Observables for spacetimes with two Killing field symmetries

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The Einstein equations for spacetimes with two commuting spacelike Killing field symmetries are studied from a Hamiltonian point of view. The complexified Ashtekar canonical variables are used, and the symmetry reduction is performed directly in the Hamiltonian theory. The reduced system corresponds to the field equations of the SL(2,R) chiral model with additional constraints. On the classical phase space, a method of obtaining an infinite number of constants of motion, or observables, is given. The procedure involves writing the Hamiltonian evolution equations as a single "zero curvature" equation and then employing techniques used in the study of two-dimensional integrable models. Two infinite sets of observables are obtained explicitly as functionals of the phase space variables. One set carries sl(2,R) Lie algebra indices and forms an infinite-dimensional Poisson algebra, while the other is formed from traces of SL(2,R) holonomies that commute with one another. The restriction of the (complex) observables to the Euclidean and Lorentzian sectors is discussed. It is also shown that the sl(2,R) observables can be associated with a solution-generating technique which is linked to that given by Geroch.

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I. INTRODUCTION

In classical general relativity one of the important questions is that of finding exact solutions and extracting their properties. This is hindered by the complexity of Einstein's equations, and the discovery of a new solution is rare.

It is therefore usual to simplify the problem by seeking solutions that have certain symmetries. These are normally specified by requiring the metric to have a number of Killing vector fields, which leads to a simplified set of equations to solve.

One such set of reduced equations is obtained by requiring the metric to have two commuting vector fields. This simplification leads to a two-dimensional field theory, and has the advantage that it still leaves the gravitational field with two local degrees of freedom (unlike, for example, the minisuperspace reductions, where only a finite number of degrees of freedom remain). This symmetry reduction was first studied in detail by Geroch [1], who found that the resulting Einstein equations have an infinite-dimensional "hidden" symmetry. These symmetry transformations of the equations provide a solution-generating technique, whereby, given one solution with two commuting Killing fields, a new family of solutions can be generated. The solution-generating technique was later presented from other points of view [2–4]. These equations have also been studied using the inverse-scattering method [5] to obtain solitonic solutions.

The question of exact solutions is related to that of conserved quantities. It is expected, as for any dynamical system, that exact solutions will be labeled by values of the conserved quantities. In general relativity, for spacetimes with compact spacelike hypersurfaces, the latter are also referred to as observables. This is because if conserved quantities can be written explicitly as functionals of the phase space variables (which should always be possible), they would also be the fully gauge-invariant variables.

It is useful to have phase space observables for the classical theory, in particular, in attempts to prove integrability. For example, in all the known two-dimensional integrable models such as the Korteweg–de Vries (KdV) and sine-Gordon equations, an explicit generating procedure for observables may be used to prove integrability [6,7].

Apart from the classical questions, in attempts to construct a canonical quantum theory starting from general relativity, a complete set of such classical variables is a prerequisite for certain quantization schemes, where the quantum theory is to be obtained as a representation of the Poisson algebra of observables [8–10]. This method has been under study for the nonperturbative approach to quantum gravity using the Ashtekar variables [9,11] and the related loop space representation [12]. It has been successful for the quantization of 2+1 gravity [13].

For the *full* Einstein equations, it is known that the only "hidden" symmetries, apart from diffeomorphisms, are constant rescalings of the metric [14]. From this result it follows that no observables can be built as integrals of local functions of the initial data [15]. However, from the works mentioned above, the two Killing field reduced equations are known to have an associated infinite-dimensional symmetry group other than diffeomorphims. It is then natural to ask what are the conserved quantities associated with these symmetries, and in particular, what they are as functionals of the phase space variables.

In some recent work [16], a procedure based on methods used for finding conservation laws for soliton equations has been applied to the two Killing field reduced Einstein equations. The starting point in this work was a particular form of the metric with two commuting space-

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like Killing fields. The dynamical Einstein equations following from this were then studied using ideas from two-dimensional integrable models. If these quantities can be written as phase space functionals, one would have an infinite number of observables for this sector of Einstein gravity. However it is not clear from this work how the conserved quantities can be rewritten in terms of the Arnowitt-Deser-Misner (ADM) phase space variables.

This paper addresses the question of obtaining observables for two Killing field reduced Einstein gravity. The main result presented below is an explicit construction of an infinite number of phase space observables for spacetimes with two commuting spacelike Killing fields, and with compact spatial hypersurfaces. The observables are obtained for complexified gravity (i.e., complex phase space variables on a real manifold). The reality conditions are then discussed for the Euclidean and Lorentzian restrictions.

The natural starting point is the Hamiltonian form of the Einstein equations. The Ashtekar Hamiltonian formulation [9,11] is used for this, and in the next section the two Killing field symmetry is imposed in these variables to obtain a reduced first class Hamiltonian system, which still has 2 local degrees of freedom. This reduction corresponds to the Gowdy cosmological models [17], and has been studied earlier by the author and Smolin [18]. In the third section the reduced system is fully gauge fixed, with the gauge fixing conditions chosen to put the Hamiltonian evolution equations in a suggestive form. This is discussed further in the following section, where a zero curvature form of the evolution equations is given. The fifth section gives the procedure for obtaining the observables, and is based on methods used in two dimensional integrable models. There is also a discussion of the Poisson algebra of the observables. The sixth section describes a solution generating technique for this sector of the Einstein equations using the observables, as well as its connection with the Geroch method. The paper ends with a summary and outlook for the quantization of this sector of gravity.

