Fully electrified Neugebauer spacetimes

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Generalizing a method presented in an earlier paper, we express the complex potentials $\mathcal{E}$ and Φ of all stationary axisymmetric electrovac spacetimes that correspond to axis data of the form $\mathcal{E}(z,0) = (U - W)/(U + W)$, $\Phi(z,0) = V/(U + W)$, where $U = z^2 + U_1 z + U_2$, $V = V_1 z + V_2$, $W = W_1 z + W_2$, in terms of the complex parameters U_1, V_1, W_1, U_2, V_2, and W_2, that are directly associated with the various multipole moments.

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I. INTRODUCTION

This is the second paper of a short series of papers, all of which are concerned with those stationary axisymmetric solutions of the Einstein and the Einstein-Maxwell equations that are characterized by axis data of the form

$$\mathcal{E} = \frac{U - W}{U + W}, \quad \Phi = \frac{V}{U + W} , \tag{1.1}$$

with

$$U = \sum_{a=0}^{n} U_a z^{n-a} , \tag{1.2a}$$

$$V = \sum_{a=1}^{n} V_a z^{n-a} , \tag{1.2b}$$

$$W = \sum_{a=1}^{n} W_a z^{n-a} , \tag{1.2c}$$

where $\mathcal{E}$ and Φ are the complex potentials of Ernst, z is the Weyl canonical coordinate, and the coefficients in the polynomials are complex constants.

In the first paper [1] of this series, to which we shall henceforth refer as Ref. [1], we found that it was extraordinarily simple to construct the general solution of the vacuum problem ($V = 0$) for the case $n = 2$. That is, we described a procedure that allows one to express the complex potential $\mathcal{E}$ and the metrical fields ω and γ of an exact vacuum solution ($V = 0$) directly in terms of the complex parameters $U_a, W_a (a = 1, \ldots, n)$ with $U_0 = 1$. In particular, we showed that the resulting family of solutions contained as a special case the vacuum limit of an electrovac solution published recently by Manko *et al.* [2].

In the present paper, we undertake the extension of the methods of Ref. [1] to electrovac fields ($V \neq 0$). This extension is far from trivial, and to construct it we found that it was necessary to attain first an understanding of why everything worked out so handily in the vacuum case.

An additional complexity of the electrovac problem arises from the fact that, while all the vacuum solutions could be constructed from Minkowski space by applying successive quadruple-Neugebauer Bäcklund transformations [3], or double-Harrison Bäcklund transformations [4], there was no single known Kinnersley-Chitre transformation that could yield all the electrovac solutions with axis data of the type we are considering. This forced us to consider what we call a *complexified Cosgrove transformation*, which will be defined in this paper.

II. THE AXIS RELATION

Using the Hauser-Ernst axis relation [5,6], one can always identify a Kinnersley-Chitre transformation that will, in principle, produce from Minkowski space a spacetime with any specified axis data. Of course, it may be difficult to solve in closed form the associated homogeneous Hilbert problem (HHP).

A. The vacuum case

Let us first review the application of the axis relation within the context of vacuum spacetimes, where we know that the quadruple-Neugebauer (double-Harrison) transformation does the job. It will suffice to consider the generation of the Kerr metric, for which

$$\mathcal{E} = 1 - \frac{2m}{r - ia\cos\theta} . \tag{2.1}$$

On the axis, where $\theta = 0$, we have $\cos\theta = 1$. On the other hand, the Weyl canonical coordinates z, ρ are given by

$$\rho^2 = (r^2 + a^2 - 2mr)\sin^2\theta, z = (r - m)\cos\theta . \tag{2.2}$$

Therefore, the axis data for the Kerr metric assumes the form

$$\mathcal{E}(z,0) = \frac{z - m - ia}{z + m - ia} , \tag{2.3}$$

i.e., $U(z,0) = z - ia$ and $W(z,0) = m$.

Now, the axis relation says [5]

$$-i\mathcal{E}(\tau,0)=\frac{-i\mathcal{E}^{(0)}(\tau,0)v^{UL}(\tau)+v^{UR}(\tau)}{-i\mathcal{E}^{(0)}(\tau,0)v^{LL}(\tau)+v^{LR}(\tau)}\,, \tag{2.4}$$

where $\mathcal{E}^{(0)}$ is the complex potential of the seed metric. For Minkowski space, $\mathcal{E}^{(0)}=1$. Plugging in $\mathcal{E}(\tau,0)$ for the Kerr metric, we are tempted to try

$$v(\tau)=\begin{pmatrix}\tau-m & -a\\ a & \tau+m\end{pmatrix}.$$

However, $v(\tau)$ should be a matrix of the group SU(1,1)= SL$(2,R)$. Since the determinant has the value

$$\det v(\tau)=\tau^2-m^2+a^2\,,$$

we should divide by the square root of this determinant, and select

$$v(\tau)=\frac{1}{\sqrt{\tau^2-m^2+a^2}}\begin{pmatrix}\tau-m & -a\\ a & \tau+m\end{pmatrix}. \tag{2.5}$$

Suppose, for the moment, that $a^2<m^2$. Then, using the identities

$$\begin{aligned}\tau&=\tfrac{1}{2}[(\tau+\sqrt{m^2-a^2})+(\tau-\sqrt{m^2-a^2})]\,,\\ 1&=\tfrac{1}{2}[(\tau+\sqrt{m^2-a^2})-(\tau-\sqrt{m^2-a^2})]\,,\end{aligned} \tag{2.6}$$

we can cast our expression for $v(\tau)$ into the form

$$\begin{aligned}v(\tau)=&\frac{1}{2}(1+J)\sqrt{\frac{\tau+\sqrt{m^2-a^2}}{\tau-\sqrt{m^2-a^2}}}\\ &+\frac{1}{2}(I-J)\sqrt{\frac{\tau-\sqrt{m^2-a^2}}{\tau+\sqrt{m^2-a^2}}}\,,\end{aligned} \tag{2.7}$$

where the matrix

$$J:=\frac{1}{\sqrt{m^2-a^2}}\begin{pmatrix}-m & -a\\ a & m\end{pmatrix} \tag{2.8}$$

satisfies

$$\mathrm{tr}J=0, J^2=I\,. \tag{2.9}$$

This result can also be expressed in the exponential form

$$v(\tau)=\exp[J\eta(\tau)]\,, \tag{2.10}$$

where

$$e^{\eta(\tau)}=\sqrt{\frac{\tau+\sqrt{m^2-a^2}}{\tau-\sqrt{m^2-a^2}}}\,. \tag{2.11}$$

If, on the other hand, $a^2>m^2$, this procedure would have yielded

$$\begin{aligned}v(\tau)=&\frac{1}{2}(I+J)\sqrt{\frac{\tau+i\sqrt{a^2-m^2}}{\tau-i\sqrt{a^2-m^2}}}\\ &+\frac{1}{2}(I-J)\sqrt{\frac{\tau-i\sqrt{a^2-m^2}}{\tau+i\sqrt{a^2-m^2}}}\,,\end{aligned} \tag{2.12}$$

where

$$iJ:=\frac{1}{\sqrt{a^2-m^2}}\begin{pmatrix}-m & -a\\ a & m\end{pmatrix}. \tag{2.13}$$

In either case, $v(\tau)$ is an SU(1,1)= SL$(2,R)$ matrix. In the case $a^2>m^2$ we probably would have used the symbol J for the real matrix iJ and then would have had $J^2=-1$ for that case.

