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Skyrmion quantization and the decay of the Δ

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We present the complete solution to the so-called "Yukawa problem" of the Skyrme model. This refers to the perceived difficulty of reproducing, purely from soliton physics, the usual pseudovector pion-nucleon coupling, echoed by pion coupling to the higher-spin/isospin baryons ($I = J = \frac{3}{2}, \frac{5}{2}, \ldots, N_c/2$) in a manner fixed by large-N_c group theory. The solution involves surprisingly elegant interplay between the classical and quantum properties of a new configuration: the ***rotationally improved Skyrmion.*** This is the near-hedgehog solution obtained by minimizing the usual Skyrmion mass functional augmented by an all-important (iso)rotational kinetic term. The numerics are pleasing: a Δ decay width within a few MeV of its measured value, and, furthermore, the higher-spin baryons ($I = J \geq \frac{5}{2}$) with widths so large ($\Gamma > 800$ MeV) that these undesirable large-N_c artifacts effectively drop out of the spectrum, and pose no phenomenological problem. Beyond these specific results, we ground the Skyrme model in the Feynman path integral, and set up a transparent collective coordinate formalism that makes maximal use of the $1/N_c$ expansion. This approach elucidates the connection between Skyrmions on the one hand, and Feynman diagrams in an effective field theory on the other.

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I. OVERVIEW

Since its reinvention a decade ago by Adkins, Nappi, and Witten [1], and despite many phenomenological successes [2], the Skyrme model [3] has suffered from too many competing, often conflicting, formalisms. Particular confusion surrounds those problems that involve interactions between Skyrmions (read: baryons) and the elementary quanta of the theory (read: mesons). In fact, the most basic such question one can pose—how does the Skyrme model generate the correct pion-baryon three-point coupling?—has not been satisfactorily resolved.

In this paper, we present the complete solution to this so-called "Yukawa problem," as it has come to be known in the literature. Both the problem and the solution are detailed in this expanded introductory section, which also sets forth some general principles that we think are important, and underappreciated.

This is the second in a series of papers intended to clarify the nature of the meson-Skyrmion interactions, by grounding the Skyrme model in the Feynman path integral (FPI). The small parameter of the Skyrme model is $1/N_c$ (N_c being the number of colors of the underlying gauge theory), which is tied to the loop expansion natural to the FPI by appearing in the combination $\hbar/N_c$. In Ref. [4] we focused on a "toy" version of the Skyrme model, in which space-time is 1+1 dimensional, and the internal flavor symmetry is U(1) rather than chiral SU(2)×SU(2). The process we analyzed, while simple, proved illuminating: the decay of a soliton in its nth excited state, to its next-lower state, by emission of a single charged meson. Here we will focus not only on the analogous physical decay $\Delta \to N\pi$, but also on virtual processes such as $N \to N\pi$ and $\Delta \to \Delta\pi$ that are building blocks for more complicated diagrams, and likewise for all the higher-spin/isospin baryons ($I = J = \frac{5}{2}, \frac{7}{2}$, etc.) that emerge as rotational excitations of the hedgehog Skyrmion. While numerics are not our primary goal at the present, it is pleasing that the width of the Δ in the Skyrme model works out to 114 MeV versus 120 ± 5 MeV experimentally (a result not original to us, but rather confirming a large-N_c ansatz in Ref. [1]), while the higher-spin baryons are so broad (>800 MeV) that they would not normally be classified as "particles"—here again, by their absence from the spectrum, in agreement with nature.[1]

By virtue of the delicate interplay between its classical and quantum properties, the Skyrme model will be seen to be richer and more elegant than the U(1) toy model. Yet the three main points of Ref. [4], which one might characterize as a *caveat*, a *prescription*, and a *moral*, hold

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[1]See Sec. VI below for width calculations.

here as well, and provide useful guideposts for the development below. We review them accordingly.

