ARTICLES

Small scale integrated Sachs-Wolfe effect

Wayne Hu*
Departments of Astronomy and Physics, University of California, Berkeley, California 94720

Naoshi Sugiyama†
Departments of Astronomy and Physics, University of California, Berkeley, California 94720
and Department of Physics, Faculty of Science, The University of Tokyo, Tokyo, 113, Japan

(Received 27 October 1993)

The integrated Sachs-Wolfe (ISW) effect can be an important factor in the generation of cosmic microwave background anisotropies *on all scales,* especially in a reionized curvature- or Λ-dominated universe. We present a *simple* analytic treatment of the ISW effect, which is analogous to thick last scattering surface techniques for the Doppler effect, that compares quite well with the full numerical calculations. The power spectrum of temperature fluctuations due to the small scale ISW effect has a wave number dependence k^{-5} times that of the matter power spectrum.

PACS number(s): 98.70.Vc, 98.80.Hw

I. INTRODUCTION

It is well known that several effects contribute to cosmic microwave background (CMB) fluctuations, e.g., the difference in the gravitational potential between the last scattering surface (LSS) and the present combined with the intrinsic fluctuation due to adiabatic or isocurvature initial conditions (the ordinary Sachs-Wolfe effect) [1,2] and Doppler shifts from moving electrons on the LSS [3]. The gravitational redshift induced by a time-varying potential, i.e., the *integrated Sachs-Wolfe* (ISW) effect [1], gives another important contribution to CMB fluctuations, although it vanishes in linear theory for an $\Omega_0=1$ flat universe. This effect is dominant on very large scales [4], but only the nonlinear effect is usually considered important on small scales [5]. For example, Kofman and Starobinskii [6] perform a comparative study at large scales for a Λ-dominated adiabatic Harrison-Zel'dovich spectrum, but do not consider the *relative* contribution of the ISW effect at small scales. This is particularly significant in curvature or Λ-dominated models with early reionization, since the Doppler contributions are severely canceled on scales below the thickness of the LSS [7]. Such early reionization is actually required for open primeval isocurvature baryon (PIB) models [8,9]. We provide a simple, unified, and model-independent analytic method of treating the cancellation behavior of all three effects. Furthermore, we prove in Sec. II that the ISW effect contributes temperature fluctuations with the *same* scale dependence and amplitude as the canceled Doppler effect in reionized models. In Sec. III, we show that the analytic treatment compares quite favorably with exact numerical solutions.

II. ANALYTIC FORMALISM

The full *first-order* Boltzmann equation [2,10] for the evolution of temperature perturbations $\Theta(\mathbf{x},\gamma,\eta)$ in gauge-invariant notation [11] is given by

$$\frac{d}{d\eta}[\Theta+\Psi](\mathbf{x},\gamma,\eta) = \dot{\Psi}-\dot{\Phi}+\dot{\tau}(\Theta_0-\Theta+\gamma^i v_i+\tfrac{1}{16}\gamma^i\gamma^j\Pi_{ij}) , \quad (1)$$

where v_i is the electron velocity ($c=1$), Π_{ij} is the anisotropic stress perturbation, γ_i ($=dx_i/d\eta$) are the direction cosines for the photon momentum, overdots represent derivatives with respect to conformal time $\eta=\int dt/a$, Θ_0 is the monopole component of Θ, and Ψ is the Newtonian potential. The last term in Eq. (1) accounts for Compton scattering, where $\dot{\tau}=x_e n_e \sigma_T a$ is the differential optical depth, with x_e the ionization fraction, n_e the electron number density, and σ_T the Thomson cross section. The perturbation to the intrinsic spatial curvature Φ is given by the Poisson equation

$$(\nabla^2-3K)\Phi=4\pi G a^2\rho\Delta_T , \\ =\tfrac{3}{2}\Omega_0 H_0^2(\Delta_T/a) , \quad (2)$$

where $a=1/(1+z)$, the curvature constant $K=-H_0^2(1-\Omega_0-\Omega_\Lambda)$, the Hubble constant $H_0=100h$ km s^{-1} Mpc^{-1}, the scaled cosmological constant $\Omega_\Lambda=\Lambda/3H_0^2$, and the total density fluctuation, taken here

