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Consistent derivation of the quark-antiquark and three-quark potentials in a Wilson loop context

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We give a new derivation of the quark-antiquark potential in the Wilson loop context. This makes more explicit the approximations involved and enables an immediate extension to the three-quark case. In the $q\bar{q}$ case we find the same semirelativistic potential obtained in preceding papers but for a question of ordering. In the $3q$ case we find a spin-dependent potential identical to that already derived in the literature from the *ad hoc* and incorrect assumption of scalar confinement. Furthermore we obtain the correct form of the spin-independent potential up to the $1/m^2$ order.

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I. INTRODUCTION

The aim of this paper is twofold. First we give a simplified derivation of the quark-antiquark potential in the context of the so-called Wilson loop approach [1] in which the basic assumptions, the conditions for the validity of a potential description and the relation with the flux tube model [2], can be better appreciated. Second we show how the procedure can be extended to the three-quark system [3] obtaining consistently not only the static part (stat) of the potential but also the spin-dependent (SD) and the velocity-dependent (VD) ones at the $1/m^2$ order.

For what concerns the $q\bar{q}$ potential, the result is identical to that reported in [4,5] (see [6] for the spin dependent potential) except for a problem of ordering of minor phenomenological interest:

$$V^{q\bar{q}} = V^{q\bar{q}}_{\text{stat}} + V^{q\bar{q}}_{\text{SD}} + V^{q\bar{q}}_{\text{VD}}, \tag{1.1}$$

where

$$V^{q\bar{q}}_{\text{stat}} = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r\,, \tag{1.2}$$

$$\begin{aligned} V^{q\bar{q}}_{\text{SD}} = {} & \frac{1}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\nabla^2\left(-\frac{4}{3}\frac{\alpha_s}{r}+\sigma r\right) + \frac{1}{2}\left(\frac{4}{3}\frac{\alpha_s}{r^3}-\frac{\sigma}{r}\right)\left[\frac{1}{m_1^2}\mathbf{S}_1\cdot(\mathbf{r}\times\mathbf{p}_1) - \frac{1}{m_2^2}\mathbf{S}_2\cdot(\mathbf{r}\times\mathbf{p}_2)\right] \\ & + \frac{1}{m_1 m_2}\frac{4}{3}\frac{\alpha_s}{r^3}[\mathbf{S}_2\cdot(\mathbf{r}\times\mathbf{p}_1) - \mathbf{S}_1\cdot(\mathbf{r}\times\mathbf{p}_2)] \\ & + \frac{1}{m_1 m_2}\frac{4}{3}\alpha_s\left\{\frac{1}{r^3}\left[\frac{3}{r^2}(\mathbf{S}_1\cdot\mathbf{r})(\mathbf{S}_2\cdot\mathbf{r}) - \mathbf{S}_1\cdot\mathbf{S}_2\right] + \frac{8\pi}{3}\delta^3(\mathbf{r})\mathbf{S}_1\cdot\mathbf{S}_2\right\}, \end{aligned} \tag{1.3}$$

$$V^{q\bar{q}}_{\text{VD}} = \frac{1}{2m_1 m_2}\left\{\frac{4}{3}\frac{\alpha_s}{r}(\delta^{hk}+\hat{r}^h\hat{r}^k)p_1^h p_2^k\right\}_W - \sum_{j=1}^{2}\frac{1}{6m_j^2}\{\sigma\, r\, \mathbf{p}_{jT}^2\}_W - \frac{1}{6m_1 m_2}\{\sigma\, r\, \mathbf{p}_{1T}\cdot\mathbf{p}_{2T}\}_W\,. \tag{1.4}$$

Obviously in Eqs. (1.2)–(1.4) $\mathbf{r} = \mathbf{z}_1 - \mathbf{z}_2$ denotes the relative position of the quark and the antiquark and $\mathbf{p}_{jT}$ the transversal momentum of the particle j, $p^h_{jT} = (\delta^{hk} - \hat{r}^h\hat{r}^k)p^k_j$ where $\hat{\mathbf{r}} = (\mathbf{r}/r)$; the symbol $\{\ \}_W$ stands for the Weyl ordering prescription among momentum and position variables (see Sec. IV). Furthermore, in comparison with [5] the terms in the zero point energy C have been omitted, since they should be reabsorbed in a redefinition of the masses in a full relativistic treatment.

For the $3q$ potential the result is

$$V^{3q} = V^{3q}_{\text{stat}} + V^{3q}_{\text{SD}} + V^{3q}_{\text{VD}} \tag{1.5}$$

with

$$V^{3q}_{\text{stat}} = \sum_{j<l}\left(-\frac{2}{3}\frac{\alpha_s}{r_{jl}}\right) + \sigma(r_1+r_2+r_3)\,, \tag{1.6}$$

$$
\begin{aligned}
V_{\rm SD}^{3q} = {} & \frac{1}{8m_1^2}\nabla^2_{(1)}\left(-\frac{2}{3}\frac{\alpha_s}{r_{12}} - \frac{2}{3}\frac{\alpha_s}{r_{31}} + \sigma r_1\right) \\
& + \left\{ \frac{1}{2m_1^2}\mathbf{S}_1 \cdot \left[(\mathbf{r}_{12}\times \mathbf{p}_1)\left(\frac{2}{3}\frac{\alpha_s}{r_{12}^3}\right) + (\mathbf{r}_{31}\times \mathbf{p}_1)\left(-\frac{2}{3}\frac{\alpha_s}{r_{31}^3}\right) - \frac{\sigma}{r_1}(\mathbf{r}_1\times \mathbf{p}_1)\right] \right. \\
& \left. + \frac{1}{m_1 m_2}\mathbf{S}_1\cdot(\mathbf{r}_{12}\times \mathbf{p}_2)\left(-\frac{2}{3}\frac{\alpha_s}{r_{12}^3}\right) + \frac{1}{m_1 m_3}\mathbf{S}_1\cdot(\mathbf{r}_{31}\times \mathbf{p}_3)\left(\frac{2}{3}\frac{\alpha_s}{r_{31}^3}\right)\right\} \\
& + \frac{1}{m_1 m_2}\frac{2}{3}\alpha_s\left\{\frac{1}{r_{12}^3}\left[\frac{3}{r_{12}^2}(\mathbf{S}_1\cdot \mathbf{r}_{12})(\mathbf{S}_2\cdot \mathbf{r}_{12}) - \mathbf{S}_1\cdot \mathbf{S}_2\right] + \frac{8\pi}{3}\delta^3(\mathbf{r}_{12})\mathbf{S}_1\cdot \mathbf{S}_2\right\} + \text{cyclic permutations}\,, \qquad (1.7)
\end{aligned}
$$

$$
\begin{aligned}
V_{\rm VD}^{3q} = {} & \sum_{j<l}\frac{1}{2m_j m_l}\left\{\frac{2}{3}\frac{\alpha_s}{r_{jl}}(\delta^{hk} + \hat{r}_{jl}^h\hat{r}_{jl}^k)p_j^h p_l^k\right\}_W - \sum_{j=1}^{3}\frac{1}{6m_j^2}\{\sigma\, r_j\, \mathbf{p}_{jT_j}^2\}_W \\
& - \sum_{j=1}^{3}\frac{1}{6}\{\sigma\, r_j\, \dot{\mathbf{z}}_{MT_j}^2\}_W - \sum_{j=1}^{3}\frac{1}{6m_j}\{\sigma\, r_j\, \mathbf{p}_{jT_j}\cdot\dot{\mathbf{z}}_{MT_j}\}_W\,. \qquad (1.8)
\end{aligned}
$$

Again $\mathbf{r}_{jl} = \mathbf{z}_j - \mathbf{z}_l$ denotes the relative position of the quark j with respect to the quark l $(j,l = 1,2,3)$ and $\mathbf{r}_j = \mathbf{z}_j - \mathbf{z}_M$ the position of the quark j with respect to a common point M such that $\sum_{j=1}^{3} r_j$ is minimum. As well known, if no angle in the triangle made by the quarks exceeds 120°, the three lines that connect the quarks with M meet at this point with equal angles of 120° like a Mercedes star [type-I configuration, see Fig. 1(a)]. If one of the angles is $\geq$ 120°, then M coincides with the respective vertex and the potential becomes a two-body one [type-II configuration, see Fig. 1(b)]. Furthermore, $\nabla^2_{(j)}$ is the Laplacian with respect to the variable z_j and now $v^h_{T_j} = (\delta^{hk} - \hat{r}^h_j\hat{r}^k_j)v^k$, where $\hat{\mathbf{r}}_j = (\mathbf{r}_j/r_j)$. The quantity $\dot{\mathbf{z}}_M$ in (1.8) is given by

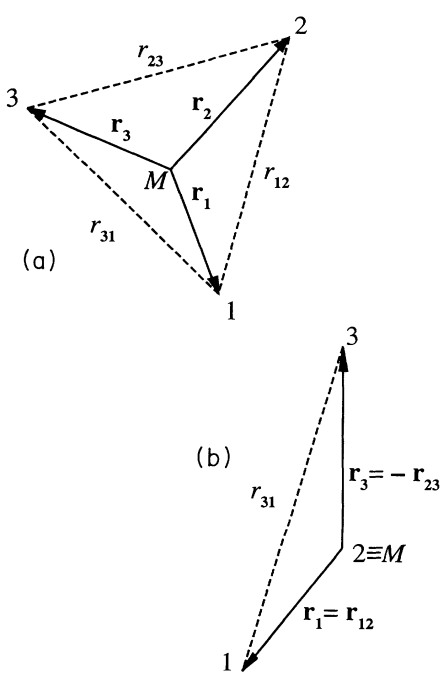

FIG. 1. The two types of configurations of the three-quark system.

$$
\dot{\mathbf{z}}_M = \begin{cases} N^{-1}\sum_{j=1}^{3}(\mathbf{p}_{jT_j}/r_j m_j) & \text{type-I configuration,} \\ \mathbf{p}_l/m_l \quad \text{type-II configuration} : \mathbf{z}_M \equiv \mathbf{z}_l\,, \end{cases} \qquad (1.9)
$$

N being a matrix with elements

$$
N^{hk} = \sum_{j=1}^{3}(1/r_j)(\delta^{hk} - \hat{r}^h_j\hat{r}^k_j).
$$

Finally Eq. (1.7) properly refers to the I configuration case. In general, one should write

$$
V_{\rm SD}^{\rm LR} = -\sum_{j=1}^{3}\frac{1}{2m_j^2}\mathbf{S}_j\cdot\boldsymbol{\nabla}_{(j)}V_{\rm stat}^{\rm LR}\times\mathbf{p}_j\,. \qquad (1.10)
$$

In comparing (1.10) with (1.7) one should keep in mind that the partial derivatives in $\mathbf{z}_M$ of $V_{\rm stat}^{\rm LR}$ vanish due to the definition of M.

