\partial x} + b_2\frac{\partial\beta_1}{\partial x} - 3\beta_1\frac{\partial b_2}{\partial x} - 3a_2\frac{\partial\alpha_1}{\partial x} - \alpha_1\frac{\partial b_2}{\partial y} - a_2\frac{\partial\beta_1}{\partial y}$$
$$+3\beta_1\frac{\partial a_2}{\partial y} - 3b_2\frac{\partial\alpha_1}{\partial y} + 4p_0^{-1}\alpha_1\left(a_2\frac{\partial p_0}{\partial x} + b_2\frac{\partial p_0}{\partial y}\right) + 2p_0^{-2}(a_2^2 - b_2^2) = \frac{\partial a_3}{\partial x} - \frac{\partial b_3}{\partial y}, \quad (2.15)$$

and

$$8\dot{\beta}_3 - 4m\beta_1 - 20\alpha_1\dot{\alpha}_1\beta_1 - 4\alpha_1^2\dot{\beta}_1 - 8\beta_1^2\dot{\beta}_1 - \beta_1\frac{\partial a_2}{\partial x} - b_2\frac{\partial\alpha_1}{\partial x} + 3\alpha_1\frac{\partial b_2}{\partial x} - 3a_2\frac{\partial\beta_1}{\partial x} - \beta_1\frac{\partial b_2}{\partial y} + a_2\frac{\partial\alpha_1}{\partial y}$$
$$-3\alpha_1\frac{\partial a_2}{\partial y} - 3b_2\frac{\partial\beta_1}{\partial y} + 4p_0^{-1}\beta_1\left(a_2\frac{\partial p_0}{\partial x} + b_2\frac{\partial p_0}{\partial y}\right) + 4p_0^{-2}a_2b_2 = \frac{\partial a_3}{\partial y} + \frac{\partial b_3}{\partial x}. \quad (2.16)$$

A helpful guide to verifying these calculations using the computer algebra system REDUCE has been given in the lecture notes of McCrea [11].

The curvature tensor components, in Newman-Penrose notation, for the space-time with line element given by (2.1)–(2.3) take the form

$$\Psi_0 = -\frac{1}{r^5}\left\{6(\alpha_3 + i\beta_3) - \frac{3}{2}(\gamma + \bar{\gamma})^2(\gamma - \bar{\gamma}) - 2\bar{\gamma}^3\right\} + \cdots, \quad (2.17a)$$

$$\Psi_1 = -\frac{1}{r^4\sqrt{2}}\left\{\frac{3}{2}p_0^{-1}(a_3 + ib_3) + 3p_0^3\gamma\frac{\partial}{\partial\bar{z}}(p_0^{-2}\bar{\gamma})\right\} + \cdots, \quad (2.17b)$$

$$\Psi_2 = -\frac{1}{r^3}\left\{M + \gamma\frac{\partial\bar{\gamma}}{\partial u} + 2p_0^2\frac{\partial}{\partial\bar{z}}\left(p_0^2\frac{\partial}{\partial\bar{z}}(p_0^{-2}\bar{\gamma})\right)\right\} + \cdots, \quad (2.17c)$$

$$\Psi_3 = -\frac{2}{r^2\sqrt{2}}p_0^2\frac{\partial}{\partial u}\left(p_0\frac{\partial}{\partial\bar{z}}(p_0^{-2}\bar{\gamma})\right) + \cdots, \quad (2.17d)$$

$$\Psi_4 = -\frac{1}{r}\frac{\partial^2\bar{\gamma}}{\partial u^2} + \cdots, \quad (2.17e)$$

where $z = x + iy$.