Alternatively, we can unify these two cases by considering members of the larger group SL$(2,C)$, temporarily setting aside the reality condition on $v(\tau)$ and allowing general complex values for the parameters. It turns out [5] that the Hauser-Ernst homogeneous Hilbert problem works for members of SL$(2,C)$ as well as SU(1,1)= SL$(2,R)$, so there is no need to solve the HHP twice. The solutions for both $a^2<m^2$ and $a^2>m^2$ can be inferred from the complexified spacetime that results from an application of the SL$(2,C)$ transformation.

B. The electrovac case

Now let us turn our attention to the charged Kerr metric, where

$$\mathcal{E}=1-\frac{2m}{r-ia\cos\theta},\Phi=\frac{e}{r-ia\cos\theta}\,. \tag{2.14}$$

Thus,

$$\mathcal{E}(\tau,0)=\frac{\tau-m-ia}{\tau+m-ia},\quad \Phi(\tau,0)=\frac{e}{\tau+m-ia}\,. \tag{2.15}$$

In the electrovac case the axis relation can be expressed in the form [6]

$$X(\tau)v(\tau)Y(\tau)=0,\quad X(\tau)v(\tau)Z(\tau)=0\,, \tag{2.16}$$

where

$$X(\tau):=\left(-\frac{1}{\sqrt{2}}i\;\;\frac{1}{\sqrt{2}}\mathcal{E}(\tau,0)\;\;\Phi(\tau,0)\right)\,, \tag{2.17}$$

and

$$Y(\tau):=\begin{pmatrix}-i\mathcal{E}^{(0)}(\tau,0)\\ 1\\ 0\end{pmatrix},\quad Z(\tau):=\begin{pmatrix}-\frac{1}{\sqrt{2}}i\Phi^{(0)}(\tau,0)\\ 0\\ 1\end{pmatrix}. \tag{2.18}$$

In the present case

$$X(\tau)=\left(-\frac{1}{\sqrt{2}}i(\tau+m-ia)\;\;\frac{1}{\sqrt{2}}(\tau-m-ia)\;\;e\right)\,, \tag{2.19}$$

and

$$Y(\tau)=\begin{pmatrix}-i\\ 1\\ 0\end{pmatrix},\quad Z(\tau)=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{2.20}$$

The resulting conditions upon the SU(2,1) matrix $v(\tau)$ are not as simple as in the vacuum case, but there is a certain resemblance nevertheless. Guided by how the HHP was solved in the vacuum case, we look for a $v(\tau)$ of the form

$$v(\tau) = \exp[J\eta(\tau)] , \tag{2.21}$$

where two of the three eigenvalues of J are degenerate. Because $\mathrm{Tr}J = 0$, these two eigenvalues are λ and -2λ, respectively, where λ is a constant. If $\mathcal{P}$ is a projection operator onto the subspace corresponding to the nondegenerate eigenvector, we may express $v(\tau)$ in the form

$$v(\tau) = (I - \mathcal{P})e^{\lambda\eta(\tau)} + \mathcal{P}e^{-2\lambda\eta(\tau)} , \tag{2.22}$$

where the projection operator $\mathcal{P}$ can be written in term of the nondegenerate eigenvector h of J as

$$\mathcal{P} = (1/E)hh^\dagger\Omega, \quad E := h^\dagger\Omega h , \tag{2.23}$$

where

$$\Omega := \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.24}$$

This way of writing $\mathcal{P}$ implies that

$$\mathcal{P}^\dagger\Omega = \Omega\mathcal{P} . \tag{2.25}$$

Moreover,

$$J = \lambda(I - 3\mathcal{P}) , \tag{2.26}$$

so

$$\lambda J^\dagger\Omega - \lambda^*\Omega J = 0 . \tag{2.27}$$

On the other hand, the SU(2,1) conditions

$$\det v = 1, \quad v^\dagger\Omega v = \Omega \tag{2.28}$$

require

$$\eta(\tau)^* J^\dagger\Omega + \eta(\tau)\Omega J = 0 , \tag{2.29}$$

which is satisfied if and only if $\lambda\eta(\tau)$ is imaginary.

We may now express the axis relations in the form

$$X(\tau)[I + \mathcal{P}(e^{-3\lambda\eta(\tau)} - 1)]Y(\tau) = 0 , \tag{2.30a}$$

$$X(\tau)[I + \mathcal{P}(e^{-3\lambda\eta(\tau)} - 1)]Z(\tau) = 0 . \tag{2.30b}$$

However, in our case,

$$X(\tau)Y(\tau) = -\sqrt{2}m \ \text{ and } X(\tau)Z(\tau) = e \tag{2.31}$$

are both constants. Thus, our pair of equations reduces to

$$(e^{-3\lambda\eta(\tau)} - 1)X(\tau)hh^\dagger\Omega Y = \sqrt{2}Em , \tag{2.32a}$$

$$(e^{-3\lambda\eta(\tau)} - 1)X(\tau)hh^\dagger\Omega Z = -Ee . \tag{2.32b}$$

Because

$$h^\dagger\Omega Y = i(h_1^* + ih_2^*), \quad h^\dagger\Omega Z = h_3^* , \tag{2.33}$$

it is immediately apparent that

$$\mathcal{Q} := \sqrt{2}ih_3/(h_1^* + ih_2^*) = e/m . \tag{2.34}$$

We also find that

$$X(\tau)h = -i\left\{\left(\frac{h_1 + ih_2}{\sqrt{2}}\right)(\tau - ia) + \left(\frac{h_1 - ih_2}{\sqrt{2}}\right)m + ih_3 e\right\} . \tag{2.35}$$

Of course, we do not expect the eigenvector to be determined completely. Rather, only the ratios of its components will be determined. Therefore, we shall impose an additional condition; for example,

$$\frac{h_1 + ih_2}{\sqrt{2}} = 1 . \tag{2.36}$$

Thus, we end up with the simple result

$$X(\tau)h = -i\left\{(\tau - ia) + \frac{ih_3}{e}(e^2 - m^2)\right\} . \tag{2.37}$$