---

*Electronic address: hu@pac2.berkeley.edu
†Electronic address: sugiyama@bkyast.berkeley.edu

0556-2821/94/50(2)/627(5)/$06.00

in the matter-dominated limit, $\Delta_T=\delta\rho/\rho$. If anisotropic pressure is negligible, then $\Psi=-\Phi$ [10].

In a flat $\Omega_0=1$ with adiabatic initial conditions, the fact that $\Delta_T \propto a$ after last scattering implies $\dot\Phi=\dot\Psi=0$. In this case, the right-hand side of (1) vanishes and the quantity $\Theta+\Psi$ is conserved. This is merely the ordinary Sachs-Wolfe effect, i.e.,

$$\Theta(\mathbf{x}_0,\gamma_0,\eta_0)=\Theta(\mathbf{x}_*,\gamma_*,\eta_*)+[\Psi(\mathbf{x}_*,\eta_*)-\Psi(\mathbf{x}_0,\eta_0)] , \tag{3}$$

where the subscripts 0 and $*$ refer to the present and last scattering epochs, respectively. The temperature perturbation today has a contribution from the difference in potential between last scattering and the present and an intrinsic fluctuation on the last scattering surface. For adiabatic initial conditions at large scales, this intrinsic fluctuation is $\Theta(\mathbf{x}_*,\gamma_*,\eta_*)=-\frac{2}{3}\Psi(\mathbf{x}_*,\eta_*)$, whereas for isocurvature fluctuations, it is $\Psi(\mathbf{x}_*,\eta_*)$ [2]. We refer to the combination of intrinsic and potential difference terms as the ordinary Sachs-Wolfe effect. Since the quantity $\Theta+\Psi$ automatically includes this contribution, it is convenient to express our results in terms of this quantity.

In more general cases such as open and (flat) Λ-dominated universes, isocurvature initial conditions, *or* topological defect models, $\dot\Psi-\dot\Phi$ does not vanish and an additional effect, the ISW effect, arises. Furthermore, we must correct Eq. (3) for the Doppler effect and the finite thickness of the LSS at small scales. On small scales, we may drop the curvature terms implicit in $\dot\gamma_i$ and express Eq. (1) in terms of Fourier modes:

$$\dot\Theta+\dot\Psi+ik\mu(\Theta+\Psi) =\dot\Psi-\dot\Phi+\dot\tau[\Theta_0-\Theta+\tfrac{1}{10}\Theta_2P_2(\mu)+\mu v] , \tag{4}$$

where $\mathbf{k}\cdot\gamma=k\mu$ and the multipole moments are defined so that $\Theta(k,\mu,\eta)=\sum\Theta_l(k,\eta)P_l(\mu)$. For an analytic treatment which is applicable at large scales, see [2].

It is possible to obtain a simple approximate analytic solution to Eq. (1) by analogy to thick-LSS techniques [12,13]. Small-scale contributions to the temperature anisotropy are suppressed since contributions from opposite sides of a perturbation cancel each other. For plane wave perturbations, this implies that only if the wave vector is parallel to the line of sight $\mathbf{k}\|\gamma$, so that the fluctuation is *uniform* along the line of sight, will the contribution survive. This sort of cancellation suppresses fluctuations by $(k\eta_0)^{-1/2}$ and applies even to the ISW effect. However, since the Doppler and ordinary Sachs-Wolfe effects are proportional to μ, they suffer an additional cancellation of $(k\eta_0)^{-1}$. This is because $\mathbf{v}\|\mathbf{k}$ in linear theory, and there is no gravitational redshift if the potential is uniform along the line of sight. Only residual effects, such as the time variation of these sources across the LSS and feedback, survive. This additional cancellation in the Doppler and ordinary Sachs-Wolfe effects is the reason why the ISW effect is important at small scales.