We observe that the short-range part in Eqs. (1.6)–(1.8) is of a pure two-body type: it is identical to the electromagnetic potential among three equal charged particles except for the color group factor 2/3, and it is well known. Even the static confining potential in Eq. (1.6) is known [7,1,3]. The long-range part of Eq. (1.7) coincides with the expression obtained by Ford [8] starting from the assumption of a purely scalar Salpeter potential of the form

$$
\sigma\,(r_1 + r_2 + r_3)\,\beta_1\beta_2\beta_3\,, \qquad (1.11)
$$

but at our knowledge it has not been obtained consistently in a Wilson loop context before. Eq. (1.8) is new. It should be stressed that (1.11) corresponds to the usual assumption of scalar confinement for the quark-antiquark system. As well known from this assumption $V_{\rm stat}^{q\bar{q}}$ and $V_{\rm SD}^{q\bar{q}}$ result identical to (1.2) and (1.3), but $V_{\rm VD}^{q\bar{q}}$ turns out different from (1.4).

The important point concerning Eqs. (1.1)–(1.8) and (1.10) is that they follow from rather reasonable assumptions on the behavior of two well-known QCD objects $W_{q\bar{q}}$ and W_{3q} related to the appropriate (distorted) quark-antiquark and three-quark "Wilson loops," respectively.

For the $q\bar{q}$ case the basic object is

$$W_{q\bar{q}} = \frac{1}{3}\left\langle \mathrm{Tr}\, P \exp\left(ig \oint_\Gamma dx^\mu\, A_\mu(x)\right)\right\rangle . \quad (1.12)$$

Here the integration loop Γ is assumed to be made by a world line Γ_1 between an initial position $\mathbf{y}_1$ at the time t_i and a final one $\mathbf{x}_1$ at the time t_f for the quark ($t_i < t_f$), a similar world line Γ_2 described in the reverse direction from $\mathbf{x}_2$ at the time t_f to $\mathbf{y}_2$ at the time t_i for the antiquark and two straight lines at fixed times, which connect $\mathbf{x}_1$ to $\mathbf{x}_2$, $\mathbf{y}_2$ to $\mathbf{y}_1$, and close the contour (Fig. 2). As usual $A_\mu(x) = \frac{1}{2}\lambda_a A^a_\mu(x)$, P prescribes the ordering of the color matrices (from right to left) according to the direction fixed on the loop and the angular brackets denote the functional integration on the gauge fields.

The quantity $i \ln W_{q\bar{q}}$ is written as the sum of a short-range (SR) contribution and of a long-range (LR) one: $i\ln W_{q\bar{q}} = i\ln W^{\mathrm{SR}}_{q\bar{q}} + i\ln W^{\mathrm{LR}}_{q\bar{q}}$. Then it is assumed that the first term is given by the ordinary perturbation theory, that is, at the lowest order,

$$i\ln W^{\mathrm{SR}}_{q\bar{q}} = -\frac{4}{3}g^2 \int_{\Gamma_1} dx_1^\mu \int_{\Gamma_2} dx_2^\nu i D_{\mu\nu}(x_1 - x_2) \quad (1.13)$$

($D_{\mu\nu}$ being the usual gluon propagator and $\alpha_s = g^2/4\pi$ the strong interaction constant) and the second term by the so-called "area law" [9,1,4]

$$i\ln W^{\mathrm{LR}}_{q\bar{q}} = \sigma S_{\min} , \quad (1.14)$$

where $S_{\min}$ denotes the minimal surface enclosed by the loop (σ is the string tension). Obviously Eq. (1.13) is justified by asymptotic freedom, Eq. (1.14) is suggested by lattice theory, numerical simulation, string models, and other types of arguments.

Up to the $1/m^2$ order, the minimal surface can be identified with the surface spanned by the straight line joining $(t, \mathbf{z}_1(t))$ to $(t, \mathbf{z}_2(t))$ with $t_i \le t \le t_f$; the generic point of this surface is [4]

$$u^0_{\min} = t, \qquad \mathbf{u}^{\min} = s\mathbf{z}_1(t) + (1-s)\mathbf{z}_2(t) \quad (1.15)$$

with $0 \le s \le 1$ and $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ being the positions of the quark and the antiquark at the time t.

We further perform the so-called instantaneous approximation in (1.13), consisting in replacing

$$D_{\mu\nu}(x) \longrightarrow D^{\mathrm{inst}}_{\mu\nu}(x) = \delta(t)\int_{-\infty}^{+\infty} d\tau\, D_{\mu\nu}(\tau, \mathbf{x}) \quad (1.16)$$

and use (1.15) and (1.16) at an early stage in the derivation procedure. In this way we shall obtain Eqs. (1.1)–(1.4) in a much more direct way and without the need of assuming *a priori* the existence of a potential as done in [4]. So, once that Eqs. (1.13) and (1.14) have been written, Eqs. (1.15) and (1.16) give the conditions under which a description in terms of a potential actually holds.

Notice that, while (1.12), (1.14), and even (1.13) in the limit of large $t_f - t_i$, are gauge invariant quantities, the error introduced by (1.16) is strongly gauge dependent. The best choice of the gauge at the lowest order in perturbation theory is the Coulomb gauge for which the above error is minimum. To this choice Eq. (1.4) does refer.

For the three-quark case the quantity analogous to (1.12) is

$$W_{3q} = \frac{1}{3!}\left\langle \varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3}\left[P\exp\left(ig\int_{\overline{\Gamma}_1} dx^{\mu_1} A_{\mu_1}(x)\right)\right]^{a_1b_1} \times \left[P\exp\left(ig\int_{\overline{\Gamma}_2} dx^{\mu_2} A_{\mu_2}(x)\right)\right]^{a_2b_2} \times \left[P\exp\left(ig\int_{\overline{\Gamma}_3} dx^{\mu_3} A_{\mu_3}(x)\right)\right]^{a_3b_3}\right\rangle . \quad (1.17)$$

Here a_j, b_j are color indices; $\overline{\Gamma}_j$ denotes a curve made by a world line Γ_j for the quark j between the times t_i and t_f ($t_i < t_f$), a straight line on the surface $t = t_i$ merging from a point I (whose coordinate we denote by y_M) and connected to the world line, another straight line on the surface $t = t_f$ connecting the world line to a point F with coordinate x_M (Fig. 3). The positions of the two points I and F are determined by the same rules which determine the point M above.

The assumption corresponding to (1.13) and (1.14) is

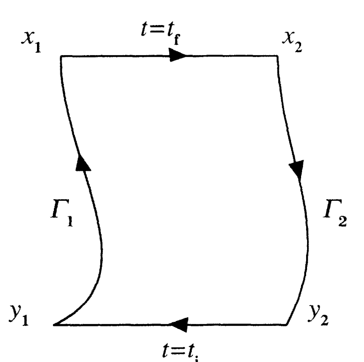

FIG. 2. Generalized Wilson loop for the quark-antiquark system.

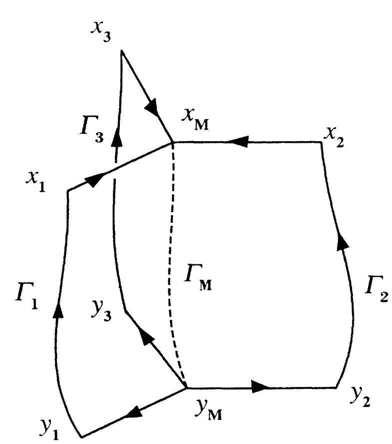

FIG. 3. The analogous of the Wilson loop for the three-quark system.

then

$$i \ln W_{3q} = \frac{2}{3} g^2 \sum_{i<j} \int_{\Gamma_i} dx_i^\mu \int_{\Gamma_j} dx_j^\nu \, i D_{\mu\nu}(x_i - x_j) + \sigma S_{\min} \,, \tag{1.18}$$

where now $S_{\min}$ denotes the minimum among the surfaces made by three sheets having the curves $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, and $\overline{\Gamma}_3$ as contours and joining on a line Γ_M connecting I with F.

We shall see that Eqs. (1.5)–(1.8) follow if we substitute (1.16) in (1.18) and again replace $S_{\min}$ with the surface spanned by the straight lines

$$u^0_{j\,\min} = t, \qquad \mathbf{u}_j^{\min} = s\mathbf{z}_j(t) + (1-s)\mathbf{z}_M(t) \tag{1.19}$$

with $j = 1,2,3$, $s \in [0,1]$, $\mathbf{z}_M(t)$ being again the point for which $\sum_{j=1}^3 |\mathbf{z}_j(t) - \mathbf{z}_M(t)|$ is minimum.

The plan of the paper is the following one. In Sec. II we shall report the simplified derivation of the quark-antiquark potential as sketched above. In Sec. III we shall report the derivation of the three-quark potential. In Sec. IV we shall make some remarks and discuss the connection with the flux tube model.

II. QUARK-ANTIQUARK POTENTIAL

As usual the starting point is the gauge invariant quark-antiquark $(q_1, \bar{q}_2)$ Green function (for the moment we assume the quark and the antiquark to have different flavors)

$$\begin{aligned} G(x_1,x_2,y_1,y_2) &= \frac{1}{3}\langle 0|T\psi_2^c(x_2)U(x_2,x_1)\psi_1(x_1)\overline{\psi}_1(y_1)U(y_1,y_2)\overline{\psi}_2^c(y_2)|0\rangle \\ &= \frac{1}{3}\mathrm{Tr}\langle U(x_2,x_1)S_1^F(x_1,y_1|A)U(y_1,y_2)C^{-1}S_2^F(y_2,x_2|A)C\rangle \,. \end{aligned} \tag{2.1}$$

Here c denotes the charge-conjugate fields, C is the charge-conjugation matrix, U the path-ordered gauge string

$$U(b,a) = P\exp\left(ig\int_a^b dx^\mu \, A_\mu(x)\right) \tag{2.2}$$

(the integration path being the straight line joining a to b), S_1^F and S_2^F the quark propagators in an external gauge field A^μ; furthermore in principle the angular brackets should be defined as

$$\langle f[A]\rangle = \frac{\int \mathcal{D}[A]M_f(A)f[A]e^{iS[A]}}{\int \mathcal{D}[A]M_f(A)e^{iS[A]}} \,, \tag{2.3}$$

$S[A]$ being the pure gauge field action and $M_f(A)$ the determinant resulting from the explicit integration on the fermionic fields. In practice assuming (1.14) corresponds to take $M_f(A) = 1$ (quenched approximation).

Summarizing the first part of the procedure followed in Ref. [4] (see such paper for details) first we assume $x_1^0 = x_2^0 = t_f$, $y_1^0 = y_2^0 = t_i$ with $\tau = t_f - t_i > 0$ and note that S_j^F are 4×4 Dirac indices matrices type. Then performing a Foldy-Wouthuysen transformation on G we can replace S_j^F with a Pauli propagator K_j (a 2×2 matrix in the spin indices) and obtain a two-particle Pauli-type Green function K. We shall show that in the described approximations this function satisfies a Schrödinger-like equation with the potential (1.1)–(1.4).