A caveat. Whether in the U(1) model where the elementary boson is the real scalar doublet $\boldsymbol{\phi}$, or in the Skyrme model where it is the pion field $\boldsymbol{\pi}$, or in other soliton theories, it is customary to split up the total field into classical and fluctuating parts, $\boldsymbol{\phi} = \boldsymbol{\phi}_{\rm cl} + \delta\boldsymbol{\phi}$. Here $\boldsymbol{\phi}_{\rm cl}$ is the classical soliton, whose zero modes are the "baryon degrees of freedom," while the fluctuating field $\delta\boldsymbol{\phi}$, properly orthogonalized to these zero modes, is often said to represent the "meson degrees of freedom." Unfortunately, this commonly held distinction between "baryon" and "meson" degrees of freedom is *false*, and leads to incorrect results.[2] While it may be convenient mathematically to split up the field in this way at an intermediate stage in the calculation (as we ourselves do in Secs. II and III below), physically meaningful Green's functions $\mathcal{G}$ must be formed from the reconstituted *total* field, $\boldsymbol{\phi}$ or $\boldsymbol{\pi}$. The presence, or absence, of asymptotic states containing n physical mesons with four-momenta $q_1, \ldots, q_n$ can then be gleaned from the analytic properties of $\mathcal{G}$, as per the Lehmann-Symanzik-Zimmermann (LSZ) amputation procedure.[3] Specifically, one looks for simultaneous poles on the meson mass shell, $\mathcal{G} \sim \prod_{i=1}^{n}(q_i^2 - m_\pi^2 + i\epsilon)^{-1} + \cdots$, and identifies the residue with the S matrix element. As for the baryons/Skyrmions, the LSZ procedure is even simpler, since in large N_c baryons are very heavy (masses $\sim N_c$) and can for present purposes be treated in a nonrelativistic, first-quantized manner. To reiterate, we concentrate on these analytic properties, not because we are infatuated with formalism, but because we want to avoid wrong answers.

A prescription. As stated above, N_c enters into the problem in the combination $N_c/\hbar$. An economical formalism should exploit this fact, and map a leading-order calculation in $1/N_c$ onto a *zeroth* order expression, i.e., a saddle-point, in the semiclassical expansion (and not some "higher order effect" in a naive perturbative expansion as often appears in the Skyrmion literature when the author elects to split up the total pion field). Framed in these terms, the problem is that, for purposes of LSZ, $\boldsymbol{\phi}_{\rm cl}$ is the *wrong approximate saddle point.* By definition, $\boldsymbol{\phi}_{\rm cl}$ solves the static Euler-Lagrange equation $0{=}\delta M_s/\delta\boldsymbol{\phi}$ where $M_s[\boldsymbol{\phi}, \partial_x\boldsymbol{\phi}]$ is the soliton mass functional (the integrated Hamiltonian). The prescription we proved in Ref. [4] is to solve, instead, the static equation $0 = \delta(M_s + P^2/2\mathcal{I})/\delta\boldsymbol{\phi}$, where P is the momentum conjugate to the soliton's U(1) collective coordinate θ, and $\mathcal{I}[\boldsymbol{\phi}]$ is the soliton's moment of inertia. Likewise, in the Skyrme model, we will show that the right semiclassical starting point is the solution to

$$0 = \frac{\delta}{\delta\boldsymbol{\pi}}(M_s + \tfrac{1}{2}J^m\mathcal{I}^{-1}_{mn}J^n) \ , \tag{1.1}$$