After Compton drag has become negligible, $z<z_d\simeq160(\Omega_0h^2)^{1/5}x_e^{-2/5}$, the Sachs-Wolfe and Doppler sources can be solved independently of the radiation and merely follow the linear theory predictions

$$\begin{aligned}\Delta_T(k,\eta)&=D(\eta)\Delta_T(k,\eta_0) ,\\ v(k,\eta)&=\frac{i}{k}\dot D(\eta)\Delta_T(k,\eta_0) ,\\ \Psi(k,\eta)&=-\frac{3}{2}\frac{1}{ak^2}\Omega_0H_0^2D(\eta)\Delta_T(k,\eta_0) ,\end{aligned} \tag{5}$$

where $D(\eta)$ is the growth factor in linear theory [14] such that $\Delta_T(k,\eta)=D(\eta)\Delta_T(k,\eta_0)$. In this case, Eq. (4) has the formal solution

$$[\Theta+\Psi](k,\mu,\eta)\simeq[\Theta+\Psi](k,\mu,\eta_d)e^{ik\mu(\eta_d-\eta)}e^{-\tau(\eta_d,\eta)} +\Theta_{\rm DSW}+\Theta_{\rm ISW} , \tag{6}$$

where $\tau(\eta_1,\eta_2)=\int_{\eta_1}^{\eta_2}\dot\tau(\eta)d\eta$, $\Theta_{\rm DSW}$ is the Doppler and ordinary Sachs-Wolfe (DSW) fluctuations generated on the new LSS, if last scattering occurs *after* the drag epoch as is the case in reionized scenarios, and $\Theta_{\rm ISW}$ is the ISW effect. Explicitly, these terms are given by

$$\begin{aligned}\Theta_{\rm DSW}(k,\mu,\eta)&=\int_{\eta_d}^{\eta}(\Theta_0+\Psi+\mu v)\dot\tau e^{-\tau(\eta',\eta)}e^{ik\mu(\eta'-\eta)}d\eta' ,\\ \Theta_{\rm ISW}(k,\mu,\eta)&=\int_{\eta_d}^{\eta}2\dot\Psi(k,\eta')e^{-\tau(\eta',\eta)}e^{ik\mu(\eta'-\eta)}d\eta' .\end{aligned} \tag{7}$$

We have dropped the Θ_2 quadrupole term from the angular dependence of Compton scattering since its contribution can be shown to be negligible in this limit [15].

Note that Eqs. (6) and (7) and the following argument are valid for *any* ionization history. In particular, for standard recombination $x_e(z<z_d\simeq1000)=0$ and $\tau(\eta_d,\eta_0)=0=\Theta_{\rm DSW}$; i.e., *only* the ISW effect adds to $\Theta(k,\mu,\eta_d)$, the primary temperature fluctuations after standard last scattering. On the other hand, for reionized scenarios, the primary fluctuations are damped by $e^{-\tau}$ under the horizon scale [15]. This is because streaming under the horizon scale converts radiation fluctuations to anisotropies which are destroyed by the isotropizing effect of scattering. In no-recombination scenarios, the primary fluctuations may therefore be ignored since $\tau(\eta_d,\eta_0)\gg1$.

Before we can employ Eq. (6), we have to solve for Θ_0 to determine the feedback from the isotropic effect [12,15]. Taking the isotropic component of Eq. (6), we obtain

$$[\Theta_0+\Psi](k,\eta) =\int_{\eta_d}^{\eta}d\eta' e^{-\tau(\eta',\eta)}\{[\Theta_0\dot\tau+\Psi\dot\tau+2\dot\Psi]j_0[k(\eta'-\eta)] +iv\dot\tau j_1[k(\eta'-\eta)]\} . \tag{8}$$