II. TWO KILLING VECTOR FIELD REDUCTION

The Ashtekar Hamiltonian variables for complexified general relativity are the (complex) canonically conjugate pair $(A_a^i, \tilde{E}^{ai})$ where A_a^i is an so(3) connection and $\tilde{E}^{ai}$ is a desensitized dreibein. $a, b, \ldots$ are three-dimensional spatial indices and $i, j, \ldots = 1, 2, 3$ are internal so(3) indices. The constraints of general relativity are

$$\mathcal{G}^i := D_a \tilde{E}^{ai} = 0, \tag{2.1}$$

$$\mathcal{C}_a := F_{ab}^i \tilde{E}^{ai} = 0, \tag{2.2}$$

$$\mathcal{H} := \epsilon^{ijk} F_{ab}^i \tilde{E}^{aj} \tilde{E}^{bk} = 0, \tag{2.3}$$

where $D_a\lambda^i = \partial_a \lambda^i + \epsilon^{ijk} A_a^j \lambda^k$ is the covariant derivative, and F_{ab}^i is its curvature.

Since the phase space variables are complex, reality conditions need to be imposed to obtain the Euclidean or Lorentzian sectors. These are $A_a^i = \bar{A}_a^i$, $E^{ai} = \bar{E}^{ai}$ for the former and $A_a^i + \bar{A}_a^i = 2\Gamma_a^i(E)$, $E^{ai} = \bar{E}^{ai}$ for the latter. The $\Gamma_a^i(E)$ is the connection for spatial indices and the bar denotes complex conjugation.

We now review the two commuting spacelike Killing field reduction of these constraints, which was presented in [18]. Working in spatial coordinates x, y, such that the Killing vector fields are $(\partial/\partial x)^a$ and $(\partial/\partial y)^a$ implies that the phase space variables will depend on only one of the three spatial coordinates. Specifically, we assume that the spatial topology is that of a three-torus so that the phase space variables depend on the time coordinate t and one angular coordinate θ. This situation corresponds to one of the Gowdy cosmological models [17]. (The other permitted spatial topologies for the Gowdy cosmologies are $S^1 \times S^2$ and S^3.)

In addition to these Killing field conditions, we set to zero some of the phase space variables as a part of the symmetry reduction:

$$\tilde{E}^{x3} = \tilde{E}^{y3} = \tilde{E}^{\theta 1} = \tilde{E}^{\theta 2} = 0,$$
$$A_x^3 = A_y^3 = A_\theta^1 = A_\theta^2 = 0. \tag{2.4}$$

These conditions may be viewed as implementing a partial gauge fixing and solution to some of the constraints. Details of these steps are given in Ref. [18]. The end result below (2.5)–(2.7) is a simplified set of first class constraints which describes a two-dimensional field theory on $S^1 \times R$ with 2 local degrees of freedom.

Renaming the remaining variables $A := A_\theta^3$, $E := \tilde{E}^{\theta 3}$, and A_α^I, $\tilde{E}^{\alpha I}$, where $\alpha, \beta, \ldots = x, y$ and $I, J, \ldots = 1, 2$, the reduced constraints are

$$G := \partial E + J = 0, \tag{2.5}$$

$$C := F_{\theta\alpha} E^{\alpha I} = 0, \tag{2.6}$$

$$\begin{aligned} H &:= -2\epsilon^{IJ} F_{\theta\alpha}^I E^{\alpha J} E + F_{\alpha\beta} E^{\alpha I} E^{\beta J} \epsilon_{IJ} \\ &= -2E E^{\alpha J} \epsilon^{IJ} \partial A_\alpha^I + 2AEK - K_\alpha^\beta K_\beta^\alpha + K^2 = 0, \end{aligned} \tag{2.7}$$

where $\partial = (\partial/\partial\theta)$,

$$K_\alpha^\beta : = A_\alpha^I E^{\beta I}, \quad K := K_\alpha^\alpha, \tag{2.8}$$

$$J_\alpha^\beta : = \epsilon^{IJ} A_\alpha^I E^{\beta J}, \quad J := J_\alpha^\alpha, \tag{2.9}$$

and $\epsilon^{12} = 1 = -\epsilon^{21}$.

The SO(3) Gauss law has been reduced to U(1), and the spatial diffeomorphism constraint to Diff(S^1). This may be seen by calculating the Poisson algebra of the constraints smeared by functions Λ, V, and the lapse N (which is a density of weight -1):

$$G(\Lambda) = \int_0^{2\pi} d\theta\, \Lambda G, \tag{2.10}$$

$$C(V) = \int_0^{2\pi} d\theta\, VC, \tag{2.11}$$

$$H(N) = \int_0^{2\pi} d\theta\, NH, \tag{2.12}$$

$$\{G(\Lambda), G(\Lambda')\} = \{G(\Lambda), H(N)\} = 0, \tag{2.13}$$

$$\{C(V), C(V')\} = C(\mathcal{L}_V V'), \tag{2.14}$$

$$\{H(N), H(N')\} = C(W) - G(AW), \tag{2.15}$$

where

$$W \equiv E^2(N\partial N' - N'\partial N). \tag{2.16}$$

This shows that C generates Diff(S^1). Also we note that this reduced system still describes a sector of general relativity due to the Poisson brackets $\{H(N), H(N')\}$, which is the reduced version of that for full general relativity in the Ashtekar variables.

The variables K_α^β and J_α^β defined above will be used below in the discussion of observables. Here we note their properties. They are invariant under the reduced Gauss law (2.5), transform as densities of weight +1 under the Diff(S^1) generated by C, and form the Poisson algebra

$$\{K_\alpha^\beta, K_\gamma^\sigma\} = \delta_\alpha^\sigma K_\gamma^\beta - \delta_\gamma^\beta K_\alpha^\sigma, \tag{2.17}$$

$$\{J_\alpha^\beta, J_\gamma^\sigma\} = -\delta_\alpha^\sigma K_\gamma^\beta + \delta_\gamma^\beta K_\alpha^\sigma, \tag{2.18}$$

$$\{K_\alpha^\beta, J_\gamma^\sigma\} = \delta_\alpha^\sigma J_\gamma^\beta - \delta_\gamma^\beta J_\alpha^\sigma. \tag{2.19}$$

This shows that K_α^β form the gl(2) Lie algebra, and hence generate gl(2) rotations on variables with indices $\alpha, \beta, \ldots = x, y$.