Moreover,

$$E = 1 + (|h_3|/e)^2(e^2 - m^2) , \tag{2.38}$$

so the axis relation yields

$$(e^{-3\lambda\eta(\tau)} - 1)\left[\tau - ia + \frac{ih_3}{e}(e^2 - m^2)\right] = -\left[\frac{ie}{h_3^*} + \frac{ih_3}{e}(e^2 - m^2)\right] , \tag{2.39}$$

or

$$e^{3\lambda\eta(\tau)} = \frac{\tau - ia + i(h_3/e)(e^2 - m^2)}{\tau - ia - i(e/h_3^*)} . \tag{2.40}$$

To complete the discussion we must fully determine the eigenvector h, and hence the projection operator $\mathcal{P}$. Only h_3 remains to be determined, for once h_3 is known, we shall also know $(h_1 - ih_2)/\sqrt{2}$, and $(h_1 + ih_2)/\sqrt{2} = 1$ by our convention for the selection of the representative eigenvector h. Suppose now that e/h_3 is a root of the equation

$$(e/h_3)^2 + 2a(e/h_3) - (e^2 - m^2) = 0 . \tag{2.41}$$

If $a^2 + e^2 > m^2$, there are two real roots

$$e/h_3 = -a \pm \sqrt{a^2 + e^2 - m^2} , \tag{2.42}$$

and, if $a^2 + e^2 < m^2$, there are two complex conjugate roots

$$e/h_3 = -a \pm i\sqrt{m^2 - a^2 - e^2} . \tag{2.43}$$

Finally, substituting the first of these roots back into Eq. (2.40), we obtain

$$e^{3\lambda\eta(\tau)} = \frac{\tau + i\sqrt{a^2+e^2-m^2}}{\tau - i\sqrt{a^2+e^2-m^2}} , \tag{2.44}$$

and substituting the second of these roots into Eq. (2.40), we obtain

$$e^{3\lambda\eta(\tau)} = \frac{\tau + \sqrt{m^2-a^2-e^2}}{\tau - \sqrt{m^2-a^2-e^2}} . \tag{2.45}$$

In the first case, $\lambda\eta(\tau)$ is imaginary, while in the second case, $\lambda\eta(\tau)$ is real. Thus, only in the first case is the SU(2,1) condition satisfied by $v(\tau)$.

One may infer from this disappointing result that an SU(2,1) matrix $v(\tau)$ for the case $a^2+e^2<m^2$ must *not* have two degenerate eigenvectors. That complicates the identification of a suitable transformation. As far as we know, no one has yet worked out an SU(2,1) transformation matrix $v(\tau)$ that accomplishes our purposes when $a^2+e^2<m^2$, let alone solved the associated HHP. On the other hand, the SU(2,1) transformation matrix $v(\tau)$ that we found for the case $a^2+e^2<m^2$ corresponds to a transformation that was discovered by Alekseev [7] and by Cosgrove [8].

Like the double-Harrison transformation, the Cosgrove transformation can be complexified; i.e., the SU(2,1) transformation matrix $v(\tau)$ can be replaced by an SL(3,C) matrix. The solution corresponding to $a^2+e^2>m^2$ will then correspond to an obvious *real cross section* of the complexified spacetime that results from the application of the complexified Cosgrove transformation to Minkowski space. The big question is, however, can one identify another real cross section that corresponds to the case $a^2+e^2<m^2$?

In the case of the charged Kerr metric it is fairly trivial to infer the $\mathcal{E}$ and Φ potentials and the metric fields for the case $a^2+e^2<m^2$ from the corresponding potentials and fields for the case $a^2+e^2>m^2$. The author has always believed that this type of construction would be possible for all electrovac spacetimes that belong to the Cosgrove family, but only recently, after recognizing the formal similarity of a five-parameter electrovac solution obtained by Manko *et al.* [2] to a particular specialization of a twelve-parameter solution that was generated by Guo, and Ernst [9] using the Cosgrove transformation, has he actually tried to prove that this is indeed possible.

III. COMPLEXIFIED COSGROVE TRANSFORMATION

The form of U, V, and W that we shall present for the spacetime that results from applying a succession of n Cosgrove transformations of Minkowski space is new, and was derived in the following way from expressions the reader can find in earlier work of Cosgrove [8], Guo, and Ernst [9], Chen, Guo, and Ernst [10], and Wang, Guo, and Wu [11].

For the first Cosgrove transformation, Guo and Ernst expressed the complex potentials in the form [12]

$$\mathcal{E} = 1 - 2i\frac{N}{D}, \Phi = -\frac{N'}{D} , \tag{3.1}$$

where D was written as the determinant of a 3×3 matrix, the columns of which were denoted by ψ_1, ψ_2, and ψ_3, and which were respectively proportional, to

$$P^{(0)}(K^*)h, P^{(0)}(K)h', \text{ and } P^{(0)}(K)h'' , \tag{3.2}$$

where

$$P^{(0)}(\tau) := \frac{1}{\sqrt{2}r(\tau)} \begin{pmatrix} -[r(\tau)-(\tau-z)] & i[r(\tau)+(\tau-z)] & 0 \\ -i & 1 & 0 \\ 0 & 0 & \sqrt{2}r(\tau) \end{pmatrix} \tag{3.3}$$

is the P potential [6] of Minkowski space,

$$r(\tau) := \sqrt{(z-\tau)^2+\rho^2} , \tag{3.4}$$

and K is a complex parameter. The elements of the vectors h are arbitrary complex parameters, but only ratios of these components are significant. The vectors h' and h'' are linearly independent vectors that are "orthogonal" to h in the sense

$$h^\dagger \Omega h' = 0 = h^\dagger \Omega h'' , \tag{3.5}$$

where

$$\Omega := \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.6}$$

Explicit expressions for the components of the vectors $\psi_i^{(k)}$ were given by Chen, Guo, and Ernst in Eqs. (8) of Ref. [10]. However, to obtain simpler expressions, we shall select h'' differently than they did. If one chooses the vectors

$$h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, h' = \begin{pmatrix} h_1^* \\ h_2^* \\ 0 \end{pmatrix}, h'' = \begin{pmatrix} h_3^* \\ ih_3^* \\ h_1^*+ih_2^* \end{pmatrix} , \tag{3.7}$$

then one can select

$$\psi_1 = \begin{pmatrix} Q_1 \\ 1 \\ Q^*S_1 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} Q_2 \\ 1 \\ 0 \end{pmatrix}, \psi_3 = \begin{pmatrix} iQ \\ 0 \\ 1 \end{pmatrix} , \tag{3.8}$$

where $K_1 := K^*, K_2 := K$,

$$Q_k := i[X_k r_k + (K_k - z)], S_k := X_k r_k , \tag{3.9}$$

and

$$X_1 : = -\frac{h_1 - ih_2}{h_1 + ih_2}, \quad X_2 := -\frac{h_1^* - ih_2^*}{h_1^* + ih_2^*} ,$$
$$\mathcal{Q} : = \sqrt{2}i\frac{h_3^*}{h_1^* + ih_2^*} . \tag{3.10}$$