For small scales, $k\eta\gg1$, and the integral only gets contributions from $\eta'\simeq\eta$. This represents the fact that, at small scales, streaming over many wavelengths of the perturbation tends to cancel out the contributions. We can therefore take the slowly varying quantities out of Eq. (8) to obtain [12]

$$[\Theta_0+\Psi](k,\eta)\simeq ik^{-1}[v\dot{\tau}+O(\Psi\dot{\tau})+O(\dot{\Psi})] \ . \tag{9}$$

Ordinarily, these feedback effects are negligible since they are proportional to the optical depth and change in potential across one wavelength. We shall see that cancellation in the Doppler term itself makes them important. Note also that the residual ordinary Sachs-Wolfe effect is much smaller than the residual Doppler effect at small scales since they are canceled in the same manner and $\Psi \propto v/k$, as seen in Eq. (5). We will hereafter drop these terms from the calculation.

Employing Eq. (9) and integrating by parts, we can now express Eq. (6) as

$$\begin{aligned}[\Theta+\Psi](k,\eta_0)&=\int_{\eta_d}^{\eta_0} d\eta[i(\dot{v}\dot{\tau}+v\ddot{\tau})/k+2\dot{\Psi}]e^{-\tau(\eta,\eta_0)}e^{ik\mu(\eta-\eta_0)}\\ &=-\frac{1}{(k\eta_0)^2}\Delta_T(k,\eta_0)e^{-ik\mu\eta_0}\int_{x_d}^{1}dx[G_{\rm DSW}(x)+G_{\rm ISW}(x)]e^{ik\mu\eta_0 x} \ ,\end{aligned} \tag{10}$$

where

$$\begin{aligned}G_{\rm DSW}(x)&=\eta_0^3(\ddot{D}\dot{\tau}+\dot{D}\ddot{\tau})e^{-\tau(\eta,\eta_0)} \ ,\\ G_{\rm ISW}(x)&=3a^{-2}\eta_0^3H_0^2\Omega_0(\dot{D}a-D\dot{a})e^{-\tau(\eta,\eta_0)} \ ,\end{aligned} \tag{11}$$

with $x=\eta/\eta_0$. Note now that both the ISW and DSW Doppler contributions have the *same* k dependence despite the fact that v and $\dot{\Psi}$ do not. This is because the Doppler effect is proportional to μ and suffers additional cancellation since it only contributes to the $\mu=0$ mode through feedback and residual effects. The ISW effect contributes directly to the $\mu=0$ mode and does not suffer this additional cancellation.

The integral in Eq. (10) is approximately a Fourier transform if the integrand is localized in between the drag epoch and the present. This is a good approximation for the DSW contributions in all cases as a result of the form of visibility function ($\dot{\tau}e^{-\tau}$). It is also a good approximation to the ISW term for curvature-dominated universes where ($\dot{\Psi}e^{-\tau}$) peaks around $(1+z_g)\simeq 1/\Omega_0-1$. The approximation technically breaks down for Λ models where $(1+z_g)\simeq[1/\Omega_0-1]^{1/3}$ since ($\dot{\Psi}e^{-\tau}$) gets significant contributions near the present epoch. We shall see, however, that because of the nature of the cancellation, the approximation remains valid even in this case. We can think of this crucial epoch z_g, or more specifically ($\dot{\Psi}e^{-\tau}$), as defining a *gravitational "last scattering" surface* (GLSS). That the LSS and GLSS are thick compared with the wavelength of the fluctuation implies that the Fourier transform in Eq. (10) has width $k\mu\eta_* \lesssim 1$ [12]. Since $k\eta_* \gg 1$, this implies that $\Theta+\Psi$ only has significant contributions in the $\mu\simeq 0$ mode, as a result of cancellation along the line of sight, as expected.