One finds (see [4]) that, up to the $1/m^2$ order, K_j satisfies the equation

$$\begin{aligned} i\frac{\partial}{\partial x^0}K_j(x,y|A) = H_{\mathrm{FW}}K_j(x,y|A) := \Bigg[& m_j + \frac{1}{2m_j}(\mathbf{p}_j - g\mathbf{A})^2 - \frac{1}{8m_j^3}(\mathbf{p}_j - g\mathbf{A})^4 - \frac{g}{m_j}\mathbf{S}_j\cdot\mathbf{B} + gA^0 \\ & - \frac{g}{8m_j^2}(\partial_i E^i - ig[A^i,E^i]) + \frac{g}{4m_j^2}\varepsilon^{ihk}S_j^k\{(p_j - gA)^i, E^h\}\Bigg] K_j(x,y|A) \end{aligned} \tag{2.4}$$

with the Cauchy condition

$$K_j(x,y|A)|_{x^0=y^0} = \delta^3(\mathbf{x}-\mathbf{y}) \tag{2.5}$$

where ε^{ihk} is the three-dimensional Ricci symbol and the summation over repeated indices is understood. By standard techniques the solution of Eq. (2.4), with the initial condition (2.5), can be expressed as a path integral in phase space:

$$K_j(x,y|A) = \int_{\mathbf{z}_j(y^0)=\mathbf{y}}^{\mathbf{z}_j(x^0)=\mathbf{x}} \mathcal{D}[\mathbf{z}_j,\mathbf{p}_j]\,T\exp\left\{i\int_{y^0}^{x^0} dt\,[\mathbf{p}_j\cdot\dot{\mathbf{z}}_j - H_{\mathrm{FW}}]\right\} \,; \tag{2.6}$$

here the time-ordering prescription T acts both on spin and gauge matrices, the trajectory of the quark j in configuration space is denoted by $\mathbf{z}_j = \mathbf{z}_j(t)$, the trajectory in momentum space by $\mathbf{p}_j = \mathbf{p}_j(t)$ and the spin by $\mathbf{S}_j$. Then, by performing the translation

$$\mathbf{p} \longrightarrow \mathbf{p} + g\mathbf{A}\,, \tag{2.7}$$

we obtain an equation containing the expression $dt\,(gA^0 - g\dot{\mathbf{z}}\cdot\mathbf{A}) \equiv g\,dx^\mu A_\mu$, which is formally covariant.[1] It is also useful to have an expression for K_j in which the tensor field $F^{\mu\nu}$ and its dual $\hat{F}^{\mu\nu}$ appear. To this end we make the further translation

$$\mathbf{p} \longrightarrow \mathbf{p} - \frac{g}{m}(\mathbf{E}\times\mathbf{S}) \tag{2.8}$$

and, apart from higher-order terms, we obtain

$$K_j(x,y|A) = \int_{\mathbf{z}_j(y^0)=\mathbf{y}}^{\mathbf{z}_j(x^0)=\mathbf{x}} \mathcal{D}[\mathbf{z}_j,\mathbf{p}_j]\, T\exp\left\{ i\int_{y^0}^{x^0} dt\,\left[\mathbf{p}_j\cdot\dot{\mathbf{z}}_j - m_j - \frac{\mathbf{p}_j^2}{2m_j} + \frac{\mathbf{p}_j^4}{8m_j^3} - gA^0 + \frac{g}{m_j}\mathbf{S}_j\cdot\mathbf{B} + \frac{g}{2m_j^2}\mathbf{S}_j\cdot(\mathbf{p}_j\times\mathbf{E}) - \frac{g}{m_j}\mathbf{S}_j\cdot(\dot{\mathbf{z}}_j\times\mathbf{E}) + g\dot{\mathbf{z}}_j\cdot\mathbf{A} + \frac{g}{8m_j^2}(\partial_i E^i - ig[A^i,E^i])\right]\right\}. \tag{2.9}$$

Thus we obtain the two-particle Pauli-type propagator K in the form of a path integral on the world lines of the two quarks:

$$\begin{aligned} K(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2;\tau) = &\int_{\mathbf{z}_1(t_i)=\mathbf{y}_1}^{\mathbf{z}_1(t_f)=\mathbf{x}_1} \mathcal{D}[\mathbf{z}_1,\mathbf{p}_1] \int_{\mathbf{z}_2(t_i)=\mathbf{y}_2}^{\mathbf{z}_2(t_f)=\mathbf{x}_2} \mathcal{D}[\mathbf{z}_2,\mathbf{p}_2] \\ &\times \exp\left\{ i\int_{t_i}^{t_f} dt\,\sum_{j=1}^{2}\left[\mathbf{p}_j\cdot\dot{\mathbf{z}}_j - m_j - \frac{\mathbf{p}_j^2}{2m_j} + \frac{\mathbf{p}_j^4}{8m_j^3}\right]\right\} \\ &\times \left\langle \frac{1}{3}\mathrm{Tr}\,T_s\,P\exp\left\{ ig\oint_\Gamma dx^\mu A_\mu(x) + \sum_{j=1}^{2}\frac{ig}{m_j}\int_{\Gamma_j} dx^\mu \right.\right. \\ &\left.\left.\times\left(S_j^l\hat{F}_{l\mu}(x) - \frac{1}{2m_j}S_j^l\varepsilon^{lkr}p_j^k F_{\mu r}(x) - \frac{1}{8m_j}D^\nu F_{\nu\mu}(x)\right)\right\}\right\rangle. \end{aligned} \tag{2.10}$$

Here T_s is the time-ordering prescription for spin matrices, P is the path-ordering prescription for gauge matrices along the loop Γ and, as usual,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu,A^\nu]\,, \tag{2.11}$$

$$\hat{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \tag{2.12}$$

$$D^\nu F_{\nu\mu} = \partial^\nu F_{\nu\mu} + ig[A^\nu, F_{\nu\mu}] \tag{2.13}$$

and $\varepsilon^{\mu\nu\rho\sigma}$ is the four-dimensional Ricci symbol.

Furthermore as in Eq. (1.12) Γ_1 denotes the path going from $(t_i,\mathbf{y}_1)$ to $(t_f,\mathbf{x}_1)$ along the quark trajectory $(t,\mathbf{z}_1(t))$, Γ_2 the path going from $(t_f,\mathbf{x}_2)$ to $(t_i,\mathbf{y}_2)$ along the antiquark trajectory $(t,\mathbf{z}_2(t))$ and Γ is the path made by Γ_1 and Γ_2 closed by the two straight lines joining $(t_i,\mathbf{y}_2)$ with $(t_i,\mathbf{y}_1)$ and $(t_f,\mathbf{x}_1)$ with $(t_f,\mathbf{x}_2)$ (see Fig. 2). Finally Tr denotes the trace on the gauge matrices. Note that the right-hand side of (2.10) is manifestly gauge invariant.

What we have to show is that the term in angular brackets in Eq. (2.10) can be expressed as the exponential of an integral function of the position, momentum and spin alone taken at the same time t:

$$\left\langle \frac{1}{3}\mathrm{Tr}\,T_s\,P\exp\{\cdots\}\right\rangle \simeq T_s\exp\left[-i\int_{t_i}^{t_f} dt\,V^{q\bar{q}}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{p}_1,\mathbf{p}_2,\mathbf{S}_1,\mathbf{S}_2)\right]\,; \tag{2.14}$$

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[1]More precisely, since the A^h are matrices, the step $\int d^3\mathbf{p}\, f(\mathbf{p} - g\mathbf{A}) = \int d^3\mathbf{p}\, f(\mathbf{p})$ can be justified by expanding $f(\mathbf{p} - g\mathbf{A})$ in powers of g; apart from the zeroth-order term, all the other terms involve derivatives of $f(\mathbf{p})$ and do not contribute to the integral.

indeed then we can conclude that

$$i\frac{\partial}{\partial t}K = \left[\sum_{j=1}^{2}\left(m_j + \frac{\mathbf{p}_j^2}{2m_j} - \frac{\mathbf{p}_j^4}{8m_j^3}\right) + V^{q\bar{q}}\right] K\,, \tag{2.15}$$

$V^{q\bar{q}}$ playing the role of a two-particle potential. To this aim, expanding the logarithm of the left-hand side of (2.14) up to $1/m^2$ order, we should have

$$\begin{aligned} i\ln W_{q\bar{q}} + i\sum_{j=1}^{2}\frac{ig}{m_j}\int_{\Gamma_j} dx^\mu \Bigg(& S_j^l\,\langle\langle \hat{F}_{l\mu}(x)\rangle\rangle - \frac{1}{2m_j}S_j^l\varepsilon^{lkr}p_j^k\,\langle\langle F_{\mu r}(x)\rangle\rangle \\ & - \frac{1}{8m_j}\,\langle\langle D^\nu F_{\nu\mu}(x)\rangle\rangle\Bigg) - \frac{1}{2}\sum_{j,j'}\frac{ig^2}{m_j m_{j'}}T_s\int_{\Gamma_j}dx^\mu\int_{\Gamma_{j'}}dx'^\sigma\, S_j^l\, S_{j'}^k \\ & \times\Bigg(\langle\langle \hat{F}_{l\mu}(x)\hat{F}_{k\sigma}(x')\rangle\rangle - \langle\langle \hat{F}_{l\mu}(x)\rangle\rangle\,\langle\langle \hat{F}_{k\sigma}(x')\rangle\rangle\Bigg) \simeq \left[\int_{t_i}^{t_f} dt\, V^{q\bar{q}}\right]\,, \end{aligned} \tag{2.16}$$

with the notation

$$\langle\langle f[A]\rangle\rangle = \frac{\frac{1}{3}\left\langle \mathrm{Tr}\,P\left\{\exp[ig\oint_\Gamma dx^\mu\,A_\mu(x)]\right\} f[A]\right\rangle}{\frac{1}{3}\left\langle \mathrm{Tr}\,P\exp(ig\oint_\Gamma dx^\mu\,A_\mu(x))\right\rangle} \tag{2.17}$$

and $W_{q\bar{q}}$ defined in Eq. (1.12).

At this point in Ref. [4] we assumed that a quantity $V^{q\bar{q}}$ satisfying (2.16) existed and derived its form. Here we no longer make such an *a priori* assumption but start directly from (1.13) and (1.14).

A. Contribution to the potential coming from $i\ln W_{q\bar{q}}$

In the Coulomb gauge we have

$$D_{00}(x) = \frac{i}{4\pi}\frac{1}{|\mathbf{x}|}\delta(t)\,, \tag{2.18}$$

$$D_{hk}(x) = (\delta_{hk} - (\nabla^2)^{-1}\partial_h\partial_k)D_{\mathrm{F}}(x) = i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\delta^{hk} - \frac{k^hk^k}{|\mathbf{k}|^2}\right)e^{-ikx}\,, \tag{2.19}$$

$$D_{0k}(x) = D_{k0}(x) = 0, \tag{2.20}$$

where $D_{\mathrm{F}}(x) = \int(d^4k/16\pi^4)(i/k^2)e^{-ikx}$.