We note, in particular, that

$$K_2 = K_1^* \text{ and } X_1^* X_2 = 1 . \tag{3.11}$$

Guo and Ernst also constructed the complex potentials $\mathcal{E}$ and Φ for the spacetime that results when Minkowski space is subjected to two successive Cosgrove transformations. Wang, Guo, and Wu then showed that D, N, and N' could be reexpressed as 6×6 determinants, while, for higher values of n, they could be expressed as $3n \times 3n$ determinants. In our present gauge, the Wang-Guo-Wu expression for D assumes the simple form

$$D = \begin{vmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & & \vdots \\ D_{n1} & \cdots & D_{nn} \end{vmatrix} , \tag{3.12}$$

where the 3×3 submatrices D_{jk} are given by

$$D_{jk} := \begin{pmatrix} (K_{2k-1})^{j-1}\mathcal{Q}_{2k-1} & (K_{2k})^{j-1}\mathcal{Q}_{2k} & i(K_{2k})^{j-1}\mathcal{Q}_k \\ (K_{2k-1})^{j-1} & (K_{2k})^{j-1} & 0 \\ (K_{2k-1})^{j-1}\mathcal{Q}_k^* S_{2k-1} & 0 & (K_{2k})^{j-1} \end{pmatrix} . \tag{3.13}$$

The determinants N and N' can be constructed from D by replacing, respectively, the $(3n)$th and $(3n-2)$th rows by

$$K_1^n \;\; K_2^n \;\; 0 \;\; \cdots \;\; K_{2n-1}^n \;\; K_{2n}^n \;\; 0 .$$

The fields U, V, and W are defined (up to a common factor) by

$$U := D - iN, V := -N', \text{ and } W := iN , \tag{3.14}$$

each of which can obviously be written as a single determinant. By using elementary row operations it can be shown that the term $i(K_k - z)$ in $\mathcal{Q}_k (k = 1, \ldots, 2n)$ contributes nothing to any of the determinants. Therefore, each $\mathcal{Q}_k$ can be replaced by iS_k. In conclusion, the complex potentials $\mathcal{E}$ and Φ of the electrovac solution that results from applying a succession of n Cosgrove transformations to Minkowski space are given by

$$\mathcal{E} = \frac{U - W}{U + W}, \Phi = \frac{V}{U + W} , \tag{3.15}$$

where (suppressing a factor i^n)U is the $3n \times 3n$ determinant

$$U = \begin{vmatrix} U_{11} & \cdots & U_{1n} \\ \vdots & & \vdots \\ U_{n1} & \cdots & U_{nn} \end{vmatrix} , \tag{3.16}$$

in which occur the 3×3 submatrices

$$U_{jk} := \begin{pmatrix} (K_{2k-1})^{j-1}X_{2k-1}r_{2k-1} & (K_{2k})^{j-1}X_{2k}r_{2k} & (K_{2k})^{j-1}\mathcal{Q}_k \\ (K_{2k-1})^{j-1} & (K_{2k})^{j-1} & 0 \\ (K_{2k-1})^{j-1}\mathcal{Q}_k^* X_{2k-1}r_{2k-1} & 0 & (K_{2k})^{j-1} \end{pmatrix} , \tag{3.17}$$

where

$$r_a := \sqrt{(z - K_a)^2 + \rho^2} . \tag{3.18}$$

The $3n \times 3n$ determinants $-V$ and W are constructed from U by replacing, respectively, the $(3n)$th and the $(3n-2)$th row of the latter determinant by

$$(K_1)^n \;\; (K_2)^n \;\; 0 \;\; \cdots \;\; (K_{2n-1})^n \;\; (K_{2n})^n \;\; 0 .$$

We have seen that the Cosgrove transformation, which is characterized by a $v(\tau)$ with one nondegenerate eigenvector h, and a pair of degenerate eigenvectors h' and h'', cannot cover all cases. A Kinnersley-Chitre transformation $v(\tau)$ with three nondegenerate eigenvectors is needed as well, but such a transformation has not yet been identified, as the associated HHP has not yet been solved.

By the *complexified Cosgrove transformation* we shall mean the SL$(3, C)$ matrix $v(\tau)$ with eigenvectors such that Eqs. (3.8) through (3.11) are valid, but X_1^*, X_2^*, and $\mathcal{Q}^*$ are complex parameters that are no longer identified as the complex conjugates of X_1, X_2, and $\mathcal{Q}$, respectively. Similarly, $K_1^* = K_2$ is no longer the complex conjugate of K_1, and $K_2^* = K_1$ is no longer the complex conjugate of K_2. The n-fold complexified Cosgrove transformation is characterized by $6n$ complex parameters, while the ordinary Cosgrove transformation is characterized by only $3n$ complex parameters. The complexified Cosgrove transformation for $n = 2$ has sufficiently many complex parameters to cover all axis data of the form

$$U(z, 0) = z^2 + U_1 z + U_2 , \tag{3.19a}$$

$$V(z, 0) = V_1 z + V_2 , \tag{3.19b}$$

$$W(z, 0) = W_1 z + W_2 , \tag{3.19c}$$

$$U^*(z, 0) = z^2 + U_1^* z + U_2^* , \tag{3.19d}$$

$$V^*(z,0) = V_1^* z + V_2^* \ , \tag{3.19e}$$

$$W^*(z,0) = W_1^* z + W_2^* \ , \tag{3.19f}$$

where the complex fields U^*, V^*, and W^* are regarded as independent of the complex fields U, V, and W, and where U_1, U_2, V_1, V_2, W_1, W_2, U_1^*, U_2^*, V_1^*, V_2^*, W_1^*, and W_2 are twelve independent complex constants. These fields, U, V, W and U^*, V^*, W^* satisfy the field equations

$$(U^*U + V^*V - W^*W)\nabla^2 \begin{pmatrix} U \\ V \\ W \end{pmatrix} = 2(U^*\nabla U + V^*\nabla V - W^*\nabla W) \cdot \nabla \begin{pmatrix} U \\ V \\ W \end{pmatrix} \ , \tag{3.20a}$$

$$(U^*U + V^*V - W^*W)\nabla^2 \begin{pmatrix} U^* \\ V^* \\ W^* \end{pmatrix} = 2(U\nabla U^* + V\nabla V^* - W\nabla W^*) \cdot \nabla \begin{pmatrix} U^* \\ V^* \\ W^* \end{pmatrix} \ . \tag{3.20b}$$

One should be aware of the fact that the metric field f, which is defined by

$$f := \frac{U^*U + V^*V - W^*W}{(U^* + W^*)(U + W)} \ , \tag{3.21}$$

is, in general, complex, since U^* is no longer the complex conjugate of U, etc. For this reason, we refer to these as *complexified spacetimes*, even though the z and ρ coordinates remain real.