It is useful to note that the complex shear σ and the expansion ϑ of the null geodesic congruence tangent to $\partial/\partial r$ are given by

$$\sigma = -\frac{\alpha_1 + i\beta_1}{r^2} - \frac{3(\alpha_3 + i\beta_3) + 2\alpha_1\beta_1^2}{r^4} + O(r^{-5}), \quad (2.18a)$$

$$\vartheta = \frac{1}{r} + \frac{2q_2}{r^3} + O(r^{-5}). \quad (2.18b)$$

III. MODEL OF IMPULSIVE WAVE

In the terminology of Sec. II we want $u = 0$ to be the history of an impulsive gravitational wave propagating through the Schwarzschild field. We take the region of space-time corresponding to $u < 0$ to be the past of the null hypersurface $u = 0$ and therefore to be, in its vacuum subregion, the Schwarzschild space-time. The region $u > 0$ lies to the future of $u = 0$ and in its vacuum subregion corresponds to the space-time left behind after the wave. To construct such a wave we take α_1, β_1 in Sec. II to have the form

$$\alpha_1 = \hat{\alpha}_1(x,y)u\vartheta(u), \tag{3.1a}$$

$$\beta_1 = \hat{\beta}_1(x,y)u\vartheta(u), \tag{3.1b}$$

where $\vartheta(u)$ is the Heaviside step function which is equal to 1 if $u > 0$ and equal to 0 if $u < 0$. The functions $\hat{\alpha}_1, \hat{\beta}_1$ are required to satisfy

$$\frac{\partial}{\partial x}(p_0^{-2}\hat{\alpha}_1) + \frac{\partial}{\partial y}(p_0^{-2}\hat{\beta}_1) = 0, \tag{3.2a}$$

$$\frac{\partial}{\partial x}(p_0^{-2}\hat{\beta}_1) - \frac{\partial}{\partial y}(p_0^{-2}\hat{\alpha}_1) = 0. \tag{3.2b}$$

A calculation helping to justify the choice of the Cauchy-Riemann equations (3.2) is given in the Appendix. Now (2.6) and (2.7) yield

$$a_2 = 0, \;\; b_2 = 0, \tag{3.3}$$

and, using (2.8), the propagation equation (2.9) reduces to

$$\dot{m} = 0. \tag{3.4}$$

To have the Schwarzschild field when $u < 0$ we must therefore have

$$m = \text{const}, \tag{3.5}$$

for all values of u. Moving on now to the propagation equations (2.12) and (2.13), we find that they simplify remarkably to

$$\dot{a}_3 = 0, \;\; \dot{b}_3 = 0, \tag{3.6}$$

respectively. To have the Schwarzschild field when $u < 0$ we must therefore have, for all values of u,

$$a_3 = 0, \;\; b_3 = 0. \tag{3.7}$$

Finally substituting into (2.15) and (2.16), the propagation equations for α_3, β_3 become

$$\dot{\alpha}_3 = \tfrac{1}{2}m\hat{\alpha}_1 u\vartheta(u) - 3\hat{\alpha}_1\hat{\beta}_1^2 u^2\vartheta(u) + \hat{\alpha}_1^3 u^2\vartheta(u), \tag{3.8a}$$

$$\dot{\beta}_3 = \tfrac{1}{2}m\hat{\beta}_1 u\vartheta(u) + 3\hat{\alpha}_1^2\hat{\beta}_1 u^2\vartheta(u) + \hat{\beta}_1^3 u^2\vartheta(u). \tag{3.8b}$$

Since α_3, β_3 must vanish for $u < 0$, to ensure a Schwarzschild vacuum when $u < 0$, these equations give

$$\alpha_3 + i\beta_3 = \tfrac{1}{4}m(\hat{\alpha}_1 + i\hat{\beta}_1)u^2\vartheta(u) + \tfrac{1}{3}\{\hat{\alpha}_1^3 - 3\hat{\alpha}_1\hat{\beta}_1^2 + 3i\hat{\alpha}_1^2\hat{\beta}_1 + i\hat{\beta}_1^3\}u^3\vartheta(u). \tag{3.9}$$

We can now calculate the curvature tensor components (2.17). The result is

$$\Psi_0 = -\frac{3m}{2r^5}(\hat{\alpha}_1 + i\hat{\beta}_1)u^2\vartheta(u) + O(r^{-6}), \tag{3.10a}$$

$$\Psi_1 = O(r^{-5}), \tag{3.10b}$$

$$\Psi_2 = -\frac{m}{r^3} + O(r^{-4}), \tag{3.10c}$$

$$\Psi_3 = O(r^{-3}), \tag{3.10d}$$

$$\Psi_4 = -\frac{\hat{\alpha}_1 - i\hat{\beta}_1}{r}\delta(u) + O(r^{-2}). \tag{3.10e}$$

The impulsive wave is described by the r^{-1} part of the field which is contained in Ψ_4 and has the required δ-function profile.