The pure temporal part $D_{00}(x)$ is already of instantaneous type, for the pure spatial part we have the instantaneous limit

$$D_{hk}^{\mathrm{inst}}(x) = \delta(t)\int_{-\infty}^{+\infty} d\tau\, D_{hk}(\tau,\mathbf{x}) = -\delta(t)\frac{i}{8\pi|\mathbf{x}|}\left(\delta^{hk} + \frac{x^hx^k}{|\mathbf{x}|^2}\right)\,. \tag{2.21}$$

Replacing (2.18), (2.20), and (2.21) in (1.13) we have

$$i\ln W_{q\bar{q}}^{\mathrm{SR}} = \int_{t_i}^{t_f} dt\left\{-\frac{4}{3}\frac{\alpha_s}{r} + \frac{1}{2}\frac{4}{3}\frac{\alpha_s}{r}(\delta^{hk} + \hat{r}^h\hat{r}^k)\dot{z}_1^h\dot{z}_2^k\right\}\,. \tag{2.22}$$

Similarly, if we denote by $u^\mu = u^\mu(s,t)$ the equation of any surface with contour Γ ($s\in[0,1]$, $t\in[t_i,t_f]$, $u^0(s,t) = t$, $\mathbf{u}(1,t) = \mathbf{z}_1(t)$, $\mathbf{u}(0,t) = \mathbf{z}_2(t)$), we can write

$$\begin{aligned} i\ln W_{q\bar{q}}^{\mathrm{LR}} = \sigma S_{\min} &= \sigma\min\int_{t_i}^{t_f} dt\int_0^1 ds\left[-\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial t}\right)\left(\frac{\partial u^\mu}{\partial s}\frac{\partial u_\mu}{\partial s}\right) + \left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial s}\right)^2\right]^{1/2} \\ &= \sigma\min\int_{t_i}^{t_f} dt\int_0^1 ds\left|\frac{\partial\mathbf{u}}{\partial s}\right|\left\{1 - \left[\left(\frac{\partial\mathbf{u}}{\partial t}\right)_T\right]^2\right\}^{1/2}\,, \end{aligned} \tag{2.23}$$

where the index T denotes the transverse part with respect to the unit vector $\hat{\mathbf{s}}$:

$$\hat{\mathbf{s}} = \frac{\partial \mathbf{u}}{\partial s} \Big/ \left| \frac{\partial \mathbf{u}}{\partial s} \right| . \tag{2.24}$$

Then in approximation (1.15) we have

$$\frac{\partial \mathbf{u}^{\min}}{\partial s} = \mathbf{z}_1(t) - \mathbf{z}_2(t) \equiv \mathbf{r}(t) , \tag{2.25}$$

$$\frac{\partial \mathbf{u}^{\min}}{\partial t} = s\dot{\mathbf{z}}_1(t) + (1-s)\dot{\mathbf{z}}_2(t), \tag{2.26}$$

and so

$$\begin{aligned} i \ln W_{q\bar{q}}^{\mathrm{LR}} &= \int_{t_i}^{t_f} dt\, \sigma r \int_0^1 ds\, [1 - (s\dot{\mathbf{z}}_{1T} + (1-s)\dot{\mathbf{z}}_{2T})^2]^{1/2} \\ &= \int_{t_i}^{t_f} dt\, \sigma r \left[1 - \frac{1}{6}(\dot{\mathbf{z}}_{1T}^2 + \dot{\mathbf{z}}_{2T}^2 + \dot{\mathbf{z}}_{1T}\cdot\dot{\mathbf{z}}_{2T}) \right. \\ &\quad \left. + \cdots \right], \end{aligned} \tag{2.27}$$

where obviously now $\hat{\mathbf{s}} = \hat{\mathbf{r}}$ and we have expanded the square root and performed the s integration explicitly.

Notice now that, at the lowest order, $\dot{\mathbf{z}}_j$ can be replaced by $\mathbf{p}_j/m_j$ in (2.22) and (2.27). Then such equations become of the correct form required by (2.16) and so does the entire $i \ln W_{q\bar{q}}$. In conclusion we have a first contribution to $V^{q\bar{q}}$ (pure Wilson loop contribution) in the form $V_{\mathrm{stat}}^{q\bar{q}} + V_{\mathrm{VD}}^{q\bar{q}}$ with $V_{\mathrm{stat}}^{q\bar{q}}$ and $V_{\mathrm{VD}}^{q\bar{q}}$ as given in (1.2) and (1.4). The ordering prescription in (1.4) shall be discussed in Sec. IV.

B. Spin-related potential

To obtain the remaining part of the potential we must evaluate the expectation values of the form (2.17) occurring in (2.16). Let us consider an arbitrary infinitesimal variation $\mathbf{z}_1(t) \longrightarrow \mathbf{z}_1(t) + \delta\mathbf{z}_1(t)$ vanishing at $t = t_f$ and $t = t_i$ and evaluate $\delta(i \ln W_{q\bar{q}})$.[2] From (1.12) we have

$$\begin{aligned} \delta W_{q\bar{q}} = -\frac{ig}{3} \Big\langle \mathrm{Tr}\, P \int_{t_i}^{t_f} \delta S^{\mu\nu}(z_1)\, F_{\mu\nu}(z_1) \\ \times \exp\left(ig \oint_\Gamma dx^\mu\, A_\mu(x) \right) \Big\rangle, \end{aligned} \tag{2.28}$$

where $\delta S^{\mu\nu}(z_1) = \frac{1}{2}(dz_1^\mu\, \delta z_1^\nu - dz_1^\nu \delta z_1^\mu)$ is the element of the surface spanned by $z_1(t)$.

Then

$$\delta(i \ln W_{q\bar{q}}) = i \frac{\delta W_{q\bar{q}}}{W_{q\bar{q}}} = g \int_{t_i}^{t_f} \delta S^{\mu\nu}(z_1)\, \langle\langle F_{\mu\nu}(z_1) \rangle\rangle \tag{2.29}$$

and we may write

$$g\, \langle\langle F_{\mu\nu}(z_1) \rangle\rangle = \frac{\delta(i \ln W_{q\bar{q}})}{\delta S^{\mu\nu}(z_1)} \tag{2.30}$$

(see Appendix A for a definition of $\delta/\delta S^{\mu\nu}(z_1)$). The computation is similar for the case of z_2 with a minus sign of difference.

Similarly it can be seen that

$$g^2\, \langle\langle F_{\mu\nu}(z_1) F_{\rho\sigma}(z_2) \rangle\rangle = \frac{1}{W_{q\bar{q}}} \frac{\delta^2 W_{q\bar{q}}}{\delta S^{\mu\nu}(z_1) \delta S^{\rho\sigma}(z_2)} \tag{2.31}$$

and therefore

$$g^2 \Big(\langle\langle F_{\mu\nu}(z_1) F_{\rho\sigma}(z_2) \rangle\rangle - \langle\langle F_{\mu\nu}(z_1) \rangle\rangle \, \langle\langle F_{\rho\sigma}(z_2) \rangle\rangle \Big) = \frac{\delta^2 \ln W_{q\bar{q}}}{\delta S^{\mu\nu}(z_1) \delta S^{\rho\sigma}(z_2)} = -ig \frac{\delta}{\delta S^{\rho\sigma}(z_2)} \langle\langle F_{\mu\nu}(z_1) \rangle\rangle . \tag{2.32}$$

From (1.13) we have then

$$g\, \langle\langle F_{\mu\nu}(z_1) \rangle\rangle^{\mathrm{SR}} = \frac{4}{3} g^2 \int_{t_i}^{t_f} dt_2\, i[\partial_\nu D_{\mu\rho}(z_1 - z_2) - \partial_\mu D_{\nu\rho}(z_1 - z_2)] \dot{z}_2^\rho \tag{2.33}$$

and

$$\begin{aligned} g^2 \Big(\langle\langle F_{\mu\nu}(z_1) F_{\rho\sigma}(z_2) \rangle\rangle - \langle\langle F_{\mu\nu}(z_1) \rangle\rangle \, \langle\langle F_{\rho\sigma}(z_2) \rangle\rangle \Big)^{\mathrm{SR}} &= \frac{4}{3} g^2 \{ \partial_\rho [\partial_\nu D_{\mu\sigma}(z_1 - z_2) - \partial_\mu D_{\nu\sigma}(z_1 - z_2)] \\ &\quad - \partial_\sigma [\partial_\nu D_{\mu\rho}(z_1 - z_2) - \partial_\mu D_{\nu\rho}(z_1 - z_2)] \} . \end{aligned} \tag{2.34}$$

Notice that obviously in the terminology of Appendix A we have $C(z_1, z_1') = 0$ and then

$$\Big(\langle\langle F_{\mu\nu}(z_1) F_{\rho\sigma}(z_1') \rangle\rangle - \langle\langle F_{\mu\nu}(z_1) \rangle\rangle \, \langle\langle F_{\rho\sigma}(z_1') \rangle\rangle \Big)^{\mathrm{SR}} = 0 . \tag{2.35}$$

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[2]The result is easily achieved in the case of an Abelian gauge theory: in fact one can freely commute the fields A_μ. In our case, on the contrary, the gauge theory is non-Abelian: one can still commute fields referring to different points because of the presence of the path-ordering operator P; however, fields at the same point do not commute and this fact must be taken into account. As a rule, one can make calculations in the Abelian case; by rewriting the final result in a gauge invariant form, an expression also valid in the non-Abelian case is obtained.