In the case of the complexified Cosgrove transformation, there are expressions for the independent complex potentials $\mathcal{E}^*$ and Φ^* that precisely parallel the expressions we have already given for $\mathcal{E}$ and Φ. We shall not state these relations explicitly, since they can be constructed quite easily by the reader. When $\mathcal{E}^*$ is the complex conjugate of $\mathcal{E}$ and Φ^* is the complex conjugate of Φ, we shall say that we have a *real cross section* of the complexified spacetime.

When the complex constants $\mathcal{Q}_k$ and $\mathcal{Q}_k^*$ all vanish, we obtain a vacuum solution, the real cross sections of which are the vacuum metrics of the Neugebauer family, the $n = 2$ exemplars of which were studied in Ref. [1].

IV. THE $N = 2$ SOLUTION

In this paper, as in Ref. [1], we shall restrict our attention to the case $n = 2$, where U, V, and W are given by

$$\begin{aligned} U = {} & (K_1 - K_4)(K_2 - K_3)[(1 - |\mathcal{Q}_1|^2)X_1 r_1 - X_2 r_2][(1 - |\mathcal{Q}_2|^2)X_3 r_3 - X_4 r_4] \\ & +(K_1 - K_2)(K_3 - K_4)[(1 - \mathcal{Q}_1^*\mathcal{Q}_2)X_1 r_1 - X_4 r_4][(1 - \mathcal{Q}_1\mathcal{Q}_2^*)X_3 r_3 - X_2 r_2] \ , \end{aligned} \tag{4.1a}$$

$$V = -\Delta(K_2, K_3, K_4)Y_1 r_1 + \Delta(K_3, K_4, K_1)Y_2 r_2 - \Delta(K_4, K_1, K_2)Y_3 r_3 + \Delta(K_1, K_2, K_3)Y_4 r_4 \ , \tag{4.1b}$$

$$W = -\Delta(K_2, K_3, K_4)Z_1 r_1 + \Delta(K_3, K_4, K_1)Z_2 r_2 - \Delta(K_4, K_1, K_2)Z_3 r_3 + \Delta(K_1, K_2, K_3)Z_4 r_4 \ , \tag{4.1c}$$

where

$$Y_1 := \left[\mathcal{Q}_2(1 - |\mathcal{Q}_1|^2) + \frac{K_1 - K_2}{K_4 - K_2}(\mathcal{Q}_1 - \mathcal{Q}_2)\right] X_1 \ , \tag{4.2a}$$

$$Y_2 := \mathcal{Q}_2 X_2 \ , \tag{4.2b}$$

$$Y_3 = \left[\mathcal{Q}_1(1 - |\mathcal{Q}_2|^2) - \frac{K_4 - K_3}{K_4 - K_2}(\mathcal{Q}_1 - \mathcal{Q}_2)\right] X_3 \ , \tag{4.2c}$$

$$Y_4 := \mathcal{Q}_1 X_4 \ , \tag{4.2d}$$

and

$$Z_1 := \left[(1-|\mathcal{Q}_1|^2) + \frac{K_1-K_2}{K_4-K_2}\mathcal{Q}_1^*(\mathcal{Q}_1-\mathcal{Q}_2)\right] X_1 \;, \tag{4.3a}$$

$$Z_2 := X_2 \;, \tag{4.3b}$$

$$Z_3 := \left[(1-|\mathcal{Q}_2|^2) - \frac{K_4-K_3}{K_4-K_2}\mathcal{Q}_2^*(\mathcal{Q}_1-\mathcal{Q}_2)\right] X_3 \;, \tag{4.3c}$$

$$Z_4 := X_4 \;, \tag{4.3d}$$

and Δ is the Vandermonde determinant introduced in Ref. [1]. (We have divided out a common actor $K_4 - K_2$ from U, V, and W.)

From the above expression for U, one easily identifies the complex constants

$$\begin{aligned} U_0 = {} & (K_1-K_4)(K_2-K_3)[(1-|\mathcal{Q}_1|^2)X_1 - X_2][(1-|\mathcal{Q}2|^2)X_3 - X_4] \\ & +(K_1-K_2)(K_3-K_4)[(1-\mathcal{Q}_1^*\mathcal{Q}_2)X_1 - X_4][(1-\mathcal{Q}_2^*)X_3 - X_2] \;, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} U_1 = {} & -(K_1-K_4)(K_2-K_3)\{[(1-|\mathcal{Q}_1|^2)X_1 - X_2][(1-|\mathcal{Q}_2|^2)K_3X_3 - K_4X_4] \\ & +[(1-|\mathcal{Q}_1|^2)K_1X_1 - K_2X_2][(1-|\mathcal{Q}_2|^2)X_3 - X_4]\} \\ & -(K_1-K_2)(K_3-K_4)\{[(1-\mathcal{Q}_1^*\mathcal{Q}_2)X_1 - X_4][(1-\mathcal{Q}_1\mathcal{Q}_2^*)K_3X_3 - K_2X_2] \\ & +[(1-\mathcal{Q}_1^*\mathcal{Q}_2)K_1X_1 - K_4X_4][(1-\mathcal{Q}_1\mathcal{Q}_2^*)X_3 - X_2]\} \;, \end{aligned} \tag{4.4b}$$

$$\begin{aligned} U_2 = {} & (K_1-K_4)(K_2-K_3)[(1-|\mathcal{Q}_1|^2)K_1X_1 - K_2X_2][(1-|\mathcal{Q}_2|^2)K_3X_3 - K_4X_4] \\ & +(K_1-K_2)(K_3-K_4)[(1-\mathcal{Q}_1^*\mathcal{Q}_2)K_1X_1 - K_4X_4][(1-\mathcal{Q}_1\mathcal{Q}_2^*)K_3X_3 - K_2X_2] \;, \end{aligned} \tag{4.4c}$$

while from the expressions for V and W one obtains the complex constants

$$V_1 = -V^{(0)}, V_2 = V^{(1)}, W_1 = -W^{(0)}, W_2 = W^{(1)} \;, \tag{4.5}$$

where

$$V^{(a)} := -\Delta(K_2,K_3,K_4)K_1^aY_1 + \Delta(K_3,K_4,K_1)K_2^aY_2 - \Delta(K_4,K_1,K_2)K_3^aY_3 + \Delta(K_1,K_2,K_3)K_4^aY_4 \;, \tag{4.6a}$$

$$W^{(a)} := -\Delta(K_2,K_3,K_4)K_1^aZ_1 + \Delta(K_3,K_4,K_1)K_2^aZ_2 - \Delta(K_4,K_1,K_2)K_3^aZ_3 + \Delta(K_1,K_2,K_3)K_4^aZ_4 \;. \tag{4.6b}$$

Our immediate objective is to determine the parameters X_a, Y_a, and $Z_a (a = 1,2,3,4)$ in terms of the axis data U_1, U_2, V_1, V_2, W_1, and W_2 (where $U_0 = 1$), and the K's.