IV. DISCUSSION

The line element of the space-time constructed in Sec. III is obtained from (2.1)–(2.3) with the appropriate substitutions. It is given by

$$ds^2 = -r^2p_0^{-2}\left\{\frac{2\hat{\alpha}_1}{r}u\vartheta(u)(dx^2 - dy^2) + \frac{4\hat{\beta}_1}{r}u\vartheta(u)dx\,dy + \frac{\hat{\alpha}_1^2 + \hat{\beta}_1^2}{r^2}u^2\vartheta(u)(dx^2 + dy^2)\right\}$$
$$-\frac{m}{2r}p_0^{-2}u^2\vartheta(u)[\hat{\alpha}_1(dx^2 - dy^2) + 2\hat{\beta}_1 dx\,dy] - r^2p_0^{-2}(dx^2 + dy^2) + 2\,du\,dr + \left(1 - \frac{2m}{r}\right)du^2 + O(r^{-2}). \tag{4.1}$$

We can write this more compactly using $z = x + iy$ and the fact that (3.2) can be equivalently written as

$$\hat{\alpha}_1 - i\hat{\beta}_1 = p_0^2 H(z), \tag{4.2}$$

where $H(z)$ is analytic function. Then (4.1) takes the form

$$ds^2 = ds_0^2 - \frac{2m}{r}\left\{du^2 + \frac{u^2\vartheta(u)}{4}(H\,dz^2 + \bar{H}\,d\bar{z}^2)\right\} + O(r^{-2}), \tag{4.3}$$

where

$$ds_0^2 = -2r^2p_0^{-2}\left|dz + \frac{u\vartheta(u)}{r}p_0^2\bar{H}(\bar{z})d\bar{z}\right|^2 + 2\,du\,dr + du^2. \tag{4.4}$$

Here (4.4) is Penrose's [6] spherical impulsive wave in flat space-time in a coordinate system in which the metric tensor components are continuous across the history of the wave (see [12] for the construction of Penrose's solution in this coordinate system). For $u < 0$, (4.3) is the Schwarzschild solution exactly [the $O(r^{-2})$ terms vanish in this case]. For $u > 0$, (4.4) is the line element of flat space-time (the necessary coordinate transformation to confirm this is given in [12]) and so (4.3) is a vacuum perturbation of flat space-time in this case.

Two further observations on these results of Sec. III are worth noting.

(1) From (4.2) we see that since $p_0 = 1 + \frac{1}{4}z\bar{z}$ the complex-valued function $\hat{\alpha}_1 \pm i\hat{\beta}_1$ is singular when $z = \infty$. On $u = 0$ this corresponds to a generator of the null hypersurface. Thus the wave we have constructed in (3.10) exhibits this directional singularity just as Penrose's wave in flat space-time does. We might speculate that the existence of this directional singularity is due to the absence of a realistic source of energy to trigger the emission of the wave in the first place. For finite positive values of u and large values of r we see from (3.10) that the field behind the wave is the Schwarzschild field, the dominant term in the curvature being Ψ_2 in this case, and thus the existence of the wave is not due to a loss of mass by the isolated matter distribution.

(2) For large positive but finite values of u at fixed r we see from (3.10) that the field behind the wave is dominated by Ψ_0. In this situation the curvature tensor is approximately Petrov-type N with propagation direction crossing u =const> 0. Hence Ψ_0 represents backscattered radiation falling on the source. Since Ψ_0 vanishes when $m = 0$, the backscattering occurs because of the presence of the isolated matter distribution (the Penrose wave in flat space-time is unaccompanied by such radiation). It is interesting to note that backscattered radiation is also found to accompany the emission of gravitational waves during the collapse of a nonperfectly spherical isolated distribution of matter, leading to the formation of a Schwarzschild black hole [13]. We finally note that the approximations we have had to make prevent us from obtaining information about the space-time to the future of $u = 0$ in the limit $u \to +\infty$.