By using the Coulomb gauge and the instantaneous approximation we have then

$$g\,\langle\langle F_{0k}(z_1)\rangle\rangle^{\rm SR} = \frac{4}{3}\alpha_s \frac{r^k}{r^3}\,, \tag{2.36}$$

$$g\,\langle\langle F_{hk}(z_1)\rangle\rangle^{\rm SR} = \frac{4}{3}\frac{\alpha_s}{m_2}\frac{1}{r^3}(r^h p_2^k - r^k p_2^h)\,, \tag{2.37}$$

$$g^2\Big(\langle\langle F_{hk}(z_1)F_{lm}(z_2)\rangle\rangle - \langle\langle F_{hk}(z_1)\rangle\rangle\,\langle\langle F_{lm}(z_2)\rangle\rangle\Big)^{\rm SR}$$
$$= -\frac{4}{3}\frac{ig^2}{8\pi}\delta(t_1-t_2)\left\{\partial_l\partial_k\left[\frac{1}{r}\left(\delta^{hm}+\hat r^h\hat r^m\right)\right] - \partial_l\partial_h\left[\frac{1}{r}\left(\delta^{km}+\hat r^k\hat r^m\right)\right]\right.$$
$$\left. -\partial_m\partial_k\left[\frac{1}{r}\left(\delta^{hl}+\hat r^h\hat r^l\right)\right] + \partial_m\partial_h\left[\frac{1}{r}\left(\delta^{kl}+\hat r^k\hat r^l\right)\right]\right\}\,. \tag{2.38}$$

In a similar way we obtain also

$$g\,\langle\langle F_{0k}(z_2)\rangle\rangle^{\rm SR} = \frac{4}{3}\alpha_s \frac{r^k}{r^3}\,, \tag{2.39}$$

$$g\,\langle\langle F_{hk}(z_2)\rangle\rangle^{\rm SR} = \frac{4}{3}\frac{\alpha_s}{m_1}\frac{1}{r^3}(r^h p_1^k - r^k p_1^h)\,. \tag{2.40}$$

Let us now consider the confinement part of $i\ln W_{q\bar q}$. From Eq. (2.23), taking into account that $u_\mu^{\rm min}$ satisfies the appropriate Euler equation, one has (see Appendix B for details)

$$g\,\langle\langle F_{\mu\nu}(z_1)\rangle\rangle^{\rm LR} = \sigma\left[\left(\frac{\partial u_\mu^{\rm min}}{\partial s}\right)_1 \dot z_{1\nu} - \left(\frac{\partial u_\nu^{\rm min}}{\partial s}\right)_1 \dot z_{1\mu}\right]\left\{-\dot z_1^2\left(\frac{\partial u^{\rm min}}{\partial s}\right)_1^2 + \left[\dot z_1\cdot\left(\frac{\partial u^{\rm min}}{\partial s}\right)_1\right]^2\right\}^{-1/2}, \tag{2.41}$$

where the subscript 1 indicates that the derivative is calculated in $s=1$. A similar formula is valid for z_2. Moreover, as in the short-range case,

$$\Big(\langle\langle F_{\mu\nu}(z_1)F_{\rho\sigma}(z_1')\rangle\rangle - \langle\langle F_{\mu\nu}(z_1)\rangle\rangle\,\langle\langle F_{\rho\sigma}(z_1')\rangle\rangle\Big)^{\rm LR} = 0\,. \tag{2.42}$$

By using the straight-line approximation one obtains

$$g\,\langle\langle F_{0k}(z_j)\rangle\rangle^{\rm LR} = \sigma\frac{r^k}{r} + O(v^2)\,, \tag{2.43}$$

$$g\,\langle\langle F_{hk}(z_j)\rangle\rangle^{\rm LR} = \frac{\sigma}{m_j}\frac{1}{r}(r^h p_j^k - r^k p_j^h) + O(v^3) \tag{2.44}$$

with $j=1,2$ and

$$g^2\Big(\langle\langle F_{hk}(z_1)F_{lm}(z_2)\rangle\rangle - \langle\langle F_{hk}(z_1)\rangle\rangle\,\langle\langle F_{lm}(z_2)\rangle\rangle\Big)^{\rm LR} = -ig\frac{\delta}{\delta S^{lm}(z_2)}\langle\langle F_{hk}(z_1)\rangle\rangle^{\rm LR} = O(v^2)\,. \tag{2.45}$$

Finally replacing (2.35)–(2.40) and (2.42)–(2.45) in (2.16) we obtain the terms in Eq. (1.3) involving explicitly spin.

As concerns the Darwin-type terms one should evaluate $g\,\langle\langle D^\nu F_{\nu\mu}(x)\rangle\rangle$ in Eq. (2.16). By using the results of this section one can obtain, by gauge invariance,

$$g\int_{\Gamma_1} dz_1^\mu\,\langle\langle D^\nu F_{\nu\mu}(z_1)\rangle\rangle = \int_{t_i}^{t_f} dt\,\dot z_1^\mu \partial^\nu\langle\langle F_{\nu\mu}(z_1)\rangle\rangle$$
$$= g\int_{t_i}^{t_f} dt\,\Big(\partial^\nu\langle\langle F_{\nu 0}(z_1)\rangle\rangle + \dot z_1^k\partial^\nu\langle\langle F_{\nu k}(z_1)\rangle\rangle\Big) = \int_{t_i}^{t_f} dt\,\partial^h\Big(-\frac{4}{3}\alpha_s\frac{r^h}{r^3} - \sigma\frac{r^h}{r} + \cdots\Big)$$
$$= \int_{t_i}^{t_f} dt\,\nabla^2\Big(-\frac{4}{3}\frac{\alpha_s}{r} + \sigma r\Big) + \cdots, \tag{2.46}$$

from which finally we have the Darwin terms as reported in Eq. (1.3).

III. THREE-QUARK POTENTIAL

Let us define the three-quark color singlet state (again for three quarks of different flavors):

$$\frac{1}{\sqrt{3!}}\varepsilon_{b_1b_2b_3}\overline{\psi}_{1d_1}(y_1)\overline{\psi}_{2d_2}(y_2)\overline{\psi}_{3d_3}(y_3)U^{d_1b_1}(y_1,y_M)U^{d_2b_2}(y_2,y_M)U^{d_3b_3}(y_3,y_M)|0\rangle, \tag{3.1}$$

where the path-ordered gauge strings U are defined in Eq. (2.2), y_M is defined as explained after Eq. (1.17), b_i and d_i for $i=1,2,3$ are the color indices, $\varepsilon_{b_1b_2b_3}$ is the completely antisymmetric tensor in the color indices, which puts the system of three quarks in a color singlet state and $1/\sqrt{3!}$ is the normalization factor.

The corresponding gauge invariant Green function can be written

$$\begin{aligned}G(x_1,x_2,x_3,y_1,y_2,y_3) = {} & \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3} \\ & \times\langle 0|T\,U^{a_3c_3}(x_M,x_3)U^{a_2c_2}(x_M,x_2)U^{a_1c_1}(x_M,x_1)\psi_{3c_3}(x_3)\psi_{2c_2}(x_2)\psi_{1c_1}(x_1) \\ & \times\overline{\psi}_{1d_1}(y_1)\overline{\psi}_{2d_2}(y_2)\overline{\psi}_{3d_3}(y_3)U^{d_1b_1}(y_1,y_M)U^{d_2b_2}(y_2,y_M)U^{d_3b_3}(y_3,y_M)|0\rangle,\end{aligned} \tag{3.2}$$

and we assume $x_1^0=x_2^0=x_3^0=x_M^0=t_f$, $y_1^0=y_2^0=y_3^0=y_M^0=t_i$; $\tau=t_f-t_i\,(\tau>0)$.

The integration over the Grassmann variables is again trivial and one can write

$$\begin{aligned}G(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3;\tau) = {} & \frac{1}{3!}\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3}\langle[U(x_M,x_1)S_1^F(x_1,y_1|A)U(y_1,y_M)]^{a_1b_1} \\ & \times[U(x_M,x_2)S_2^F(x_2,y_2|A)U(y_2,y_M)]^{a_2b_2}[U(x_M,x_3)S_3^F(x_3,y_3|A)U(y_3,y_M)]^{a_3b_3}\rangle\,;\end{aligned} \tag{3.3}$$

if some of the quarks are identical we have simply to sum over all permutations of the corresponding final variables. By performing even in this case the appropriate Foldy-Wouthuysen transformations and using (2.9) we find, in place of (2.10),

$$\begin{aligned}K(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3;\tau|\,\mathbf{x}_M,\mathbf{y}_M) = {} & \int_{\mathbf{z}_1(t_i)=\mathbf{y}_1}^{\mathbf{z}_1(t_f)=\mathbf{x}_1}\mathcal{D}[\mathbf{z}_1,\mathbf{p}_1]\int_{\mathbf{z}_2(t_i)=\mathbf{y}_2}^{\mathbf{z}_2(t_f)=\mathbf{x}_2}\mathcal{D}[\mathbf{z}_2,\mathbf{p}_2]\int_{\mathbf{z}_3(t_i)=\mathbf{y}_3}^{\mathbf{z}_3(t_f)=\mathbf{x}_3}\mathcal{D}[\mathbf{z}_3,\mathbf{p}_3] \\ & \times\exp\left\{i\int_{t_i}^{t_f}dt\sum_{j=1}^{3}\left[\mathbf{p}_j\cdot\dot{\mathbf{z}}_j-m_j-\frac{\mathbf{p}_j^2}{2m_j}+\frac{\mathbf{p}_j^4}{8m_j^3}\right]\right\} \\ & \times\left\langle\frac{1}{3!}\varepsilon\varepsilon\prod_{j=1}^{3}T_s\,P\exp\left\{ig\int_{\Gamma_j}dx^\mu\,A_\mu(x)+\frac{ig}{m_j}\int_{\Gamma_j}dx^\mu\right.\right. \\ & \times\left.\left.\left(S_j^l\hat{F}_{l\mu}(x)-\frac{1}{2m_j}S_j^l\varepsilon^{lkr}p_j^kF_{\mu r}(x)-\frac{1}{8m_j}D^\nu F_{\nu\mu}(x)\right)\right\}\right\rangle\,,\end{aligned} \tag{3.4}$$

K being now the three-quark Pauli-type Green function.

In (3.4) we have suppressed for convenience the color indices but have left trace of the tensors $\varepsilon_{a_1a_2a_3}\varepsilon_{b_1b_2b_3}$ with the notation $\varepsilon\varepsilon$. As above T_s denotes the chronological ordering for the spin matrices and P is the path-ordering prescription acting on the gauge matrices; Γ_j and $\overline{\Gamma}_j$ are defined as in Eq. (1.17) and following. Notice that the curve Γ made by the union of $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, and $\overline{\Gamma}_3$ is a closed three-branch loop, which generalizes the Wilson loop of the two-body system.