We can therefore approximate the temperature fluctuation power spectrum, using Parseval's theorem, as

$$\begin{aligned}\frac{V}{2\pi^2}k^3P_\gamma(k)&\\ &\equiv\frac{V}{2\pi^2}k^3\Theta_{\rm rms}^2\simeq\frac{V}{2\pi^2}k^3|\Theta+\Psi|_{\rm rms}^2\\ &\simeq\frac{V}{2\pi^2}k^3\frac{1}{2}\int_{-1}^{1}d\mu|\Theta+\Psi|^2\\ &\simeq\frac{1}{2\pi}\frac{V}{\eta_0^3}\frac{P(k)}{(k\eta_0)^2}\int_0^1|G_{\rm DSW}(x)+G_{\rm ISW}(x)|^2dx \ ,\end{aligned} \tag{12}$$

where $P(k)=|\Delta_T(k,\eta_0)|^2$ is the usual matter power spectrum and $\Theta_{\rm rms}(k,\eta_0)\gg\Psi(k,\eta_0)$ if $k\eta_0\gg 1$. Therefore, in reionized scenarios where the DSW term is suppressed as a result of cancellation, the ISW term not only contributes significantly at small scales, but also mimics the scale dependence of the canceled Doppler term. These two effects can actually interfere if the LSS and GLSS overlap.

III. COMPARISON WITH NUMERICAL RESULTS

For comparison, we solve Eq. (1) numerically up to the present by the method outlined in [16]. It is instructive to examine first the time evolution of the quantities in Eq. (1). In Fig. 1, we have chosen a representative no-recombination model (see caption) to illustrate the evolution of Ψ, v, and $|\Theta+\Psi|_{\rm rms}$ in (a) an open and (b) a Λ ($\Omega_\Lambda=1-\Omega_0$) model. After the drag epoch $z\lesssim 100$, v and Ψ evolve in linear theory. Since the universe at early times is nearly flat, the potential thereafter remains roughly constant until the curvature or Λ-dominated epoch. At this time, Ψ starts to decay, marking out the GLSS. More precisely, whereas the LSS is defined by the visibility function $\dot{\tau}e^{-\tau}$, the GLSS is given by $\dot{\Psi}e^{-\tau}$ (dashed lines, arbitrary normalization).

We have plotted the resultant temperature fluctuation $|\Theta+\Psi|_{\rm rms}$ with (thick line) and without (thin line) the ISW term in Eq. (1). Note that the ISW has a negligible effect during the last scattering epoch; only in the Λ- or curvature-dominated epoch *and* after suppression through the LSS does the ISW effect become important. In these no-recombination scenarios, the LSS and GLSS have comparable width, and the ISW effect is important across the spectrum.

We plot the full present-day power spectrum of temperature fluctuations with a $\sigma_8=1$ normalization for various models in Fig. 2. In panel (a), the results for no-recombination isocurvature open (top lines) and Λ models (bottom lines) of Fig. 1 are shown. The analytic calculations (dashed line) with (thick) and without (thin) ISW agree quite well with the full numerical solution on small scales. Deviations on large scales are expected since the analytic formalism is only valid below the thickness of the LSS and GLSS. We calculate the ratios of the power spectrum of temperature fluctuations with and without the ISW contribution for other baryonic no-recombination models in Table I. Note that the ISW effect gives a contribution of the same order as the Doppler effect for low-Ω_0 models.

In Fig. 2(b), we examine the ISW effect in an open (top lines) and Λ model (bottom lines) with standard recombination. Here we take an adiabatic cold dark matter (CDM) model for which this ionization history is more natural. Although this model overpredicts fluctuations if normalized to σ_8, we choose it to demonstrate that even in this extreme case the ISW effect plays no role on small scales. For standard recombination, the LSS is relatively thin, and the ISW effect is masked until the Silk damping scale. Furthermore, since the GLSS is much thicker than the LSS, cancellation of the ISW effect begins at a much larger scale and therefore is negligible at the Silk damping scale. Note, however, that the ISW effect *itself* does not depend sensitively on the ionization history. Only the damping of the other terms is affected by the ionization. Moreover, even in these models, the ISW still plays an important role on large scales (where the analytic formalism breaks down). It is interesting that the analytic formalism nevertheless predicts the correct scale for dominance of the ISW effect. Finally, we should mention that the ISW *does* contribute significantly to open or Λ adiabatic CDM models with sufficiently early reionization or no recombination.