where $\mathcal{I}$ is now a tensor. We call such solutions *rotationally improved Skyrmions*, and they are no longer precisely hedgehogs (a key point).[4] Ostensibly, the added rotational kinetic term is a small perturbation, since $\mathcal{I}^{-1} \sim N_c^{-1}$. Nevertheless, its effect on the analytic structure of the Green's functions is critical, and cannot be neglected. In the U(1) toy model it contributes a negative mass-squared, so that rather than falling off as $\exp(-m_\phi r)$ like $\boldsymbol{\phi}_{\rm cl}$, the rotationally improved soliton $\sim \exp[-(m_\phi^2 - P^2/\mathcal{I}^2)^{1/2}r]$. In momentum space this turns $(|\mathbf{q}|^2 + m_\phi^2)^{-1}$ into $(|\mathbf{q}|^2 + m_\phi^2 - P^2/\mathcal{I}^2)^{-1}$. As $P/\mathcal{I}$ can be equated to the meson energy, $\mathcal{G}(q)$ now correctly has a pole on the mass shell, and excited U(1) solitons can legally decay by meson emission. In the Skyrme model, the effect of the rotational perturbation is more interesting. Thanks to its deviation away from the hedgehog ansatz, the rotationally improved Skyrmion falls off as a superposition of two distinct exponentials, so that in momentum space

$$\mathcal{G}(q) \sim \frac{\mathcal{N}_1}{|\mathbf{q}|^2 + m_\pi^2 - J^2/\mathcal{I}^2} + \frac{\mathcal{N}_2}{|\mathbf{q}|^2 + m_\pi^2} + (\text{nonpole terms}) \ . \tag{1.2}$$

This makes perfect sense: the first pole correctly describes Skyrmion decay processes such as $\Delta \to N\pi$, while

[2] In the particular example of Yukawa couplings in the Skyrme model, we find ourselves disagreeing with Refs. [5–10], among many others, who identify as the physical pion only the fluctuating field $\delta\boldsymbol{\pi}$. Since $\delta\boldsymbol{\pi} \sim N_c^0$ while $\boldsymbol{\pi}_{\rm tot} \propto f_\pi \sim N_c^{1/2}$, it is no surprise, and has been noted by several of these authors, that the width of the Δ as may be calculated in these formalisms is down by (at least) one power of N_c from the unambiguous, model-independent leading-order answer derived from Eq. (1.3) below. Phrased the usual way, since the first variation $\delta\boldsymbol{\pi}\cdot(\delta S/\delta\boldsymbol{\pi})$ off the *static* Skyrmion vanishes by the defining equation, the first variation off the *rotating* Skyrmion (i.e., the nucleon or Δ) is suppressed in large N_c, since the Skyrmion rotates slowly ($\omega_{\rm rot} \sim N_c^{-1}$). The more complicated case of pion-Skyrmion scattering will be discussed in Sec. VII.

[3] A thorough review of the LSZ procedure is in Chap. 7 of Ref. [11]. The observation that the classical part of the field cannot be ignored, and on the contrary contributes at leading order to poles in the "meson" channels, dates back to the early literature on soliton quantization [12,13].

[4] A similarly distorted hedgehog has recently been obtained by Schroers using different methods [14]. A technical aside: It is widely believed that when the Skyrmion is not precisely a hedgehog, one must in principle introduce extra collective coordinates for *iso*rotations in addition to spatial rotations, since these are no longer equivalent, and concomitantly, an additional *iso*rotational kinetic energy term beyond the one displayed in Eq. (1.1). But for the rotationally improved Skyrmion, our FPI formalism clarifies that *this is not the case*; Eq. (1.1) suffices (see Appendix C for a discussion). In addition, the FPI approach completely obviates an ongoing dialectic about the relative merits of this or that "gauge" (meaning how one chooses to orthogonalize the fluctuating modes to the Skyrmion's zero modes). To emphasize this point, Secs. II and III below are framed in the most general (linear) gauge, and the gauge invariance of our physical results is manifest.

the second describes $N \to N\pi$, etc.[5] In either model, this pole-shift phenomenon is a variation on the old exercise of expanding a field theory about the wrong mass, and treating the mass shift in perturbation theory: an increasing number of two-point insertions must be summed geometrically [as accomplished implicitly in Eq. (1.1)] to move the pole in the propagator to the physical mass shell.