The following linear combinations of K_α^β form the sl(2,R) subalgebra of gl(2,R):

$$L_1 = \frac{1}{2}(K_y^x + K_x^y), \quad L_2 = \frac{1}{2}(K_x^x - K_y^y), \quad L_3 = \frac{1}{2}(K_y^x - K_x^y). \tag{2.20}$$

The Poisson brackets algebra of these is

$$\{L_i, L_j\} = C_{ij}{}^k L_k, \tag{2.21}$$

where $C_{12}{}^3 = -1$, $C_{23}{}^1 = 1$, $C_{31}{}^2 = 1$ are the sl(2,R) structure constants. The corresponding linear combinations of J_α^β are denoted by J_i, $i = 1, 2, 3$. Their Poisson brackets are

$$\{L_i, J_j\} = C_{ij}{}^k J_k, \quad \{J_i, J_j\} = -C_{ij}{}^k L_k. \tag{2.22}$$

Also

$$\{J, J_i\} = \{J, L_i\} = \{K, J_i\} = \{K, L_i\} = 0. \tag{2.23}$$

For discussing observables, it will turn out to be very convenient to replace the eight canonical phase space variables $A_\alpha^I, \tilde{E}^{\alpha I}$ by the eight Gauss law invariant variables $K_\alpha^\beta, J_\alpha^\beta$.

III. GAUGE FIXING AND THE METRIC

The Dirac observables are defined as the phase space functionals $O[A, E]$ that have vanishing Poisson brackets with all the first-class constraints of the theory. This is because the first-class constraints generate local gauge transformations via Poisson brackets. The question of finding the observables can be equally well addressed prior to, or after, partial or full gauge fixing of a first class system. Each will yield observables in terms of the phase space variables.

Assuming that variables $O[A, E]$ invariant under the kinematical Gauss law and the spatial diffeomorphism constraints have already been determined (which is relatively easy), the first case would correspond to solving for $O[A, E]$ the equation

$$\{H(N), O\} \sim 0, \tag{3.1}$$

where $\sim$ denotes equality modulo the constraints. The second amounts to solving

$$\{\tilde{H}, O\} = 0, \tag{3.2}$$

where the last equality is strong and $\tilde{H}$ is a suitably gauge-fixed Hamiltonian constraint. The second procedure will be followed here since, with a particular gauge choice to be described in this section, the Hamiltonian evolution equations can be put in a rather simple form.

Full gauge fixing using the Ashtekar variables requires a careful consideration of the reality conditions on the phase space coordinate A_a^i. This is because the (complex) phase space variables depend on real coordinates. For conventional gauge fixing, where some functions of the phase space variables are chosen as the coordinates, real functions must be chosen. But since the constraints themselves are complex, two real conditions must be imposed for complete gauge fixing. Here we will gauge fix the *complex* theory by requiring that certain (complex) functions of the phase space variables vanish. This results in (complex) gauge-fixed evolution equations and second-class constraints. The reality conditions are discussed below, where the metric resulting from the gauge fixing is compared with a standard metric for this reduction, and in Sec. V, where the observables are obtained.

We start by fixing the Gauss law (2.5) by imposing the gauge-fixing condition $A = 0$. This gauge condition is second class with (2.5):

$$\{G(\Lambda), A(t, \theta)\} = \partial\Lambda(t, \theta). \tag{3.3}$$

The standard procedure now involves imposing the Gauss law strongly and then solving it [19,20]. This gives

$$E = c - \int^\theta d\theta' J(\theta'), \tag{3.4}$$

where c is an arbitrary integration constant. The canonical variable E can now be replaced everywhere by the right-hand side of (3.4), which is a function of the remaining phase space variables. We will see below, after further gauge fixing, that E eventually gets fixed to be the (nondynamical) constant c. (We note that the $A = 0$ gauge condition is analagous to the axial gauge for electromagnetism. It is in fact the axial gauge for two-dimensional electromagnetism. The difference with the present case is that our Gauss law is $\partial E + J = 0$ rather than just $\partial E = 0$.)

Substituting (3.4) and the gauge-fixing condition (which are the two strong second-class constraints), into the diffeomorphism and Hamiltonian constraints (2.6) and (2.7) gives

$$H = -2\left(c - \int^{\theta} d\theta' \; J(\theta')\right) E^{\alpha J}\epsilon^{IJ}\partial A^I_\alpha - K^\beta_\alpha K^\alpha_\beta + K^2, \tag{3.5}$$

$$C = E^{\alpha I}\partial A^I_\alpha. \tag{3.6}$$

These remaining constraints are still first class. In particular, (3.5) satisfies the Poisson brackets relation

$$\{H(N), H(N')\} = C(W), \tag{3.7}$$

with W given by (2.16), which is the usual Poisson bracket of the Hamiltonian constraint with itself. Thus (3.5) and (3.6) on the $A^I_\alpha, \tilde{E}^{\alpha I}$ phase space still describe general relativity with 2 local degrees of freedom.

We now work with the eight Gauss-law-invariant densities L_i, J_i and K, J introduced in the preceding section instead of the eight remaining phase space variables $A^I_\alpha, \tilde{E}^{\alpha I}$. The evolution equations $\dot{F} = \{F, H(N)\}$ with H from (3.5) for these variables are

$$\dot{L}_i = -2\partial\left[N\left(c - \int^{\theta} d\theta' \; J(\theta')\right) J_i\right], \tag{3.8}$$

$$\dot{J}_i = 2\partial\left[N\left(c - \int^{\theta} d\theta' \; J(\theta')\right) L_i\right] + 4NC_i^{\;jk} J_j L_k, \tag{3.9}$$

and

$$\dot{J} = 2\partial\left[N\left(c - \int^{\theta} d\theta' \; J(\theta')\right) K\right], \tag{3.10}$$

$$\dot{K} = -2\partial\left[N\left(c - \int^{\theta} d\theta' \; J(\theta')\right) J\right]. \tag{3.11}$$

We would now like to gauge fix the Hamiltonian and diffeomorphism constraints (3.5) and (3.6) in a way so as to simply these evolutions equations.