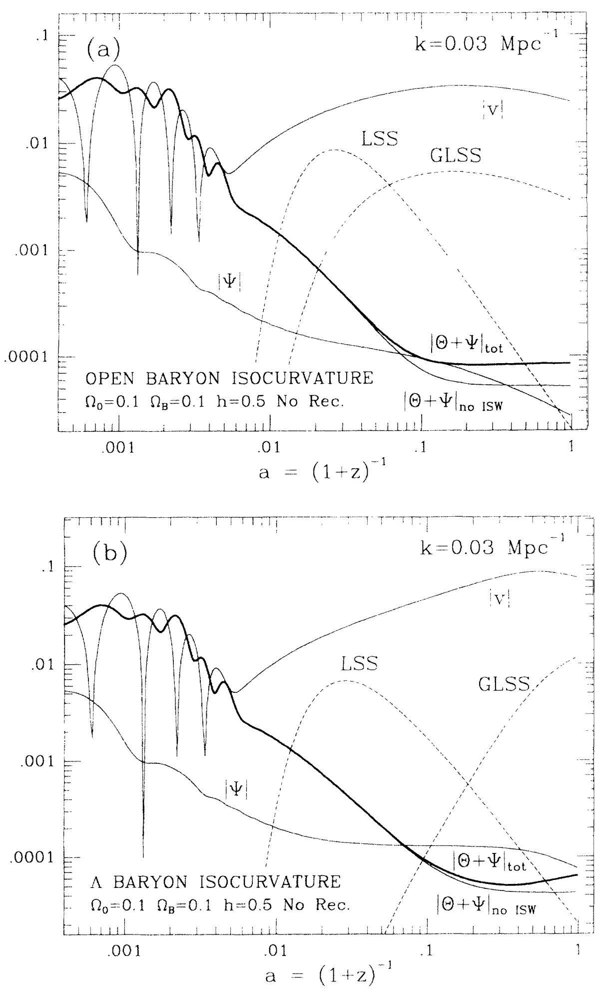

FIG. 1. Evolution of Ψ, v, and $|\Theta+\Psi|_{\rm rms}$ with and without ISW from numerical calculations for a specific small-scale mode ($k=0.03$ Mpc^{-1}) in (a) an open model ($\Omega_\Lambda=0$) and (b) a Λ model ($\Omega_\Lambda=1-\Omega_0$) for an $\Omega_0=\Omega_B=0.1$, $h=0.5$ universe with no recombination and isocurvature initial conditions. Overall normalization is arbitrary. Through the LSS, temperature fluctuations, originally of $\mathcal{O}(v)$, become increasingly suppresed as a result of cancellation and rescattering. We have also plotted the shape of the LSS [$\propto \dot\tau \exp(-\tau)$] and the GLSS [$\propto \dot\Psi \exp(-\tau)$] from linear theory for reference (dotted lines). In Λ models, the ISW effect contributes strongly near the present epoch, and the GLSS is not well defined. Nevertheless, the analytic approximations still work quite well.