From now on one can proceed strictly as in Sec. II. We shall show that from (1.18) using (1.16) and (1.19) one can write

$$\left\langle\frac{1}{3!}\varepsilon\varepsilon\prod_{j=1}^{3}T_s\,P\exp\cdots\right\rangle\simeq T_s\exp\left[-i\int_{t_i}^{t_f}dt\,V^{3q}(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3)\right], \tag{3.5}$$

and so the propagator K obeys the three-particle Schrödinger-like equation

$$i\frac{\partial}{\partial t}K=\left[\sum_{j=1}^{3}\left(m_j+\frac{\mathbf{p}_j^2}{2m_j}-\frac{\mathbf{p}_j^4}{8m_j^3}\right)+V^{3q}\right]K\,. \tag{3.6}$$

Again expanding the logarithm of the left-hand side, Eq. (3.5) turns to be equivalent to

$$i\ln W_{3q}+i\sum_{j=1}^{3}\frac{ig}{m_j}\int_{\Gamma_j}dx^\mu\Big(S_j^l\,\langle\langle\hat F_{l\mu}(x)\rangle\rangle-\frac{1}{2m_j}S_j^l\varepsilon^{lkr}p_j^k\,\langle\langle F_{\mu r}(x)\rangle\rangle-\frac{1}{8m_j}\langle\langle D^\nu F_{\nu\mu}(x)\rangle\rangle\Big)-\frac{1}{2}\sum_{j,j'}\frac{ig^2}{m_jm_{j'}}T_s\int_{\Gamma_j}dx^\mu\int_{\Gamma_{j'}}dx'^\sigma\,S_j^l\,S_{j'}^k\times\Big(\langle\langle\hat F_{l\mu}(x)\hat F_{k\sigma}(x')\rangle\rangle-\langle\langle\hat F_{l\mu}(x)\rangle\rangle\,\langle\langle\hat F_{k\sigma}(x')\rangle\rangle\Big)\simeq\left[\int_{t_i}^{t_f}dt\,V^{3q}\right]\tag{3.7}$$

with W_{3q} being defined as in Eq. (1.17) and now

$$\langle\langle f[A]\rangle\rangle=\frac{\frac{1}{3!}\langle\varepsilon\varepsilon\,\{\prod_j P[\exp(ig\int_{\Gamma_j}dx^\mu A_\mu(x))]\}f[A]\rangle}{\frac{1}{3!}\langle\varepsilon\varepsilon\,\{\prod_j P\exp(ig\int_{\Gamma_j}dx^\mu A_\mu(x))\}\rangle}\,.\tag{3.8}$$

A. Contribution to the potential coming from $i\ln W_{3q}$

Having in mind (1.19) let's evaluate $S_{\rm min}$ in the same manner as we did for the two-body case. The quantity $S_{\rm min}$ is the area made by three sheet surfaces as described in Sec. I. Let us denote by $z_M(t)$ an arbitrary world line joining y_M to x_M and by $u_j^\mu=u_j^\mu(s,t)$ the equation of an arbitrary sheet interpolating between the trajectories $z_j^\mu=z_j^\mu(t)$ and $z_M^\mu=z_M^\mu(t)$; obviously $\mathbf{u}_j(0,t)=\mathbf{z}_M(t)$, $\mathbf{u}_j(1,t)=\mathbf{z}_j(t)$ and $u_j^0(s,t)=t$.

Assuming that the minimum is taken in the choice of $u_j^\mu(s,t)$ and of $\mathbf{z}_M(t)$ we can write, in analogy with (2.23),

$$S_{\rm min}=\min\sum_{j=1}^{3}\int_{t_i}^{t_f}dt\int_0^1 ds\left|\frac{\partial\mathbf{u}_j}{\partial s}\right|\times\left\{1-\left[\left(\frac{\partial\mathbf{u}_j}{\partial t}\right)_{T_j}\right]^2\right\}^{1/2},\tag{3.9}$$

where now the index T_j stands for transverse part of a vector with respect to

$$\hat{\mathbf{s}}_j=\frac{\partial\mathbf{u}_j}{\partial s}\Big/\left|\frac{\partial\mathbf{u}_j}{\partial s}\right|\,.\tag{3.10}$$

Then, performing the straight-line approximation (1.19) we have

$$\frac{\partial\mathbf{u}_j^{\rm min}}{\partial s}=\mathbf{z}_j(t)-\mathbf{z}_M(t)\equiv\mathbf{r}_j(t)\,,\tag{3.11}$$

$$\frac{\partial\mathbf{u}_j^{\rm min}}{\partial t}=s\dot{\mathbf{z}}_j(t)+(1-s)\dot{\mathbf{z}}_M(t)\tag{3.12}$$

and expanding in the velocities

$$S_{\rm min}=\int_{t_i}^{t_f}dt\sum_{j=1}^{3}r_j\left[1-\frac{1}{6}(\dot{\mathbf{z}}_{jT_j}^2+\dot{\mathbf{z}}_{MT_j}^2+\dot{\mathbf{z}}_{jT_j}\cdot\dot{\mathbf{z}}_{MT_j})+\cdots\right]\tag{3.13}$$

with $\hat{\mathbf{s}}_j=\hat{\mathbf{r}}_j$.

Taking into account this result and introducing (2.18), (2.20), and (2.21) in (1.18),

$$i\ln W_{3q}=\int_{t_i}^{t_f}dt\left\{\sum_{j<l}\left[-\frac{2}{3}\frac{\alpha_s}{r_{jl}}+\frac{1}{2}\frac{2}{3}\frac{\alpha_s}{r_{jl}}(\delta^{hk}+\hat r_{jl}^h\hat r_{jl}^k)\dot z_j^h\dot z_l^k\right]+\sigma\sum_{j=1}^{3}r_j\left[1-\frac{1}{6}(\dot{\mathbf{z}}_{jT_j}^2+\dot{\mathbf{z}}_{MT_j}^2+\dot{\mathbf{z}}_{jT_j}\cdot\dot{\mathbf{z}}_{MT_j})\right]\right\},\tag{3.14}$$

where again $\mathbf{r}_{jl}=\mathbf{r}_j-\mathbf{r}_l\equiv\mathbf{z}_j-\mathbf{z}_l$. In the I configuration case the quantity $\dot{\mathbf{z}}_M$ can be obtained deriving the equation $\sum_{j=1}^3(\mathbf{r}_j/r_j)=0$. We get

$$\sum_{j=1}^{3}\frac{1}{r_j}(\delta^{hk}-\hat r_j^h\hat r_j^k)\dot z_j^k=\sum_{j=1}^{3}\frac{1}{r_j}(\delta^{hk}-\hat r_j^h\hat r_j^k)\dot z_M^k.$$

Obviously in the II configuration case we have $\dot{\mathbf{z}}_M=\dot{\mathbf{z}}_l$ ($M\equiv$ quark l).

Finally replacing $\dot{\mathbf{z}}_j$ by $\mathbf{p}_j/m_j$ we obtain Eq. (1.8).

B. Spin related potential

As in the $q\bar q$ case we can write

$$g\,\langle\langle F_{\mu\nu}(z_j)\rangle\rangle=\frac{\delta(i\ln W_{3q})}{\delta S^{\mu\nu}(z_j)}\,,\tag{3.15}$$

$$g^2\Big(\langle\langle F_{\mu\nu}(z_j)F_{\rho\sigma}(z_i)\rangle\rangle-\langle\langle F_{\mu\nu}(z_j)\rangle\rangle\,\langle\langle F_{\rho\sigma}(z_i)\rangle\rangle\Big)=ig\frac{\delta}{\delta S^{\rho\sigma}(z_i)}\langle\langle F_{\mu\nu}(z_j)\rangle\rangle\tag{3.16}$$

with $j, i = 1, 2, 3$ and adapt immediately the procedure used in the derivation of Eqs. (2.35)–(2.40) and (2.42)–(2.45) to the variation of each single quark world line separately.

From the short-range part of (1.18) we have then

$$g\,\langle\langle F_{\mu\nu}(z_j)\rangle\rangle^{\mathrm{SR}} = \frac{2}{3}g^2\left\{\int_{t_i}^{t_f} dt_i\, i[\partial_\nu D_{\mu\rho}(z_j - z_i) - \partial_\mu D_{\nu\rho}(z_j - z_i)]\dot{z}_i^\rho + \int_{t_i}^{t_f} dt_n\, i[\partial_\nu D_{\mu\rho}(z_j - z_n) - \partial_\mu D_{\nu\rho}(z_j - z_n)]\dot{z}_n^\rho\right\} \tag{3.17}$$

with $j, i, n =$ cyclic permutation of $1, 2, 3$ and

$$g^2\left(\langle\langle F_{\mu\nu}(z_j)F_{\rho\sigma}(z_i)\rangle\rangle - \langle\langle F_{\mu\nu}(z_j)\rangle\rangle\,\langle\langle F_{\rho\sigma}(z_i)\rangle\rangle\right)^{\mathrm{SR}} = \frac{2}{3}g^2[\partial_\sigma\partial_\nu D_{\mu\rho}(z_j - z_i) - \partial_\sigma\partial_\mu D_{\nu\rho}(z_j - z_i) - \partial_\rho\partial_\nu D_{\mu\sigma}(z_j - z_i) + \partial_\rho\partial_\mu D_{\nu\sigma}(z_j - z_i)] \tag{3.18}$$

for $j \neq i$; furthermore,

$$\left(\langle\langle F_{\mu\nu}(z_j)F_{\rho\sigma}(z_j')\rangle\rangle - \langle\langle F_{\mu\nu}(z_j)\rangle\rangle\,\langle\langle F_{\rho\sigma}(z_j')\rangle\rangle\right)^{\mathrm{SR}} = 0\,. \tag{3.19}$$

By using the Coulomb gauge and the instantaneous approximation we have, from Eqs. (3.17) and (3.18),

$$g\,\langle\langle F_{0k}(z_j)\rangle\rangle^{\mathrm{SR}} = \frac{2}{3}\alpha_s\left(\frac{r_{ji}^k}{r_{ji}^3} + \frac{r_{jn}^k}{r_{jn}^3}\right)\,, \tag{3.20}$$

$$g\,\langle\langle F_{hk}(z_j)\rangle\rangle^{\mathrm{SR}} = \frac{2}{3}\frac{\alpha_s}{m_i}\frac{1}{r_{ji}^3}(r_{ji}^h p_i^k - r_{ji}^k p_i^h) + \frac{2}{3}\frac{\alpha_s}{m_n}\frac{1}{r_{jn}^3}(r_{jn}^h p_n^k - r_{jn}^k p_n^h) \tag{3.21}$$

($j, i, n =$ cyclic permutation of $1, 2, 3$) and

$$g^2\left(\langle\langle F_{hk}(z_j)F_{lm}(z_i)\rangle\rangle - \langle\langle F_{hk}(z_j)\rangle\rangle\,\langle\langle F_{lm}(z_i)\rangle\rangle\right)^{\mathrm{SR}}$$
$$= -\frac{2}{3}\frac{ig^2}{8\pi}\delta(t_j - t_i)\left\{\partial_l^{(i)}\partial_k^{(j)}\left[\frac{1}{r_{ji}}\left(\delta^{hm} + \hat{r}_{ji}^h\hat{r}_{ji}^m\right)\right] - \partial_l^{(i)}\partial_h^{(j)}\left[\frac{1}{r_{ji}}\left(\delta^{km} + \hat{r}_{ji}^k\hat{r}_{ji}^m\right)\right] - \partial_m^{(i)}\partial_k^{(j)}\left[\frac{1}{r_{ji}}\left(\delta^{hl} + \hat{r}_{ji}^h\hat{r}_{ji}^l\right)\right] + \partial_m^{(i)}\partial_h^{(j)}\left[\frac{1}{r_{ji}}\left(\delta^{kl} + \hat{r}_{ji}^k\hat{r}_{ji}^l\right)\right]\right\}, \tag{3.22}$$

where $j \neq i$.