A moral. The moral of Ref. [4] is equally valid for the Skyrme model, namely, the order-by-order equivalence of the soliton theory to an effective relativistic quantum field theory with explicit baryon fields.[6] The eventual goal of mapping out this effective theory in full is well beyond the scope of the present paper. But by focusing on Skyrmion decay by one-meson emission, we shed light on the effective three-point meson-baryon vertex which can, in turn, be assembled into more complicated Feynman diagrams (e.g., πN scattering, or pion-exchange contributions to the NN system). In the U(1) model, the baryon/soliton states $|p\rangle$ are labeled by an integer charge. The baryon wave functions are $\psi_p(\theta) = \langle\theta|p\rangle = e^{ip\theta}$, and their effective Yukawa couplings to the charged scalars $\phi^\pm \equiv \phi_1 \pm \phi_2$ can be expressed as $g\phi^+(x)\int d\theta\, e^{i\theta}|\theta\rangle\langle\theta|$+H.c., or even more compactly as $g\phi^+(x)e^{i\hat{\theta}}$+H.c., where $\hat{\theta}|\theta\rangle = \theta|\theta\rangle$. The presence of $e^{i\theta}$ properly ensures $\Delta p = 1$, as the meson carries away one unit of charge. In 3+1 dimensions, with SU(2)×SU(2) symmetry, the form of the analogous pion-baryon effective coupling is *completely determined a priori* by the twin requirements of the chiral and large-N_c limits, and reads[7]

$$-\frac{3g_{\pi NN}}{2M_N}\partial_i\pi^a\int_{\mathrm{SU}(2)} dA\, D^{(1)}_{ai}(A)|A\rangle\langle A| + \cdots ,$$

$$D^{(1)}_{ai}(A) = \tfrac{1}{2}\mathrm{Tr}\tau_a A\tau_i A^\dagger . \tag{1.3}$$

Here $\langle A|$ stands for the superposition of *explicit* pointlike baryon fields (the nucleon field, the Δ field, and all higher spins; a better notation might be Ψ_A), any of which can be projected out using Eq. (5.7) below. The omitted terms, while subleading in $1/N_c$, are needed to form a relativistic invariant, for instance, $(g_{\pi NN}/2M_N)\partial_\mu\pi^a\bar{N}\gamma^\mu\gamma^5\tau^a N$ when Eq. (1.3) is projected onto nucleon states ($I = J = \frac{1}{2}$).

The most elegant result of our paper—and wholly unexpected, as this feature of the Skyrme model is not present in the U(1) toy model—comes from the simple requirement that Eq. (1.2) be interpretable as a Green's function in some quantum field theory (as we clarify at the end of this paragraph). This requirement resolves an operator ordering ambiguity implicit in the definition of the pole residues $\mathcal{N}_1$ and $\mathcal{N}_2$ [the noncommuting operators being **J** and $D^{(1)}_{ai}(A)$]. One finds

$$\mathcal{N}_1 = -\frac{3ig_{\pi NN}}{2M_N}q^i D^{(1)}_{ai}(A)\mathcal{P}_{\Delta J=1},$$
$$\mathcal{N}_2 = -\frac{3ig_{\pi NN}}{2M_N}q^i D^{(1)}_{ai}(A)\mathcal{P}_{\Delta J=0} . \tag{1.4}$$

Here $\mathcal{P}_{\Delta J=0}$ is the projection operator that equates the initial and final Skyrmion spin, while $\mathcal{P}_{\Delta J=1}$ requires that they differ by one unit, so that any given one-pion emission or absorption process "sees" only one of the two pole terms. In this manner, the numerators of Eq. (1.2) are brought into harmony with the LSZ interpretation of the denominators—an interesting conspiracy between the quantum and classical properties, respectively, of the rotationally improved Skyrmion. And the effective pion coupling to explicit pointlike baryon fields that is equivalent to Eq. (1.4) is *precisely* Eq. (1.3). So, this paper gives a complete solution to the "Yukawa problem" mentioned at the outset, namely, showing how Eq. (1.3) emerges directly from Skyrmion physics. In retrospect, had the $\mathcal{P}$'s *not* emerged in the numerators, then any given one-pion process would have, in addition to a pole in the right position, a spurious nearby isolated pole, violating the basic precepts of the Källen-Lehmann spectral representation of a quantum field theory [17], and dashing any possibility of proving such an equivalence.