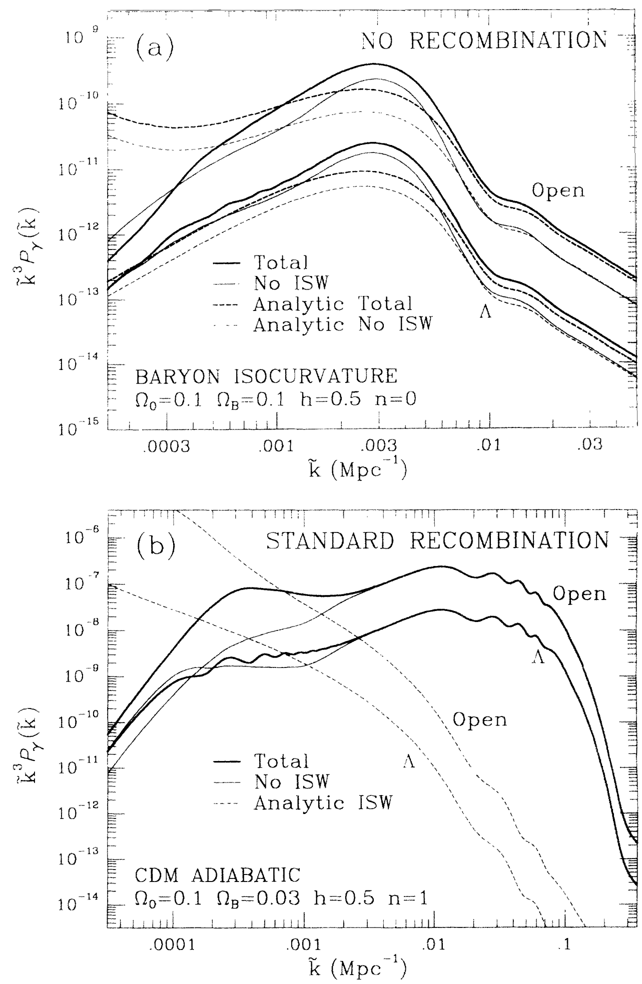

FIG. 2. Power spectra for the temperature fluctuations $P_\gamma(\tilde k)$ where $\tilde k^2 = k^2 + K$ [4] from analytic and numerical calculations: (a) no-recombination; open (upper set) and Λ (lower set) baryon isocurvature models with spectral index $n=0$ and parameters $\Omega_0=\Omega_B=0.1$, $h=0.5$ where the initial entropy fluctuations $|S(\tilde k)|^2 \propto \tilde k^n$. The solid lines represent the numerical results with (thick) and without (thin) ISW. The dashed lines represent analytic calculations in an analogous manner. (b) Standard recombination; open (upper set) and Λ (lower set) CDM models with adiabatic initial conditions $P(\tilde k)\propto \tilde k^n$ and parameters $\Omega_0=0.1$, $\Omega_B=0.03$, $h=0.5$, $n=1$. On large scales, the power spectra fall of since we have subtracted the unobservable monopole dipole.

TABLE I. Ratios of the power in temperature fluctuations with to without ISW $R \equiv P_\gamma^{(\mathrm{tot})}/P_\gamma^{(\mathrm{no\ ISW})}$ given by the numerical (R_{num}) and analytical (R_{an}) computations for an $\Omega_B = \Omega_0$, $h=0.5$ universe. R_{num} is calculated for $k = 1\Omega_0 h^2$ Mpc^{-1}, but does not vary significantly on small scales. The ISW effect is less significant in Λ models, since Λ domination occurs at a relatively late epoch. Analytic predictions agree quite well with numerical results at small scales ($\lesssim 20\%$ in power, $\lesssim 10\%$ in temperature fluctuations).

Ω_0	Ω_Λ	R_{num}	R_{an}
0.1	0.0	2.6	2.2
0.2	0.0	2.1	1.9
0.4	0.0	1.6	1.5
0.6	0.0	1.3	1.3
0.8	0.0	1.2	1.1
1.0	0.0	1.0	1.0
0.1	0.9	1.9	1.7
0.2	0.8	1.5	1.4
0.4	0.6	1.2	1.1
0.6	0.4	1.1	1.1
0.8	0.2	1.1	1.0