The normal gauge-fixing procedure for the Hamiltonian constraint [20] for closed universes involves imposing a time-dependent condition, such as setting $t = f(A, E)$, where f is some function of the phase space variables that is second class with the Hamiltonian constraint. The requirement that this condition be preserved under time evolution then gives an equation for the lapse function:

$$\dot{t} = 1 = \{f, H(N)\}. \tag{3.12}$$

The momentum conjugate to f is then identified as the negative of the *reduced* Hamiltonian, and is obtained by solving the strongly imposed Hamiltonian constraint. (An example of such a gauge-fixing is given at the end of this section.) The reduced Hamiltonian is then used to obtain the gauge-fixed evolution equations for the physical degrees of freedom.

For our purposes, there are two points to note in this procedure. The first is that for *full* general relativity (no symmetry reductions), if a time-dependent gauge condition is not chosen for closed spacelike hypersurfaces, then, with, for example, the gauge condition $f(A, E) = 0$, (3.12) becomes

$$0 = \{f, H(N)\}, \tag{3.13}$$

and *the only solution* for the lapse function is $N = 0$. This is not suitable because it gives no evolution. (An alternative way of seeing this is given in Ref. [20].) We will see below that for the two Killing field reduced equations under discussion, $N = 0$ *is not* the only solution for N. The second is that it is not necessary to first obtain a reduced Hamiltonian and *then* get the gauge-fixed evolution equations via Hamilton's equations.[1] Alternatively, one may obtain the gauge-fixed evolution equations by substituting the lapse function derived from (3.13), and the gauge-fixing condition, directly into the evolution equations [such as (3.8)–(3.11)]. If this alternative procedure is followed, a reduced Hamiltonian that gives the gauge-fixed evolution equations is not directly identified. (This in fact is exactly what is done when gauge-fixing conditions are imposed in the second-order Einstein equations: the conditions are directly substituted into the evolution and constraint equations.)

We now give a gauge-fixing procedure for the Hamiltonian constraint. In the process we will see that a time-independent gauge-fixing condition *is* possible for the two Killing field reduced system, which does not lead to $N = 0$. (It should be stressed again that this is not possible for the *full* Einstein equations, that is, the equations without any symmetry reductions.)

The gauge condition $J = 0$ is second class with the Hamiltonian constraint (3.5). Requiring that this condition be preserved in time gives an equation for the lapse function (3.10):

$$\dot{J} = 0 = 2c\partial(NK). \tag{3.14}$$

This gives for the lapse $N(\theta, t) = 1/K(\theta, t)$.

To fix the diffeomorphism constraint, we may choose the θ coordinate such that a specific real density on the circle is a constant. We choose ReK=const and Im$K = 0$. This is second class with the diffeomorphism constraint, and it fixes the shift function to be an arbitrary constant C. From (3.11), we see that this condition is also preserved under evolution as required for consistency.

The evolution equations (3.8) and (3.9) for the remaining variables, the six L_i, J_i, with the specific choices $N = 1/4$, $c = 2$ (designed to put the equations into a simple form), become

[1]The reason that a reduced Hamiltonian is normally obtained first is that one has a view to quantization. The goal in the reduction is therefore to identify the true physical degrees of freedom and the reduced Hamiltonian as a function of them. The Hamiltonian may then be converted into an operator and the Schrödinger equation written down.

$$\dot{L}_i + J_i' = 0, \tag{3.15}$$

$$\dot{J}_i - L_i' + C_i{}^{jk} L_j J_k = 0, \tag{3.16}$$

where $' \equiv \partial \equiv \partial/\partial\theta$. These, together with the strongly imposed Hamiltonian and diffeomorphism constraints,

$$\epsilon^{IJ} E^{\alpha J} \partial A^I_\alpha + L_1^2 + L_2^2 - L_3^2 = 0, \tag{3.17}$$
$$E^{\alpha I} \partial A^I_\alpha = 0, \tag{3.18}$$

form the fully gauge-fixed set of two Killing field reduced complex Einstein equations. There are $6 - 2 = 4$ local phase space degrees of freedom. We note that these are written in terms of the original phase space variables, so that Poisson brackets may still be calculated using the fundamental $(A^I_\alpha, E^{\alpha I})$ brackets. We note also that the gauge fixing has reduced the gl(2,R) Casimir term in the Hamiltonian constraint to the sl(2,R) Casimir term in (3.17).

We emphasize again that the main purpose of the gauge fixing was not to get an explicitly reduced Hamiltonian in terms of the 2 physical degrees of freedom, but rather to look at the full evolution equations (3.8) and (3.9) in a convenient gauge, which is useful for obtaining the conserved quantities. The $J = 0$ gauge is very convenient for this. One can, however, obtain a nonvanishing reduced Hamiltonian as a function of L_i, J_i that leads to the evolution equations (3.15) and (3.16). It is the Hamiltonian for the Sl(2,R) chiral model.

Since the gauge-fixed evolution equations (3.15) and (3.16) involve only J_i, L_i, the conserved charges will depend only on these. It is therefore important to check that the charges commute with the second class constraints (3.17) and (3.18). The commutation with the strong Hamiltonian constraint is guaranteed because the variables J_i, L_i commute with J, K (which are the variables fixed in the gauge choice). To see this we only need to note that

$$\dot{L}_i = \{L_i, H(N)\}|_{\substack{J=0 \\ K=\text{const}}} = \{L_i, H(N)|_{\substack{J=0 \\ K=\text{const}}}\}, \tag{3.19}$$

with the same equation holding for J_i. This is another reason why it appears natural to separate the phase space variables into sl(2,R) variables J_i, L_i, with gauge conditions imposed on the traces J and K. As we will see below, the charges also commute with the diffeomorphism constraint (3.18) by construction, since the charges will turn out to be made from integrals of densities on the circle.

We will not solve the second-class constraints explicitly, since the goal is only to obtain the conserved quantities. The second-class constraints imply that there are two relations among the six J_i, L_i. In principle these can be substituted into the conserved quantities to rewrite them in terms of four independent reduced variables.