A. Determination of X_a $(a = 1,2,3,4)$

The simplest case is when $V_2W_1 - V_1W_2 = 0$. This corresponds to $\mathcal{Q}_1 = \mathcal{Q}_2 =: \mathcal{Q}$, where $V_1 = \mathcal{Q}W_1$ and $V_2 = \mathcal{Q}W_2$. In this case one gets the same expressions for $(1-|\mathcal{Q}|^2)X_1, X_2, (1-|\mathcal{Q}|^2)X_3$, and X_4 as one got for X_1, X_2, X_3, and X_4, respectively, in the vacuum case. This solution is merely the electrovac solution that is generated by the old electrification transformation of Harrison, as reformulated by Ernst. In this paper, we are concerned primarily with the case when $V_2W_1 - V_1W_2 \neq 0$, i.e., $\mathcal{Q}_1 \neq \mathcal{Q}_2$.

In the vacuum problem it was surprisingly easy to solve for the X's in terms of the axis data and the K's. One had to solve nothing but linear and quadratic algebraic equations. The solution itself revealed a most intriguing structure, to which the simplicity of the solution can be attributed, and a knowledge of which we found to be indispensable for solving the electrovac problem. On the one hand, one had the hierarchy of linear equations (for the X's)

$$W^{(0)} = -W_1 \;, \tag{4.7a}$$

$$W^{(1)} = W_2 \;, \tag{4.7b}$$

$$W^{(2)} = -\frac{U_1W_2 - U_2W_1}{U_0} \;, \tag{4.7c}$$

$$W^{(3)} = \frac{U_1[(U_1W_2 - U_2W_1)/U_0] - U_2W_2}{U_0} \;, \tag{4.7d}$$

and, on the other hand,

$$\Delta(K_1,K_2,K_3,K_4) = -\frac{W_1[(U_1W_2 - U_2W_1)/U_0] - W_2^2}{U_0} \;. \tag{4.8}$$

In particular, the last equation allowed one to determine U_0 in terms of the axis data and the K's, a critical step in the complete determination of the X's.

After spending a considerable amount of time trying to identify the X's and the $\mathcal{Q}$'s in the electrovac case, we abandoned that effort, and approached the problem of determining U, V, and W in a new way that avoids the determination of the X's and $\mathcal{Q}$'s (although these objects can be calculated at the very end, if they are really desired). In the electrovac case we have two sets of four linear equations:

$$V^{(0)} = -V_1 , \tag{4.9a}$$

$$V^{(1)} = V_2 , \tag{4.9b}$$

$$V^{(2)} = \frac{U_2V_1 - U_1V_2}{U_0} , \tag{4.9c}$$

$$V^{(3)} = -\frac{U_2V_2 + U_1[(U_2V_1 - U_1V_2)/U_0]}{U_0} + \frac{V_2W_1 - V_1W_2}{U_0}\left(\frac{W_1}{U_0}\right)^* , \tag{4.9d}$$

and

$$W^{(0)} = -W_1 , \tag{4.10a}$$

$$W^{(1)} = W_2 , \tag{4.10b}$$

$$W^{(2)} = \frac{U_2W_1 - U_1W_2}{U_0} , \tag{4.10c}$$

$$W^{(3)} = -\frac{U_2W_2 + U_1[(U_2W_1 - U_1W_2)/U_0]}{U_0} + \frac{V_2W_1 - V_1W_2}{U_0}\left(\frac{V_1}{U_0}\right)^* , \tag{4.10d}$$

As in the vacuum case, the right side of each of the four equations is equal to U_0 times a quantity that can be easily expressed in terms of the axis data alone, while the left side of each of the four equations is equal to a linear combination of $Y_a(a = 1,2,3,4)$ or $Z_a(a = 1,2,3,4)$.

B. Determination of U, W, and V up to a common factor

The four linear equations (4.9a)–(4.9d) for $Y_a(a = 1,2,3,4)$ and the four linear equations (4.10a)–(4.10d) for $Z_a(a = 1,2,3,4)$ are easily solved. One obtains

$$\begin{aligned}\mathcal{D}Y_1 = {} & \{U_2V_2 + U_1(U_2V_1 - U_1V_2) - (V_2W_1 - V_1W_2)W_1^*\} \\ & +(K_2 + K_3 + K_4)(U_2V_1 - U_1V_2) \\ & -(K_2K_3 + K_2K_4 + K_3K_4)V_2 - (K_2K_3K_4)V_1 ,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\mathcal{D}Z_1 = {} & \{U_2W_2 + U_1(U_2W_1 - U_1W_2) - (V_2W_1 - V_1W_2)V_1^*\} \\ & +(K_2 + K_3 + K_4)(U_2W_1 - U_1W_2) - (K_2K_3 + K_2K_4 + K_3K_4)W_2 - (K_2K_3K_4)W_1 ,\end{aligned} \tag{4.12}$$

where U, V, and W have been adjusted so that $U_0 = 1$. The expressions for Y_2, Y_3, and Y_4 can be inferred from the expression for Y_1 and the expressions for Z_2, Z_3, and Z_4 can be inferred from the expression for Z_1 by permuting indices on the K's. $\mathcal{D}$ is given by

$$\mathcal{D} := -\frac{\Delta(K_1, K_2, K_3, K_4)}{U_0} , \tag{4.13}$$

where U_0 is the *original* value of U_0, not 1.

Using Eqs. (4.1b) and (4.1c), these expressions for $\mathcal{D}Y_a, \mathcal{D}Z_a(a = 1,2,3,4)$ permit us to evaluate $\mathcal{D}V$ and $\mathcal{D}W$ without further ado, for we have

$$\begin{aligned}\mathcal{D}V = {} & -\Delta(K_2, K_3, K_4)\mathcal{D}Y_1r_1 + \Delta(K_3, K_4, K_1)\mathcal{D}Y_2r_2 \\ & -\Delta(K_4, K_1, K_2)\mathcal{D}Y_3r_3 \\ & +\Delta(K_1, K_2, K_3)\mathcal{D}Y_4r_4 ,\end{aligned} \tag{4.14a}$$

$$\begin{aligned}\mathcal{D}W = {} & -\Delta(K_2, K_3, K_4)\mathcal{D}Z_1r_1 + \Delta(K_3, K_4, K_1)\mathcal{D}Z_2r_2 \\ & -\Delta(K_4, K_1, K_2)\mathcal{D}Z_3r_3 \\ & +\Delta(K_1, K_2, K_3)\mathcal{D}Z_4r_4 .\end{aligned} \tag{4.14b}$$