In related work, by focusing on those contributions to pion-Skyrmion scattering that can be interpreted in the corresponding effective field theory as Compton graphs, the authors of Refs. [18–20] (a very important precursor being Ref. [21]) correctly deduce pseudovector coupling with $g_{\pi NN} \propto g_{\pi N\Delta} \sim N_c^{3/2}$. While we have yet to reconcile the operator Hamiltonian formalism of Ref. [20] with our own more pedestrian FPI approach, it appears that Refs. [18–20] are a major step towards the complete solution, presented here, of the "Yukawa problem." Another interesting idea is to extract the effective Yukawa couplings from the Skyrmion-Skyrmion potential [22]. Additional proposed fixes to the Yukawa problem may be found in Refs. [23–25].

The remainder of this paper is organized as follows. In Sec. II, the nonlinear σ model is formulated as a phase-space FPI. The baryon-number-unity sector is then selected using a natural extension of the collective coordinate method developed long ago by Gervais, Jevicki, and Sakita [26]. These authors quantized the translational mode of a one-dimensional kink, whereas we extend the methodology to internal symmetries. The upshot of Sec.

[5]Very roughly speaking, the arithmetic works as follows. The pion energy ω_π must equal the difference of the initial and final Skyrmion energies, namely $J_i(J_i+1)/2\mathcal{I} - J_f(J_f+1)/2\mathcal{I}$. When $J_f = J_i - 1$, this difference is $J_i/\mathcal{I}$, whereas when $J_f = J_i$ it is zero, consistent with the two pole locations in Eq. (1.2), respectively. The actual analysis of Sec. V is not quite so simple: operator ordering ambiguities must be resolved.

[6]Significant progress in the reverse direction, from field theory to Skyrmions, can be found in Refs. [15] and [16], in which the Skyrme model (or variants thereof) is conjectured to emerge as an ultraviolet renormalization group fixed point of a class of effective meson-baryon Lagrangian field theories.

[7]This coupling, which we review in Appendix D, was first written down in Sec. 5 of Ref. [1], with no explicit input from or connection to Skyrmion physics.

II is SU(2) quantum mechanics coupled to a quantum field theory. The latter is treated in saddle-point approximation in Sec. III, leading to the formal derivation of the rotationally improved Euler-Lagrange equation (1.1). While variations on this equation have been posited by other authors as a preferred starting point [27–31], it is reassuring to see it grounded firmly in the FPI. As Sec. III is a little technical, the reader who is already happy with Eq. (1.1) is encouraged to skip directly to Secs. IV and V on a first reading, as these are the heart of our paper. In Sec. IV we extract the large-distance behavior of the rotationally improved Skyrmion, and confirm the two distinct poles of Eq. (1.2), while in Sec. V we describe the operator ordering solution (1.4) for $\mathcal{N}_1$ and $\mathcal{N}_2$. In Sec. VI the width of the Δ and of the higher-spin large-N_c $I=J$ baryons are calculated. The application of rotationally improved Skyrmions to πN scattering [18–20,32,33], and some concluding comments, can be found in Sec. VII.

We also include four Appendices. Appendix A revisits the U(1) toy model of Ref. [4] in a manner that more closely parallels, in a simpler pedagogical setting, the development in Secs. II and III. Appendix B contains a hand-waving justification of the rotationally enhanced Euler-Lagrange equation (1.1), and might therefore substitute for Sec. III on a first pass. Appendix C discusses the effect of the Faddeev-Popov constraints needed in the approach of Gervais *et al.*, while Appendix D reviews the properties of the effective large-N_c pion-baryon coupling (1.3).