For comparison with the usual metric variables, it is useful to see what form of the metric arises from the gauge choices made above. A standard line element with two commuting spacelike Killing fields is of the form [17]

$$ds^2 = e^{2F}(-dt^2 + d\theta^2) + g_{\alpha\beta} dx^\alpha dx^\beta, \tag{3.20}$$

where the four functions F and $g_{\alpha\beta}$ are four functions of t, θ only. On the other hand, the gauge choices made above lead to the line element

$$ds^2 = -\left(\frac{1}{16} + \frac{1}{2\sqrt{q}}C^2\right) dt^2 + \frac{C}{\sqrt{q}} dt d\theta + \frac{1}{2\sqrt{q}} d\theta^2 + \frac{2}{\sqrt{q}} q_{\alpha\beta} dx^\alpha dx^\beta, \tag{3.21}$$

where $q_{\alpha\beta}$ is the matrix inverse of $\tilde{E}^{\alpha I}\tilde{E}^{\beta I}$, $q = \det q_{\alpha\beta}$, and C is a constant (the shift).

In arriving at (3.21), E^{ai} has been fixed to be real (reality condition), and the lapse and shift chosen to be real constants. Note that while the reality conditions on A^i_a have not been imposed, this does not affect the general form (3.21) of the Lorentzian metric that will result.

That these two line elements are related by a coordinate transformation and a gauge choice may be seen in the following way. Since any two-dimensional metric is conformally flat, there is a coordinate transformation of t, θ that puts the t-θ part of the metric (3.21) into conformally flat form, at least locally. The conformal factor so identified will be a function of the three functions $q_{\alpha\beta}$. The identification with the standard form is made complete by setting e^{2F} in (3.20) to be this factor. This is the gauge choice. (The explicit form of the coordinate transformation is given by a set of coupled partial differential equations, solutions of which may be shown to exist [21].)

We now point out another way of viewing the derivation of the simplified equations (3.15) and (3.16) from the full equations (3.8)–(3.11). This is to simply consider the former as an explicit by hand simplification of the latter by setting $J = 0$, $\dot{J} = 0$, which imply constant N and K. In this way the simplified equations would be viewed as a particular subset of the two Killing field reduced Einstein equations, and all the analysis in the following sections would amount to the derivation of conservation laws for this subset. However, as we have seen above, there *is* a canonical gauge-fixing procedure which leads to the simplified equations (3.15) and (3.16), and so they still represent the full two Killing field reduced sector under consideration.

We now note an alternative natural gauge fixing which may also be useful for this system but will not be used in this paper. The Hamiltonian constraint (2.7) contains the product AE, and E transforms like a scalar under the reduced diffeomorphism constraint (2.6). This suggests the (partial) gauge fixing $\mathrm{Re}E = t$, $\mathrm{Im}E = 0$, which gives $H_R := -A$ as the (complex) reduced Hamiltonian. Substituting this gauge condition into (2.5)–(2.7) gives the first-class constraints

$$J = 0, \tag{3.22}$$
$$E^{\alpha I} \partial A^I_\alpha = 0, \tag{3.23}$$

and the time-dependent reduced Hamiltonian

$$H_R = -\frac{1}{K} E^{\alpha J} \epsilon^{IJ} \partial A^I_\alpha + \frac{1}{2Kt}(K^2 - K^\beta_\alpha K^\alpha_\beta). \tag{3.24}$$

The time dependence in H_R is associated only with the ultralocal part, which is also the gl(2,R) Casimir invariant. This suggests that for small times the ultralocal piece dominates the dynamics and that a perturbation theory in t may be possible. The reality conditions on the A's still need to be applied.

IV. EVOLUTION EQUATIONS AS A ZERO-CURVATURE CONDITION

The evolution equations (3.15) and (3.16) derived in the preceding section can be rewritten in a compact form using the sl(2,R) matrix generators

$$g_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.1}$$

$$g_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which satisfy the relations $[g_i, g_j] = C_{ij}{}^k g_k$ and $g_i g_j = \frac{1}{2}C_{ij}{}^k g_k$. Defining the matrices

$$A_0 := L_i g_i, \qquad A_1 := J_i g_i, \tag{4.2}$$

the evolution equations (3.15) and (3.16) become

$$\partial_0 A_0 + \partial_1 A_1 = 0, \tag{4.3}$$

$$\partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = 0. \tag{4.4}$$

Equations (4.3) and (4.4) are the first order form of the SL(2,R) chiral model field equations.

The two evolution equations, (4.3) and (4.4), may be rewritten as a single equation in the following way. Define, for a real parameter λ,

$$a_0 := \frac{1}{1+\lambda^2}(A_0 - \lambda A_1), \quad a_1 := \frac{1}{1+\lambda^2}(\lambda A_0 + A_1). \tag{4.5}$$

Then Eqs. (4.3) and (4.4) follow from the single "zero-curvature" equation

$$\dot{a_1} - a_0' + [a_0, a_1] = 0. \tag{4.6}$$

This equation, together with the strong constraints (3.17) and (3.18), form the two spacelike commuting Killing field reduction. The dynamical equation (4.6) is used in the following section to obtain the conserved charges.

V. OBSERVABLES

The field equations for all the known two-dimensional integrable models have zero curvature formulations analagous to that given in the last section. This is a direct consequence of the existence of two distinct symplectic forms on the phase spaces of the models [6], which is also the geometric way of viewing the Lax pair formulation. Another consequence of the zero-curvature formulation is a procedure for generating an infinite number of conserved charges. We now apply this to the dynamical equation (4.6) arising from the two Killing field reduction. The resulting observables will be for complex gravity and the reality conditions on them will be discussed at the end of the section.