But what about U, which is given by Eq. (4.1a)? Interestingly, U can be expressed directly in terms of the Y's and Z's as

$$U = -\frac{1}{2}\left(\frac{K_4 - K_2}{\mathcal{Q}_1 - \mathcal{Q}_2}\right)\sum_{i,j,k,l}\epsilon_{ijkl}(K_i - K_j)Z_kY_lr_kr_l , \tag{4.15}$$

where ϵ_{ijkl} is the Levi-Civita permutation symbol. On the other hand,

$$\begin{aligned}\frac{K_4 - K_2}{\mathcal{Q}_1 - \mathcal{Q}_2} &= -\frac{\Delta(K_1, K_2, K_3, K_4)U_0}{V_2W_1 - V_1W_2} \\ &= \mathcal{D}\left(\frac{U_0^2}{V_2W_1 - V_1W_2}\right) .\end{aligned} \tag{4.16}$$

Hence, with $U_0 = 1, \mathcal{D}U$ can be expressed in the final form

$$\mathcal{D}U = -\frac{1}{2(V_2W_1 - V_1W_2)} \times \sum_{i,j,k,l}\epsilon_{ijkl}(K_i - K_j)(\mathcal{D}Z_k)(\mathcal{D}Y_l)r_kr_l . \tag{4.17}$$

This is most remarkable, since it means that $\mathcal{D}U$ involves only the axis data, the known $\mathcal{D}Y$'s and $\mathcal{D}Z$'s and the K's. Since one is only interested in ratios of U, V, and W, it suffices to know $\mathcal{D}U$, $\mathcal{D}V$, and $\mathcal{D}W$. One does not have to evaluate $\mathcal{D}$ itself.

C. Determination of K_a ($a = 1, 2, 3, 4$)

Using Eqs. (3.11) and defining $|Z|^2 := Z^* Z$ even when Z^* is not just the complex conjugate of Z, we find that, on the symmetry axis,

$$|U(z,0)|^2 + |V(z,0)|^2 - |W(z,0)|^2 = |U_0|^2(K_1 - z)(K_2 - z)(K_3 - z)(K_4 - z) \ , \quad (4.18)$$

from which it follows that each $K_a (a = 1, 2, 3, 4)$ satisfies the quartic equation

$$0 = K_a^4 + 2(\text{Re}U_1)K_a^3 + (|U_1|^2 + |V_1|^2 - |W_1|^2 + 2\,\text{Re}U_2)K_a^2 + 2\,\text{Re}(U_2U_1^* + V_2V_1^* - W_2W_1^*)K_a + (|U_2|^2 + |V_2|^2 - |W_2|^2) \ , \quad (4.19)$$

where $\text{Re}Z := (Z + Z^*)/2$ even when Z^* is not just the complex conjugate of Z.

Assuming that z has been chosen so that $\text{Re}U_1 = 0$ (or, equivalently, $K_1 + K_2 + K_3 + K_4 = 0$), we have

$$K_a^4 - AK_a^2 - BK_a + C = 0 \ , \quad (4.20)$$

where

$$A = |W_1|^2 - |V_1|^2 - |U_1|^2 - 2\,\text{Re}U_2 \ , \quad (4.21a)$$

$$B = -2\,\text{Re}(U_2U_1^* + V_2V_1^* - W_2W_1^*) \ , \quad (4.21b)$$

$$C = |U_2|^2 + |V_2|^2 - |W_2|^2 \ . \quad (4.21c)$$

The general solution $K_a (a = 1, 3, 3, 4)$ of this quartic equation is given by Eqs. (2.9a)–(2.9d) or Eqs. (2.14a)–(2.14d) of Ref. [1].

In conclusion, the determination of the K's is no more difficult in the electrovac case than it was in the vacuum case. Of course, when $U(z,0)^*$, $V(z,0)^*$, and $W(z,0)^*$ are the complex conjugates of $U(z,0)$, $V(z,0)$, and $W(z,0)$, respectively, the quartic equation, like its vacuum analog, has solutions in which the K's are real, rather than occurring in complex conjugate pairs. Such K's cannot be used with the ordinary Cosgrove transformation. It is instead necessary to employ the complexified Cosgrove transformation.

We would be the first to admit that the approach we have described requires further refinement, which we hope to supply in a future paper. However, we shall turn now to an application that already demonstrates the practical value of this approach.

V. A SIMPLE BUT CONVINCING APPLICATION

Suppose we select the axis data

$$U_1 = -ia,\ U_2 = b,\ V_1 = e,\ V_2 = ic,\ W_1 = m,\ W_2 = 0 \ , \\ U_1^* = ia,\ U_2^* = b,\ V_1^* = e,\ V_2^* = -ic, W_1^* = m \ , \\ W_2^* = 0 \ , \quad (5.1)$$

where the parameters a, b, e, c, m are real. In this case, one has

$$U_2U_1^* + V_2V_1^* - W_2W_1^* = b(ia) + (ic)(e) - 0(m) = i(ab + ce) \ , \quad (5.2)$$

so $\text{Re}(U_2U_1^* + V_2V_1^* - W_2W_1^*) = 0$. Therefore, as in Ref. [1] we may write

$$K_1 = -K_2 = \tfrac{1}{2}(\kappa_+ + \kappa_-),\ K_3 = -K_4 = \tfrac{1}{2}(\kappa_+ - \kappa_-) \ , \quad (5.3)$$

where κ_+ and κ_- are given by

$$\kappa_\pm := \sqrt{m^2 - a^2 - e^2 + 2(\pm d - b)}, d := \sqrt{b^2 + c^2} \ . \quad (5.4)$$

Note, in particular, that κ_+^2 and κ_-^2 are real.

With the selected axis data, Eqs. (4.11), (4.12) and their analogs reduce to

$$\mathcal{D}Y_1 = (e/m)\mathcal{D}Z_1 - \tfrac{1}{2}ic[(m^2 - a^2 - e^2) + \kappa_+\kappa_- + ia(\kappa_+ + \kappa_-)] \ , \quad (5.5a)$$

$$\mathcal{D}Y_2 = (e/m)\mathcal{D}Z_2 - \tfrac{1}{2}ic[(m^2 - a^2 - e^2) + \kappa_+\kappa_- - ia(\kappa_+ + \kappa_-)] \ , \quad (5.5b)$$

$$\mathcal{D}Y_3 = (e/m)\mathcal{D}Z_3 - \tfrac{1}{2}ic[(m^2 - a^2 - e^2) - \kappa_+\kappa_- + ia(\kappa_+ - \kappa_-)] \ , \quad (5.5c)$$

$$\mathcal{D}Y_4 = (e/m)\mathcal{D}Z_4 - \tfrac{1}{2}ic[(m^2 - a^2 - e^2) - \kappa_+\kappa_- - ia(\kappa_+ - \kappa_-)] \ , \quad (5.5d)$$

and

$$\mathcal{D}Z_1 = -m\{i(ab + ce) + \tfrac{1}{2}[\kappa_+(d + b) - \kappa_-(d - b)]\} \ , \quad (5.6a)$$