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APPENDIX: A WAVE FRONT EXPANSION

To motivate the choice of the Cauchy-Riemann equations (3.2) we examine the space-time to the future of the history of the impulsive wave (i.e., for $u \geq 0$) not in terms of the asymptotic expansions (2.3) but in terms of expansions of the functions $\alpha, \beta, a, b, c, p$ in powers of u about $u = 0$. The appropriate wave-front expansion will be obtained if we assume (1) when $u = 0$,

$$\alpha = 0,\ \ \beta = 0,\ \ a = 0,\ \ b = 0,$$

$$c = \left(1 - \frac{2m}{r}\right),\ \ p = p_0 = 1 + \tfrac{1}{4}(x^2 + y^2), \qquad \text{(A1)}$$

where m is a constant, (2) with an overdot denoting differentiation with respect to u we require at $u = 0$,

$$\dot{\alpha} = A(x,y,r),\ \ \dot{\beta} = B(x,y,r),\ \ \dot{a} = 0,\ \ \dot{b} = 0,$$

$$\dot{c} = 0,\ \ \dot{p} = 0, \qquad \text{(A2)}$$

and (3) at $u = 0$ the second and subsequent derivatives of $\alpha, \beta, a, b, c, p$ with respect to u are not necessarily zero. Assumptions (1)–(3) ensure the appearance of Penrose's impulsive wave [Eq. (4.4)] in the special case $m = 0$. We now require that at $u = 0$ the Ricci tensor and all its derivatives with respect to u should vanish. Denoting by R_{ab} the components of the Ricci tensor on the half-null tetrad defined via the one-forms (2.2) we find that, at $u = 0$,

$$R_{33} \equiv 0,\ \ R_{A3} \equiv 0,\ \ R_{11} + R_{22} \equiv 0, \qquad \text{(A3)}$$

for $A = 1, 2$. The vanishing of $R_{11} - R_{22}$ and R_{12} at $u = 0$ is equivalent to

$$\frac{\partial A}{\partial r} + \frac{1}{r}A = 0, \qquad \text{(A4)}$$

$$\frac{\partial B}{\partial r} + \frac{1}{r}B = 0, \qquad \text{(A5)}$$

respectively. We solve these with

$$A = \frac{\hat{\alpha}(x,y)}{r},\ \ B = \frac{\hat{\beta}(x,y)}{r}. \qquad \text{(A6)}$$

Now R_{34} vanishes identically at $u = 0$ while the vanishing of R_{A4} for $A = 1, 2$ at $u = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial}{\partial x}(p_0^{-2}\hat{\alpha}_1) + \frac{\partial}{\partial y}(p_0^{-2}\hat{\beta}_1) = 0, \qquad \text{(A7)}$$

$$\frac{\partial}{\partial x}(p_0^{-2}\hat{\beta}_1) - \frac{\partial}{\partial y}(p_0^{-2}\hat{\alpha}_1) = 0, \qquad \text{(A8)}$$

used in Sec. III of the text. The vanishing of the remaining Ricci tensor component R_{44} at $u = 0$ gives the information

$$\ddot{p}(u = 0, x, y) = p_0 r^{-2}(\hat{\alpha}_1^2 + \hat{\beta}_1^2). \qquad \text{(A9)}$$

This process can readily be continued yielding more information about the derivatives of $\alpha, \beta, a, b, c, p$ with respect to u at $u = 0$. We see at this early stage though that we can write

$$\alpha = \frac{\hat{\alpha}_1(x,y)}{r}u + O(u^2), \qquad \text{(A10)}$$

$$\beta = \frac{\hat{\beta}_1(x,y)}{r}u + O(u^2), \qquad \text{(A11)}$$

for $u \geq 0$ with $\hat{\alpha}_1, \hat{\beta}_1$ satisfying (A7) and (A8). To solve the (characteristic) initial-value problem for the system of equations (2.9), (2.12), (2.13), (2.15), and (2.16) we can first specify the two functions of three variables α_1, β_1. This we have done in (3.1) to ensure that we will obtain an impulsive wave on $u = 0$. To have the choice of (3.1) consistent with the wave front expansion results (A10) and (A11) with (A7) and (A8) we have imposed the conditions (3.2).

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