Let us now consider the long-range part of Eq. (1.18). Notice that, because of its definition, an infinitesimal variation of the intermediate point world line $\mathbf{z}_M = \mathbf{z}_M(t)$ leaves the quantity $i \ln W_{3q}$ unchanged. Then, in the evaluation of the functional derivatives (3.15) and (3.16), we can treat such world line as fixed. From Eq. (3.9), taking into account that $u_{j\mu}^{\min}$ satisfies the appropriate Euler equation, one has

$$g\,\langle\langle F_{\mu\nu}(z_j)\rangle\rangle^{\mathrm{LR}} = \sigma\left[\left(\frac{\partial u_{j\mu}^{\min}}{\partial s_j}\right)_1 \dot{z}_{j\nu} - \left(\frac{\partial u_{j\nu}^{\min}}{\partial s_j}\right)_1 \dot{z}_{j\mu}\right]\left\{-\dot{z}_j^2\left(\frac{\partial u_j^{\min}}{\partial s_j}\right)_1^2 + \left[\dot{z}_j\left(\frac{\partial u_j^{\min}}{\partial s_j}\right)_1\right]^2\right\}^{-1/2} \tag{3.23}$$

where the subscript 1 indicates that the derivative is calculated in $s = 1$. Moreover,

$$\left(\langle\langle F_{\mu\nu}(z_j)F_{\rho\sigma}(z_j')\rangle\rangle - \langle\langle F_{\mu\nu}(z_j)\rangle\rangle\,\langle\langle F_{\rho\sigma}(z_j')\rangle\rangle\right)^{\mathrm{LR}} = 0\,. \tag{3.24}$$

By using the straight-line approximation one obtains

$$g\,\langle\langle F_{0k}(z_j)\rangle\rangle^{\mathrm{LR}} = \sigma\frac{r_j^k}{r_j}\,, \tag{3.25}$$

$$g\,\langle\langle F_{hk}(z_j)\rangle\rangle^{\rm LR} = \frac{\sigma}{m_j}\frac{1}{r_j}(r_j^h p_j^k - r_j^k p_j^h) \tag{3.26}$$

with $j = 1, 2, 3$ and

$$g^2\left(\langle\langle F_{hk}(z_j)F_{lm}(z_i)\rangle\rangle - \langle\langle F_{hk}(z_j)\rangle\rangle\,\langle\langle F_{lm}(z_i)\rangle\rangle\right)^{\rm LR} = ig\frac{\delta}{\delta S^{lm}(z_i)}\langle\langle F_{hk}(z_j)\rangle\rangle^{\rm LR} = O(v^2) \tag{3.27}$$

with $j \neq i$.

As concerns the Darwin-type terms we must evaluate $g\,\langle\langle D^{(j)\nu}F_{\nu\mu}(z_j)\rangle\rangle$ for $j = 1, 2, 3$ with $D^{(j)\nu} = \partial^{(j)\nu} + igA^\nu$. This can be done by using the same line of derivation as for the two-body case.

At this point we may derive the spin-dependent potential: after some calculations using Eqs. (3.19)–(3.22) and (3.24)–(3.27) in Eq. (3.7) we get Eq. (1.7).

IV. ADDITIONAL CONSIDERATIONS

To complete the work some additional considerations and remarks are necessary.

(1) In Sec. III we have assumed different flavors for the three quarks. If two or three quarks are identical we must identify the corresponding operators in Eq. (3.1) and add the appropriate normalization factor $1/\sqrt{2!}$ or $1/\sqrt{3!}$. Correspondingly we have to replace the right-hand side in (3.3) with the corresponding sum over the permutations of the final identical particles divided by the combinatorial factor (2! or 3!). For what concerns the potential this means only that we have to equate the masses of such particles in (1.7) and (1.8). Similarly if in Sec. II the quark and the antiquark have the same flavor, a new term must be added to the last member of (2.1), which is obtained by interchanging the roles of y_1 and x_2. Such "annihilation" term is not properly of potential type but can be treated perturbatively [10].

(2) Concerning the ordering problem in (1.4) and (1.8) we recall that there are two independent possible prescriptions for a quantity quadratic in the momenta: the Weyl prescription

$$\{f(\mathbf{r})p^hp^k\}_W = \frac{1}{4}\{\{f(\mathbf{r}),p^h\},p^k\} \tag{4.1}$$

and the symmetric one

$$\{f(\mathbf{r})p^hp^k\}_S = \frac{1}{2}\{f(\mathbf{r}),p^hp^k\}\,. \tag{4.2}$$

As well known, in the path integral formalism the ordering prescription corresponds to the specific discretization rules used in the definition of it.

In Ref. [5] we adopted a somewhat *ad hoc* rule corresponding to the ordering $\{\ \}_{\rm ord} = \frac{2}{3}\{\ \}_W + \frac{1}{3}\{\ \}_S$. Such rule was motivated by the fact that it enables a by part integration at the discrete level which was necessary to eliminate a dependence of the potential from the acceleration.

Notice, however, that the limit procedure used for the definition of the path integrals in (2.9) or (2.10) is not at our choice, but it is a consequence of the corresponding procedure used in the definition of the field functional integration in (2.1). To see what is the correct prescription, let us assume a definite lattice with spacing ε in the time direction and a spacing a in the space directions. Let us consider the corresponding discrete counterpart of (2.1), written according to the usual rules for gauge theories [11] and perform the integration of the fermionic fields at this discrete level. Then in place of (2.10) we arrive at an equation in which not only the time integral is replaced by a sum over the appropriate discrete times t_s, but for every t_s even the integrals on $\mathbf{z}_1$ and $\mathbf{z}_2$ are replaced by the sum over all the sites of the lattice corresponding to that time coordinate. Finally Eq. (1.12) has to be interpreted as

$$W_{q\bar{q}} \simeq \int \prod_{\{n',n\}} dU_{n'n}\; e^{iS[U]}\,\mathrm{Tr}\,P\prod_{\{r,r'\}\in\Gamma} U_{r'r}, \tag{4.3}$$

where $U_{n'n}$ denotes the element of the color group associated to the link between the contiguous sites n and n' and the product is extended to all such links or to all links laying on the curve Γ. Since in turn $U_{n'n}$ can be interpreted as

$$\exp\left[ig(x_{n'} - x_n)^\mu A_\mu\left(\frac{x_{n'} + x_n}{2}\right)\right],$$

we see that, after having explicitly integrated over U [and so used (1.13) and (1.14)] and performed the limit $a \to 0$, we are left with the discrete form of an ordinary path integral with

$$X^{hk}(\mathbf{r})p_i^hp_j^k \longrightarrow X^{hk}\left(\frac{\mathbf{r}_s + \mathbf{r}_{s-1}}{2}\right)p_{is}^hp_{js}^k\,. \tag{4.4}$$

Equation (4.4) does correspond to the Weyl ordering, as indicated in (1.4) [12]. Notice, however, that a different ordering in (1.4) would bring simply to additional Darwin-like terms, which, in practice, can be nearly completely compensated by a readjustment of the potential parameters. Similar arguments apply to Eq. (1.8) in the $3q$ case.

(3) To clarify the connection between the $q\bar{q}$ potential and the *relativistic flux tube model* [2] it is convenient to neglect the spin-dependent terms in (2.10) and replace the $1/m^2$ expansion by its exact relativistic expression. We have

$$K(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2;t_f-t_i)=\int\mathcal{D}[\mathbf{z}_1,\mathbf{p}_1]\int\mathcal{D}[\mathbf{z}_2,\mathbf{p}_2]\exp\left\{i\left[\int_{t_i}^{t_f}dt\sum_{j=1}^{2}(\mathbf{p}_j\cdot\dot{\mathbf{z}}_j-\sqrt{m_j^2+\mathbf{p}_j^2})\right]+\ln W_{q\bar{q}}\right\}. \tag{4.5}$$

Further by taking advantage of (2.22) and of the first step in (2.27), after expanding again the exponential in (4.5) around the values $\mathbf{p}_j=m\dot{\mathbf{z}}_j/\sqrt{1-\dot{\mathbf{z}}_j^2}$ and performing the integration in the Gaussian approximation (semiclassical approximation), we can write (see, however, in this connection Ref. [12])

$$K(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_1,\mathbf{y}_2;t_f-t_i)=\int\mathcal{D}[\mathbf{z}_1]\Delta[\mathbf{z}_1]\int\mathcal{D}[\mathbf{z}_2]\Delta[\mathbf{z}_2]\exp\left\{i\int_{t_i}^{t_f}dtL(\mathbf{z}_1,\mathbf{z}_2,\dot{\mathbf{z}}_1,\dot{\mathbf{z}}_2)\right\} \tag{4.6}$$

with

$$L=-\sum_{j=1}^{2}m_j\sqrt{1-\dot{\mathbf{z}}_j^2}+\frac{4}{3}\frac{\alpha_s}{r}\left[1-\frac{1}{2}(\delta^{hk}+\hat{r}^h\hat{r}^k)\dot{z}_1^h\dot{z}_2^k\right]-\sigma r\int_0^1 ds[1-(s\dot{\mathbf{z}}_{1T}+(1-s)\dot{\mathbf{z}}_{2T})^2]^{1/2}. \tag{4.7}$$

In (4.6) $\mathcal{D}[\mathbf{z}]$ denotes the usual nonrelativistic configurational measure and $\Delta[\mathbf{z}]$ is a determinantal factor, which has to be be considered part of the relativistic measure. Formally we can write

$$\mathcal{D}[\mathbf{z}]=\left(\frac{m}{2\pi i\epsilon}\right)^{3/2}\prod_t\left[\left(\frac{m}{2\pi i\epsilon}\right)^{3/2}d^3\mathbf{z}(t)\right] \tag{4.8}$$

(ϵ = time lattice spacing) and

$$\Delta[\mathbf{z}]=\left\{\prod_t\det\left[\frac{1}{m}(1-\dot{\mathbf{z}}^2)^{1/2}(\delta^{hk}-\dot{z}^h\dot{z}^k)\right]\right\}^{-1/2}. \tag{4.9}$$

What we want to stress is that Eq. (4.7) is identical to the center of mass Lagrangian for the relativistic flux tube model. This is consistent with what is already observed at the $1/m^2$ order in Ref. [2].

(4) In phenomenological analysis the following long-range static potential of two-body type has been often adopted [7] for the $3q$ case:

$$V_{\text{stat}}^{\text{LR}}=\frac{1}{2}\sigma\,(r_{12}+r_{23}+r_{31}) \tag{4.10}$$

with a corresponding spin-dependent potential again of the form (1.10),

$$V_{\text{SD}}^{\text{LR}}=-\sum_{j=1}^{3}\frac{1}{2m_j^2}\mathbf{S}_j\cdot\boldsymbol{\nabla}_{(j)}V_{\text{stat}}^{\text{LR}}\times\mathbf{p}_j\,. \tag{4.11}$$

The factor 1/2 in Eq. (4.10) is motivated by the fact that when two quarks collapse they become equivalent from the color point of view to an antiquark and V_{stat}^{3q} and V_{SD}^{3q} must reduce to $V_{\text{stat}}^{q\bar{q}}$ and $V_{\text{SD}}^{q\bar{q}}$. It has been shown that (4.10) and (4.11) produce a spectrum very close to that obtained from (1.6) and (1.10). From a numerical point of view the use of (1.6) amounts to replace the factor 1/2 in (4.10) by a factor of the order of 0.54–0.55 [7,8]. Notice, however, that from a fundamental point of view (4.10) has no clear basis.