The transfer matrix used in the study of two-dimensional models is analagous to the Wilson loop. For the present case, it is the path ordered exponential associated with the matrix a_1:

$$\begin{aligned} U[A_0, A_1](0,\theta) : &= \lim_{\substack{N\to\infty \\ \Delta\theta\to 0}} \prod_{i=0}^{N} [1 + a_1(\theta_i)\Delta\theta] \\ &\equiv P\exp\left(\int_0^\theta a_1(A_0, A_1, \lambda)\, d\theta'\right). \end{aligned} \tag{5.1}$$

The trace of the transfer matrix is preserved under time evolution as may be seen by deriving its equation of motion using Eq. (4.6). We note first that

$$U'(0,\theta) = U(0,\theta)a_1(\theta), \qquad U'(\theta, 2\pi) = -a_1(\theta)U(\theta, 2\pi). \tag{5.2}$$

The time derivative of the first gives

$$\begin{aligned} \dot{U}'(0,\theta) &= \dot{U}(0,\theta)a_1 + U(0,\theta)\dot{a}_1 \\ &= \dot{U}(0,\theta)a_1 + U(0,\theta)(a_0' - [a_0, a_1]), \end{aligned}$$

which may be rewritten as

$$[\dot{U}(0,\theta) - U(0,\theta)a_0]' = [\dot{U}(0,\theta) - U(0,\theta)a_0]a_1. \tag{5.3}$$

Thus, since $\dot{U}(0,\theta) - U(0,\theta)a_0$ satisfies the same equation as $U(0,\theta)$, we get the equation of motion

$$\dot{U}(0,\theta) = U(0,\theta)a_0(\theta) - a_0(0)U(0,\theta). \tag{5.4}$$

From this it follows that

$$M[A_0, A_1](\lambda) := \mathrm{Tr}U(0, 2\pi) \tag{5.5}$$

is conserved in time. The conservation of this trace follows in basically the same way as the conservation of the Wilson loop observable when there is a zero-curvature constraint on the phase space, such as in 2+1 gravity [13]. That M is a spatial diffeomorphism invariant follows from noting that a_1 transforms like a density under the Diff(S^1) generated by (3.6):

$$\{C(V), a_1\} = -\partial(V a_1), \tag{5.6}$$

from which it follows that $\{C(V), M\} = 0$.

Expanding M in a power series in λ gives explicitly the phase space observables, which are the coefficients of powers of λ: The first three observables are

$$Q^0 := M|_{\lambda=0} = \mathrm{Tr} P \exp\left(\int_0^{2\pi} d\theta\ A_1\right) =: \mathrm{Tr} V(0,2\pi), \tag{5.7}$$

$$Q^1 := \frac{\partial M}{\partial \lambda}\bigg|_{\lambda=0} = \int_0^{2\pi} d\theta\ \ \mathrm{Tr}[V(0,\theta)A_0(\theta)V(\theta,2\pi)], \tag{5.8}$$

and

$$\begin{aligned}\frac{\partial^2 M}{\partial \lambda^2}\bigg|_{\lambda=0} &= -2\int_0^{2\pi} d\theta \mathrm{Tr}[V(0,\theta)A_1(\theta)V(\theta,2\pi)] \\ &\quad + \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2\ \Theta(\theta_2-\theta_1) \\ &\quad \times \mathrm{Tr}[V(0,\theta_1)A_0(\theta_1)V(\theta_1,\theta_2)A_0(\theta_2) \\ &\quad \times V(\theta_2,2\pi)],\end{aligned} \tag{5.9}$$

where $\Theta(\theta-\theta') = 1, \theta \geq \theta'$ and zero otherwise. It is straightforward to verify directly the conservation of these functionals using the equations of motion (4.3) and (4.4).

The structure of the general observable can now be seen and we can write down the observable with n insertions of A_0 in the holonomies V:

$$Q^n := \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \Theta(\theta_n-\theta_{n-1})\cdots\Theta(\theta_2-\theta_1)\mathrm{Tr}[V(0,\theta_1)A_0(\theta_1)V(\theta_1,\theta_2)A_0(\theta_2)\cdots A_0(\theta_n)V(\theta_n,2\pi)]. \tag{5.10}$$

This has a remarkable resemblance to the T variables used in 3+1 gravity [12]:

$$T^{a_1\cdots a_n}[A_a^i,\tilde{E}^{ai}](x_1,\ldots,x_n;\alpha) := \mathrm{Tr}[U_\alpha(x_0,x_1)\tilde{E}^{a_1}(x_1)U_\alpha(x_1,x_2)\cdots\tilde{E}^{a_n}(x_n)U_\alpha(x_n,x_0)], \tag{5.11}$$

where the holonomies U_α are based on the loop, α are made from Ashtekar's connection A_a^i, and the insertions in the product of holonomies are the conjugate momenta $\tilde{E}^{ai}$ instead of A_0. The other difference is that in Eq. (5.10) there is an integration over all the point insertions of A_0 [which gives invariance under the remaining spatial diffeomorphisms Diff(S^1) in the present reduction].

Another set of observables is obtained by looking at the first term in (5.9) where there is an insertion of A_1 in the holonomies instead of A_0. The general observables of this type is similar to (5.10) but with n insertions of A_1:

$$P^n := \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \Theta(\theta_n-\theta_{n-1})\cdots\Theta(\theta_2-\theta_1)\mathrm{Tr}[V(0,\theta_1)A_1(\theta_1)V(\theta_1,\theta_2)A_1(\theta_2)\cdots A_1(\theta_n)V(\theta_n,2\pi)]. \tag{5.12}$$

The Poisson algebra of the observables (5.10)–(5.12) may be calculated using the trace identity

$$\mathrm{Tr}[Xg^i]\mathrm{Tr}[Yg^j] = C_k{}^{ij}(\mathrm{Tr}[Xg^kY] - \mathrm{Tr}[Yg^kX]), \tag{5.13}$$

for SL(2,R) matrices X, Y and 2×2 generators g^i. We find

$$\{P^m,P^n\} = 0, \tag{5.14}$$

$$\{Q^m,Q^n\} \sim Q^{m+n-1} + Q^{m+n-1}, \quad m,n>1, \tag{5.15}$$

$$\{Q^0,Q^m\} \sim Q^{m+1}. \tag{5.16}$$

There is another method for generating conserved charges for two-dimensional chiral models [22], which can be applied here to generate observables. This is useful for comparison with the above procedure. Also, as discussed below, the resulting observables for the Killing field reduction give a solution-generating technique which may be viewed as the Hamiltonian analog of Geroch method [1]. This procedure for obtaining observables has also been applied to self-dual gravity [23,24].