$$\mathcal{D}Z_2 = -m\{i(ab + ce) - \tfrac{1}{2}[\kappa_+(d + b) - \kappa_-(d - b)]\} \ , \quad (5.6b)$$

$$\mathcal{D}Z_3 = -m\{i(ab+ce) + \tfrac{1}{2}[\kappa_+(d+b) + \kappa_-(d-b)]\} , \tag{5.6c}$$

$$\mathcal{D}Z_4 = -m\{i(ab+ce) - \tfrac{1}{2}[\kappa_+(d+b) + \kappa_-(d-b)]\} , \tag{5.6d}$$

respectively. Equations (4.17), (4.1b), and (4.1c) then yield the following expressions for $\mathcal{D}U$, $\mathcal{D}V$, and $\mathcal{D}W$:

$$\begin{aligned}\mathcal{D}U = {} & \kappa_-^2\{[d(m^2-a^2-e^2)+c^2-a(ab+ce)](r_1r_3+r_2r_4) + i\kappa_+(ab+ce+ad)(r_1r_3-r_2r_4)\} \\ & +\kappa_+^2\{[d(m^2-a^2-e^2)-c^2+a(ab+ce)](r_1r_4+r_2r_3) + i\kappa_-(ab+ce-ad)(r_2r_3-r_1r_4)\} \\ & -4d[b(m^2-e^2)+c(ae+c)](r_2r_1+r_4r_3) ,\end{aligned} \tag{5.7a}$$

$$\begin{aligned}\mathcal{D}V = {} & \kappa_+\kappa_-\{d[e(m^2-a^2-e^2)-2ac](r_4+r_3-r_2-r_1) + de\kappa_+\kappa_-(r_2+r_1+r_4+r_3) \\ & +icd[(\kappa_++\kappa_-)(r_2-r_1)+(\kappa_+-\kappa_-)(r_3-r_4)] \\ & +i[e(ab+ce)+bc][(\kappa_++\kappa_-)(r_3-r_4)+(\kappa_+-\kappa_-)(r_2-r_1)]\} ,\end{aligned} \tag{5.7b}$$

$$\begin{aligned}\mathcal{D}W = {} & m\kappa_+\kappa_-\{d[(m^2-a^2-e^2)(r_4+r_3-r_2-r_1)+\kappa_+\kappa_-(r_2+r_1+r_4+r_3)] \\ & +i(ab+ce)[(\kappa_++\kappa_-)(r_3-r_4)+(\kappa_+-\kappa_-)(r_2-r_1)]\} .\end{aligned} \tag{5.7c}$$

Of course, there are similar expressions for $(\mathcal{D}U)^* := \mathcal{D}^*U^*$, etc., and, for arbitrary real a, b, e, c, m, the latter expressions turn out to be equal to the complex conjugates of $(\mathcal{D}U)$, etc.

The solution given in Eqs. (5.7a)–(5.7c) is, therefore, valid not only when $0 > \kappa_+^2 > \kappa_-^2$, but for other values of κ_+ and κ_- as well. When both κ_+ and κ_- are real, the solution is identical to the five-parameter electrovac solution published recently by Manko *et al.* [2] in which the parameters m, a, b, e, c were associated, respectively, with the mass, the rotation, the mass quadrupole moment, the electric charge and the magnetic dipole moment.

It should be observed that $(\mathcal{D}U)^*$, $(\mathcal{D}V)^*$, and $(\mathcal{D}W)^*$ have the same functional form if $\kappa_\pm$ and $r_a (a = 1,2,3,4)$ are treated as real as they have if $\kappa_\pm$ are treated as imaginary, with $r_1^* = r_2$ and $r_3^* = r_4$. This means that the expression obtained by Manko *et al.* for the metric fields f, γ, and ω in

$$ds^2 = f^{-1}\{e^{2\gamma}(dz^2+d\rho^2)+\rho^2 d\varphi^2\} - f(dt-\omega d\varphi)^2 \tag{5.8}$$

will hold for the other cases as well. In a later paper concerned with the complexified Cosgrove transformation, we shall develop a completely general formula for the field ω. At this time, we merely remark that the field γ is given by [13]

$$e^{2\gamma} = \frac{|U|^2+|V|^2-|W|^2}{|U_0|^2 r_1r_2r_3r_4} , \tag{5.9}$$

and the field f is given by

$$f = \mathrm{Re}\mathcal{E} + |\Phi|^2 = \frac{|U|^2+|V|^2-|W|^2}{|U+W|^2} . \tag{5.10}$$

Thus, the infinite red shift surface corresponds to

$$|U|^2+|V|^2-|W|^2 = 0 , \tag{5.11}$$

and the curvature singularities occur at $U + W = 0$.

VI. TOWARD A PURELY ALGEBRAIC DERIVATION

It was Kinnersley [14] who first pointed out that U, V, and W could always be selected so that the field equations

$$(|U|^2+|V|^2-|W|^2)\nabla^2\begin{pmatrix}U\\V\\W\end{pmatrix} = 2(W^*\nabla U + V^*\nabla V - W^*\nabla W)\cdot\nabla\begin{pmatrix}U\\V\\W\end{pmatrix} \tag{6.1}$$

are satisfied. The reader will find it instructive to work out the $n = 1$ solution of these equations, where

$$U = \sum_i u_i r_i , \tag{6.2a}$$

$$V = v , \tag{6.2b}$$

$$W = w , \tag{6.2c}$$

and u_1, u_2, v, and w are complex constants. This is not difficult to do, if one observes that

$$\nabla^2 r_i = \frac{2}{r_i} \tag{6.3}$$

and

$$\nabla r_i \cdot \nabla r_j = \frac{r_i^2 + r_j^2 - (K_i - K_j)^2}{2r_ir_j} , \tag{6.4}$$

and one uses the relation [15]

$$|U(z,0)|^2 + |V(z,0)|^2 - |W(z,0)|^2 = |U_0|^2\Pi_{a=1}^{2n}(z-K_a) . \tag{6.5}$$

In the present paper we have been interested in $n = 2$ solutions, in which

$$U = \sum_{i<j} u_{ij} r_i r_j \ , \tag{6.6a}$$

$$V = \sum_{i} v_i r_i \ , \tag{6.6b}$$

$$W = \sum_{i} w_i r_i \ , \tag{6.6c}$$

where u_{ij}, v_i, and w_i are complex constants. For all values of n the mechanism of solution remains the same as that illustrated by the $n = 1$ case, but the algebra becomes increasingly more difficult as n increases.

It would be nice if one could formulate a simple strictly algebraic derivation of the general solution of Eqs. (6.1) corresponding to rational axis data by using Eqs. (6.3)–(6.5). We shall postpone further consideration of this approach until a later paper, where we shall be concerned primarily with $n > 2$.

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