(5) As we already mentioned in lattice gauge theory the area law is obtained under the approximation in which the quantity $M_f(A)$ is replaced by 1, the so-called quenched approximation. Our entire treatment has to be understood in this perspective. Then the effect of virtual quark-antiquark creation should be introduced as a correction at a later stage. Various attempts have been done in this direction: see Refs. [13,14].

(6) In Eqs. (1.6)–(1.8) V_{SD}^{3q} has been written in the coordinates $\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3$ and $\mathbf{r}_{12},\mathbf{r}_{23},\mathbf{r}_{31}$, which provides a symmetric form of the potential. For numerical computations it may be useful to express such variables in terms of a system of independent coordinates like the Jacobi coordinates (see [3] and [8]).

V. CONCLUSIONS

In conclusion we have strongly simplified the derivation of the quark-antiquark potential as given in [4,5]. We have shown that, once the assumptions (1.13) and (1.14) are done, the existence of a potential follows when one performs the instantaneous approximation (1.16) and the straight-line approximation (1.15). We have also corrected the ordering prescription.

Because of the above simplifications, the method has been extended without difficulty to the three-quark case, where the relevant assumptions and approximations are (1.18), (1.16), and (1.19). As a result a $3q$ spin-dependent potential has been consistently obtained in the Wilson loop context. This coincides with the one already proposed by Ford under an assumption of scalar confinement. It has also been evaluated the $O(1/m^2)$ $3q$ velocity-dependent potential, which is new at our knowledge.

Notice that in both the $q\bar{q}$ and $3q$ cases the spin-independent relativistic corrections obtained by us differ from those resulting from the mentioned assumption of scalar confinement and seems to agree better with the data [14,10] (for the difficulties of the scalar confinement hypothesis see also [15,16]).

Finally we have seen that if in Eq. (2.10) we replace the kinetic terms by the exact relativistic expressions and neglect the spin-dependent terms, if further we use the first step in Eq. (2.27) (without any velocity expansion) and perform the momentum integration in the semiclassical approximation, we obtain the Lagrangian for the relativistic flux tube model [2].

APPENDIX A

Concerning the definition of the symbol $\delta/\delta S^{\mu\nu}(z)$, let us consider a functional of the world line γ of the form

$$F[\gamma] = A + \int_\gamma dz^\mu B_\mu(z) + \frac{1}{2}\int_\gamma dz^\mu \int_\gamma dz'^\rho C_{\mu\rho}(z,z') + \cdots \tag{A1}$$

with $C_{\mu\rho}(z,z') = C_{\rho\mu}(z,z')$, etc. Let us denote by $z = z(\lambda)$ the parametric equation of γ and consider an infinitesimal variation of such curve $z(\lambda) \longrightarrow z(\lambda) + \delta z(\lambda)$. We have

$$\begin{aligned}\delta F &= \int_\gamma \left(\delta dz^\mu B_\mu(z) + dz^\mu \frac{\partial B_\mu(z)}{\partial z^\nu}\delta z^\nu\right) + \int_\gamma dz'^\rho \int_\gamma \left(\delta dz^\mu C_{\mu\rho}(z,z') + dz^\mu \frac{\partial C_{\mu\rho}(z,z')}{\partial z^\nu}\delta z^\nu\right) + \cdots \\ &= \frac{1}{2}\int_\gamma (dz^\mu \delta z^\nu - dz^\nu \delta z^\mu)\left[\left(\frac{\partial B_\mu(z)}{\partial z^\nu} - \frac{\partial B_\nu(z)}{\partial z^\mu}\right) + \int_\gamma dz'^\rho \left(\frac{\partial C_{\mu\rho}(z,z')}{\partial z^\nu} - \frac{\partial C_{\nu\rho}(z,z')}{\partial z^\mu}\right) + \cdots\right],\end{aligned} \tag{A2}$$

where a partial integration has been performed at the second step. Assuming $\delta z(\lambda)$ different from zero only in a small neighborhood of a specific value $\bar\lambda$ of λ we write

$$\frac{\delta F}{\delta S^{\mu\nu}(\bar z)} = \frac{\partial B_\mu(\bar z)}{\partial z^\nu} - \frac{\partial B_\nu(\bar z)}{\partial z^\mu} + \int_\gamma dz'^\rho \left(\frac{\partial C_{\mu\rho}(\bar z,z')}{\partial z^\nu} - \frac{\partial C_{\nu\rho}(\bar z,z')}{\partial z^\mu}\right) + \cdots \tag{A3}$$

with $\bar z = z(\bar\lambda)$. Furthermore, if we consider a second variation $\delta z(\lambda)$ different from zero only in a small neighborhood of a second value $\bar\lambda' \neq \bar\lambda$ we have

$$\begin{aligned}\delta \frac{\delta F}{\delta S^{\mu\nu}(\bar z)} = \frac{1}{2}\int_\gamma (dz'^\rho \delta z'^\sigma - dz'^\sigma \delta z'^\rho)\Bigg[&\frac{\partial}{\partial z'^\sigma}\left(\frac{\partial C_{\mu\rho}(\bar z,z')}{\partial z^\nu} - \frac{\partial C_{\nu\rho}(\bar z,z')}{\partial z^\mu}\right) \\ &- \frac{\partial}{\partial z'^\rho}\left(\frac{\partial C_{\mu\sigma}(\bar z,z')}{\partial z^\nu} - \frac{\partial C_{\nu\sigma}(\bar z,z')}{\partial z^\mu}\right) + \cdots\Bigg]\end{aligned} \tag{A4}$$

and

$$\frac{\delta^2 F}{\delta S^{\rho\sigma}(\bar z')\delta S^{\mu\nu}(\bar z)} = \frac{\partial^2 C_{\mu\rho}(\bar z,\bar z')}{\partial z'^\sigma \partial z^\nu} - \frac{\partial^2 C_{\nu\rho}(\bar z,\bar z')}{\partial z'^\sigma \partial z^\mu} - \frac{\partial^2 C_{\mu\sigma}(\bar z,\bar z')}{\partial z'^\rho \partial z^\nu} + \frac{\partial^2 C_{\nu\sigma}(\bar z,\bar z')}{\partial z'^\rho \partial z^\mu} + \cdots. \tag{A5}$$

Notice that the assumption $\bar\lambda' \neq \bar\lambda$ is essential to make the definitions unambiguous. The case in which a $\delta(z-z')$ term occurs in $C(z,z')$ must be treated as a limit one. In practice this amounts to say that (A3) and (A5) hold true even in this case.

APPENDIX B

In applying (A3) and (A5) to $i\ln W_{q\bar q}$ [17] and $i\ln W_{3q}$ and specifically to Eqs. (2.23) and (3.9) it is convenient to think of t as an arbitrary parameter on the same foot of λ. It is obviously understood that at the end t is identified with the ordinary time by setting $z_j^0(t) = t$ and $u^0(s,t) = t$ (this in the $q\bar q$ case). In particular rewriting Eq. (2.23) as

$$i\ln W_{q\bar q}^{\mathrm{LR}} = \sigma S_{\min} = \sigma \int_{t_i}^{t_f} dt \int_0^1 ds\, \mathcal{S}(u^{\min}) \tag{B1}$$

with

$$\mathcal{S}(u) = \left[-\left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial t}\right)\left(\frac{\partial u^\mu}{\partial s}\frac{\partial u_\mu}{\partial s}\right) + \left(\frac{\partial u^\mu}{\partial t}\frac{\partial u_\mu}{\partial s}\right)^2\right]^{1/2} \tag{B2}$$

one can notice that the equation of the minimal surface $u^\mu = u^\mu_{\min}(s,t)$ is the solution of the Euler equations

$$\frac{\partial}{\partial s}\frac{\partial \mathcal{S}}{\partial\left(\frac{\partial u^\mu}{\partial s}\right)} + \frac{\partial}{\partial t}\frac{\partial \mathcal{S}}{\partial\left(\frac{\partial u^\mu}{\partial t}\right)} = 0 \tag{B3}$$

satisfying the contour conditions $u^\mu(1,t) = z_1^\mu(t)$, $u^\mu(0,t) = z_2^\mu(t)$. Then considering an infinitesimal variation of the world line of the quark 1, $z_1^\mu(t) \longrightarrow z_1^\mu(t) + \delta z_1^\mu(t)$, even $u^\mu_{\rm min}(s,t)$ must change, $u^\mu_{\rm min}(s,t) \longrightarrow u^\mu_{\rm min}(s,t) + \delta u^\mu(s,t)$ and one has

$$\delta(i \ln W_{q\bar{q}}^{\rm LR}) = \sigma \int_{t_i}^{t_f} dt \int_0^1 ds \left[\frac{\partial S}{\partial(\partial u^\mu/\partial s)} \frac{\partial}{\partial s} \delta u^\mu + \frac{\partial S}{\partial(\partial u^\mu/\partial t)} \frac{\partial}{\partial t} \delta u^\mu \right]_{u=u_{\rm min}} = \sigma \int_{t_i}^{t_f} dt \left[\frac{\partial S}{\partial(\partial u^\mu/\partial s)} \delta u^\mu \right]_{s=1}, \tag{B4}$$

where $\delta z_1^\mu(t)$ was assumed to vanish out of a small neighborhood of a specific value of t. Finally taking into account that

$$\delta u^\mu(1,t) = \delta z_1^\mu(t)\,, \qquad \frac{\partial u^\mu_{\rm min}(1,t)}{\partial t} = \dot{z}_1^\mu(t) \tag{B5}$$

one obtains

$$\begin{aligned} \delta(i \ln W_{q\bar{q}}^{\rm LR}) &= \sigma \int_{t_i}^{t_f} dt \frac{1}{[S]_{s=1}} \left[-\dot{z}_1^2 \left(\frac{\partial u_\nu^{\rm min}}{\partial s} \right)_1 + \left(\frac{\partial u_\mu^{\rm min}}{\partial s} \right)_1 \dot{z}_1^\mu \dot{z}_{1\nu} \right] \delta z_1^\nu \\ &= \frac{1}{2} \sigma \int_{t_i}^{t_f} dt \, (dz_1^\mu \delta z_1^\nu - dz_1^\nu \delta z_1^\mu) \left[\left(\frac{\partial u_\mu^{\rm min}}{\partial s} \right)_1 \dot{z}_{1\nu} - \left(\frac{\partial u_\nu^{\rm min}}{\partial s} \right)_1 \dot{z}_{1\mu} \right] \\ &\quad \times \left\{ -\dot{z}_1^2 \left(\frac{\partial u^{\rm min}}{\partial s} \right)_1^2 + \left[\dot{z}_1 \cdot \left(\frac{\partial u^{\rm min}}{\partial s} \right)_1 \right]^2 \right\}^{-1/2}, \end{aligned} \tag{B6}$$

and so Eq. (2.41).

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