The starting point is the dynamical equations (4.3) and (4.4). We note that (4.3) is already like a conservation law and so the first conserved charge is

$$q^{(1)} = \int_0^{2\pi} d\theta\ A_0 = \int_0^{2\pi} d\theta\ L_i g_i, \tag{5.17}$$

which gives the three sl(2,R) charges

$$q_i^{(1)} = \int_0^{2\pi} d\theta\ L_i. \tag{5.18}$$

These observables were obtained earlier in [18].

The current $\mathcal{J}_\mu^{(1)} := A_\mu$ $(\mu,\nu,\ldots = 0,1)$ is conserved so there exists a (matrix) function $f^{(1)}(t,\theta)$ such that

$$\mathcal{J}_\mu^{(1)} = \epsilon_\mu{}^\nu \partial_\nu f^{(1)}. \tag{5.19}$$

We now define the second current by

$$\mathcal{J}_\mu^{(2)} := D_\mu f^{(1)} \equiv \partial_\mu f^{(1)} + A_\mu f^{(1)}. \tag{5.20}$$

With this definition of a derivative operator, the equation of motion (4.4) may be rewritten as $[D_0,D_1]=0$. The conservation of $\mathcal{J}_\mu^{(2)}$ is easy to show:

$$\delta^{\mu\nu}\partial_\nu \mathcal{J}^{(2)}_\mu = \delta^{\mu\nu}\partial_\nu D_\mu f^{(1)} = \delta^{\mu\nu} D_\mu \partial_\nu f^{(1)}$$
$$= \delta^{\mu\nu}\epsilon^{\mu\alpha} D_\mu \mathcal{J}^{(1)}_\alpha = \delta^{\mu\nu}\epsilon^{\mu\alpha} D_\mu D_\alpha f^{(0)} = 0, \tag{5.21}$$

where the last equality follows because $\mathcal{J}^{(1)}_\mu = D_\mu f^{(0)} = A_\mu$, where $f^{(0)}$ is the identity matrix, and $[D_0, D_1] = 0$ by the equation of motion (4.4). This procedure generalizes, and it is straightforward to give an inductive proof that $\mathcal{J}^{(n+1)} := D_\mu f^{(n)}$ is conserved, assuming $\mathcal{J}^{(n)}_\mu$ is conserved. The observables are

$$q^{(n)} := \int_0^{2\pi} d\theta \; \mathcal{J}^{(n)}_0. \tag{5.22}$$

The second conserved charge is

$$q^{(2)} : = \int_0^{2\pi} d\theta \; D_0 f^{(1)}(\theta, t)$$
$$= \int_0^{2\pi} d\theta \left(-A_1(\theta,t) + A_0(\theta,t) \int^\theta d\theta' \; A_0(\theta',t) \right). \tag{5.23}$$

In terms of the sl(2,R) phase space functions this is

$$q^{(2)}_i = \int_0^{2\pi} d\theta \left(-J_i + \frac{1}{2} C_i{}^{jk} L_j \int^\theta d\theta' \; L_k \right). \tag{5.24}$$

The conservation of this may be checked directly using (3.15) and (3.16):

$$\dot{q}^{(2)}_i = \int_0^{2\pi} d\theta \left[-L'_i + C_i{}^{jk} L_j J_k - \frac{1}{2} C_i{}^{jk} \left(J'_j \int^\theta d\theta' \; L_k + L_j \int^\theta d\theta' \; J'_k \right) \right]$$
$$= \int_0^{2\pi} d\theta \left(C_i{}^{jk} L_j J_k + \frac{1}{2} C_i{}^{jk} (J_j L_k - L_j J_k) \right) = 0. \tag{5.25}$$

The Poisson brackets of the first two charges is

$$\{q^{(1)}_i, q^{(2)}_j\} = C_{ij}{}^k q^{(2)}_k. \tag{5.26}$$

Since $q^{(1)}_i$ form an sl(2,R)$\sim$so(2,1) Lie algebra, it follows that all the observables $q^{(n)}_i$ with sl(2,R) indices will have the Poisson algebra

$$\{q^{(1)}_i, q^{(n)}_j\} = C_{ij}{}^k q^{(n)}_k. \tag{5.27}$$

The Poisson algebra of the higher observables $q^{(n)}_i$ with themselves is more involved and there are in general, nonlinear combinations of observables on the right-hand sides. We note that given the first two observables $q^{(1)}_i, q^{(2)}_j$, the remaining observables may also be generated by taking Poisson brackets of these with themselves. Another feature of this set is that they are sl(2,R) Lie algebra valued whereas the first set obtained above, using M (5.5), are traces of SL(2,R) group elements.

In the steps above, we have obtained a gauge-fixed version of complex two Killing field reduced gravity, and given two methods for obtaining observables. The observables are for the complexified theory and reality conditions must be imposed on them to obtain their restrictions on the Euclidean or Lorentzian sections.