IV. CONCLUSIONS

We have found that the ISW effect is important at *all scales* in a reionized open or Λ universe including the so-called Doppler peak and below. Furthermore, we have developed an analytic formalism which compares quite well with detailed numerical solutions. From this formalism, we find that the power spectrum of temperature fluctuations behaves on small scales as $P_\gamma(k) \propto k^{-5}P(k)$ for *both* the Doppler and ISW contributions, despite the fact that the scale dependence of v and $\dot{\Psi}$ themselves are not similar, as a result of a difference in their cancellation behavior. The amplitude of the temperature fluctuations can be predicted to 10% accuracy by the analytic approximation. In standard recombination models, the small-scale ISW effect is masked since the LSS is relatively thin compared to the GLSS, and thus cancellation of the Doppler effect does not occur until a very small scale. The analytic approach derived here is a potentially powerful yet simple tool for understanding and calculating the ISW effect. It is independent of the detailed model and is thus applicable to all scenarios which have a time-varying potential, e.g., even topological defect models [17] for which the ISW effect is essentially the only important contribution to CMB fluctuations.

ACKNOWLEDGMENTS

We would like to thank F. Atrio-Barandela, D. Scott, and J. Silk for useful comments and discussions. W. H. has been partially supported by the NSF. N.S. acknowledges financial support from the JSPS for research abroad.

[1] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967). Note that what is here called the "ordinary" and "integrated" Sachs-Wolfe effects are both contained in the general formulas of this paper.
[2] W. Hu and N. Sugiyama (unpublished); W. Hu, in Proceedings of the Case Western CMB Meeting, edited by L. Krauss (unpublished). We provide a unified treatment comparing and contrasting the physical basis of each effect for isocurvature versus adiabatic initial conditions.
[3] R. A. Sunyaev and Ya. B. Zel'dovich, Astrophys. Space Sci. **7**, 3 (1970).
[4] M. L. Wilson, Astrophys. J. **273**, 2 (1983); L. F. Abbott and R. K. Schaefer, *ibid.* **308**, 546 (1986); N. Gouda, N. Sugiyama, and M. Sasaki, Prog. Theor. Phys. **85**, 1023 (1991).
[5] E. Martinez-Gonzalez, J. L. Sanz, and J. Silk, Phys. Rev. D **46**, 4193 (1992).
[6] L. Kofman and A. Starobinskii, Sov. Astron. Lett. **11**, 271 (1985).
[7] J. P. Ostriker and E. Vishniac, Astrophys. J. **306**, 51 (1986); E. Vishniac, *ibid.* **322**, 597 (1987). The Vishniac effect dominates over ISW in many models for $k\eta_0 \gtrsim 10^5$.
[8] P. J. E. Peebles, Astrophys. J. **315**, L73 (1987); G. Efstathiou and J. R. Bond, Mon. Not. R. Astron. Soc. **227**, 33 (1987).
[9] W. Hu and N. Sugiyama, Astrophys. J. (to be published).
[10] H. Kodama and M. Sasaki, Int. J. Mod. Phys. **1**, 265 (1986), gives the full Boltzmann equation in their appendix ($\frac{1}{6}$ should be replaced with $\frac{1}{16}$); N. Gouda, M. Sasaki, and Y. Suto, Astrophys. J. **341**, 557 (1989), gives the collisionless Boltzmann equation.
[11] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1982); H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984). Gauge choice is, of course, irrelevant for our small-scale calculation.
[12] N. Kaiser, Astrophys. J. **282**, 374 (1984).
[13] G. Efstathiou, in *Large Scale Motions in the Universe: A Vatican Study Week,* edited by V. C. Rubin and G. V. Coyne (Princeton University Press, Princeton, 1988), p. 299.
[14] P. J. E. Peebles, *Large Scale Structure of the Universe* (Princeton University Press, Princeton, 1980).
[15] W. Hu, D. Scott, and J. Silk, Phys. Rev. D **49**, 648 (1994).
[16] N. Sugiyama and N. Gouda, Prog. Theor. Phys. **88**, 803 (1992).
[17] U. Pen, D. N. Spergel, and N. Turok, Phys. Rev. D **49**, 692 (1994); A. Jaffe, A. Stebbins, and J. A. Frieman, Astrophys. J. **490**, 9 (1994); D. Coulson *et al.*, Nature (London) **368**, 27 (1994).