The restriction to the Euclidean section involves just setting the L_i, J_i to be real. The Lorentzian restriction requires setting the triads to be real, and imposing $A^I_\alpha + \bar{A}^I_\alpha = 2\Gamma^I_\alpha(E)$. This reality condition implies that the complex conjugate of the observables are also observables. Therefore when the triads are set to be real, if $Q[A,E]$ is an observable, so is $Q[\bar{A},E]$. Thus $Q[A,E] + Q[\bar{A},E]$ is a *real* observable for the *complex* theory. The real observables for the Lorentzian section in terms of the original phase space variables may then be obtained as

$$(Q[A,E] + Q[\bar{A},E])|_{\bar{A}=2\Gamma - A}. \tag{5.28}$$

The observables found in this section are expressed in terms of the spatial triad $E^{\alpha I}$ and the Ashtekar connection A^I_α. They may be reexpressed in terms of the triad and the extrinsic curvature of the ADM phase space variables by recalling that [9]

$$A^I_\alpha = \Gamma^I_\alpha(E) + i\mathcal{K}^I_\alpha, \tag{5.29}$$

where $\mathcal{K}^I_\alpha = \mathcal{K}_{\alpha\beta} E^{\beta I}$ and $\mathcal{K}_{\alpha\beta}$ is the extrinsic curvature. The sl(2,R) variables K_i, J_i used above are therefore contractions on the internal indices of (5.29) with $E^{\alpha I}$, using δ^{IJ} or ϵ^{IJ}: $K \sim E \circ (\Gamma + i\mathcal{K})$ and $J \sim E \times (\Gamma + i\mathcal{K})$. Thus the observables may be easily written in terms of the triad ADM variables.

We do not know the physical or spacetime geometric interpretation of the observables presented above. This in fact is also an open question for integrable systems such as the KdV equation, where the physical interpretations of most of the conserved quantities are not known, except for the few associated with the (manifest) Galilean invariance of this equation [6]. In general, it is easy to find physical interpretations of observables associated with *manifest* symmetries of equations. The task is much more difficult for infinite dimensional *hidden* symmetries, as is the case here.

VI. SOLUTION-GENERATING TECHNIQUE

In this section we discuss the relation between the second set of observables obtained above and the solution-generating technique for spacetimes with two commuting Killing fields given by Geroch [1]. We note only the general features of the method, which are unchanged by the reality conditions.

A solution of the Einstein equation with two commuting spacelike Killing fields is a phase space trajectory labeled by values of the conserved quantities $q^{(n)}_i$. A new solution can be generated from a given one by considering the Hamiltonian flow of the phase space variables L_i, J_i generated by the observables $q^{(n)}_i$. This flow may be parametrized by a parameter s, and specified by giving three "shift" functions $F^i(s)$:

$$\frac{dL_i(t,\theta;s)}{ds} = \{L_i(t,\theta;s), F^k(s)q_k^{(n)}\},$$
$$\frac{dJ_i(t,\theta;s)}{ds} = \{J_i(t,\theta;s), F^k(s)q_k^{(n)}\}. \qquad (6.1)$$

Integration of these equations, with the initial condition that $L_i(t,\theta;s=0), J_i(t,\theta;s=0)$ lie on the given solution, gives the values of these variables on the new solution at, say, $s = 1$.

We therefore see that a new exact solution of the Einstein equations may be constructed from a given one by specifying a curve $\gamma(s)$ $(0 \leq s \leq 1)$ in a three-dimensional vector space with tangent vector $F^i(s)$, and with $\gamma(0)$ at the origin. But these are precisely the conditions given by Geroch for generating new solutions from a given one [1]. In particular, the intermediate equations (6.1) that need to be integrated as a part of the procedure are of exactly the same form as those present in Ref. [1]. Thus the infinite number of sl(2,R) observables obtained in the preceding section may be viewed as the phase space analogs of the generators of Geroch's transformation.

VII. DISCUSSION

The main result given in this paper is the explicit construction of an infinite number of phase space observables for spacetimes with two commuting spacelike Killing vector fields. The previous studies of this reduction of the Einstein equations, in particular Geroch's work, provided strong indications of the existence of such observables.

Our approach involved rewriting the Hamiltonian evolution equations using the Ashtekar variables, and then choosing a particular gauge fixing which allowed these equations to be rewritten as those of the SL(2,R) chiral model [(4.3) and (4.4)]. From this form of the equations, two known methods were used to obtain the observables. The first made use of the conservation of the trace of the monodromy matrix M (5.5), which acts as the generating functional for the observables. The second made use of a recursive procedure given by Brezin *et al.* [22] to calculate nonlocal conserved charges in two-dimensional models.

One set of observables obtained from the monodromy matrix have a structure similar to that of the loop observables that have been used to study the quantization of full 3+1 gravity [12]. This is interesting and suggests that it should be possible to obtain the quantized two-Killing-field reduction directly from the the full 3+1 observables.

The second set have an infinite-dimensional algebra, which does not appear to have a simple form. However, as discussed in Sec. VI, these observables can be used to give a solution-generating method for this sector of the Einstein equations. In particular, the solution-generating procedure has exactly the same ingredients as Geroch's, which indicates that ours is the phase space analog of his.

One of main reasons for addressing the observables problem is that it provides one way to address the quantization issue. For generally covariant theories the observables are also the fully gauge-invariant phase space variables. A quantum theory may be constructed by finding a representation of the Poisson algebra of a complete set of classical observables. From the results given above, the second set of observables $q_i^{(n)}$ may be suitable for this provided their Poisson algebra can be put into a more manageable form. Previous work [4] on a simpler method of obtaining the Geroch procedure provides a hint that this Poisson algebra may actually be an SL(2,R) Kac-Moody (affine) algebra. The task is then to see if the $q_i^{(n)}$ can be replaced by some functions of them such that the Poisson algebra simplifies to this. This is under investigation.

A further question regarding the observables that has not been addressed is the question of completeness: Can any invariant phase space variable be expressed as a sum of products of the observables obtained here? In particular, is there any relation between the observables obtained using the two different methods? These questions are important for studying quantization, which has been previously studied in the loop space representation in Ref. [18]. It was found that there are an infinite number of observables in the quantum theory that form a gl(2) loop algebra. However, surprisingly the classical counterparts of these observables was not known. It is likely that the observables given here form a subset of these quantum observables, and the correspondence merits further study.

A question related to completeness of the observables is that of integrability. While we have given an infinite set of conserved quantities, we have not shown that there are an infinite number of sums of products of them that are in involution. Therefore integrability of this system remains to be shown.

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