II. THE SKYRME MODEL AS A CONSTRAINED PHASE-SPACE PATH INTEGRAL

The two-flavor massive Skyrme model is defined by the Lagrangian [3]

$$\mathcal{L} = \frac{f_\pi^2}{16}\mathrm{Tr}\partial_\mu U^\dagger \partial^\mu U + \frac{1}{32e^2}\mathrm{Tr}([U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2) + \frac{m_\pi^2 f_\pi^2}{8}\mathrm{Tr}(U-1) \qquad (2.1)$$

with U an SU(2) matrix. The two most popular representations of the pion field are

$$U = \exp(2i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/f_\pi) \ , \qquad (2.2)$$

or alternatively

$$U = u_0 + i\mathbf{u}\cdot\boldsymbol{\tau}, \ \ u_0^2 + \mathbf{u}^2 = 1, \ \ \mathbf{u} = 2\boldsymbol{\pi}/f_\pi \ . \qquad (2.3)$$

While these presumably define different quantum theories at $O(\hbar^2)$ (an underappreciated possibility[8]), they are equivalent for our present purposes, and we will not need to choose between them. For $m_\pi \neq 0$ the chiral symmetries $U \to AUB^\dagger$ are broken explicitly to isospin, that is $B = A$, or equivalently $\pi^i \to D_{ij}^{(1)}(A)\pi^j$.

[8]This is an interesting side story in itself; see Ref. [34] and references therein.

The parameter N_c enters the theory implicitly through the assignments $f_\pi^2 \sim e^{-2} \sim N_c$. Likewise, the coefficients of any desired higher derivative terms should also scale like N_c, so that $N_c/\hbar$ effectively sits outside the action. This observation justifies not only the specific saddle-point calculation of Sec. III to follow, but also illustrates the semiclassical picture of the large-N_c world in general [35,21,15,16]. In contrast, the large-N_c scaling behavior of m_π is somewhat arbitrary. While meson masses generically scale like N_c^0, in the special case of the pion this depends on whether one elects to link the chiral and large-N_c limits. Since, for a reasonable resemblance to nature, we would like the Δ to be able to decay to $N\pi$ in our theory, and since the N-Δ mass difference $\sim 1/N_c$, we will need to take the chiral limit *at least* as fast as the $1/N_c$ limit: $m_\pi \sim N_c^{-\nu}$ with $\nu \geq 1$. For technical reasons our optimal choice turns out to be

$$m_\pi \sim \frac{1}{N_c} \ , \qquad (2.4)$$

which is the convention we adopt from now on.

We remind the reader that Skyrme's choice of four-derivative term in (2.1) is the unique four-derivative construction that is at most second order in time derivatives [3]. This restriction is always invoked to justify an operator quantum mechanics approach to the model. For our present purposes, it is important as it allows us to work in a phase-space ("Hamiltonian") FPI formalism, following Ref. [26]. Modulo this important restriction, we generalize the Lagrangian (2.1) to *all* isospin-invariant models of the form

$$\mathcal{L} = \tfrac{1}{2}\dot{\pi}^i g_{ij}(\boldsymbol{\pi})\dot{\pi}^j - V(\boldsymbol{\pi}, \partial_i\boldsymbol{\pi}) \qquad (2.5)$$

which admit a hedgehog soliton. The Hamiltonian is then

$$\mathcal{H} = \tfrac{1}{2}\zeta^i g_{ij}^{-1}\zeta^j + V(\boldsymbol{\pi}, \partial_i\boldsymbol{\pi}) \ , \qquad (2.6)$$

where we have introduced the conjugate momenta

$$\zeta^i = \partial\mathcal{L}/\partial\dot{\pi}^i = g_{ij}\dot{\pi}^j \ . \qquad (2.7)$$

The phase-space formalism is the logically prior version of the FPI in which one integrates over both the generalized momenta and the generalized coordinates of the theory [36]. Accordingly, the transition amplitudes between initial and final states Ψ_i and Ψ_f, at times $t=-T$ and $t=+T$, in the presence of an external source $\mathcal{J}(x)$, are expressed as

$$T_{fi}[\mathcal{J}] = \int \mathcal{D}\boldsymbol{\pi}\,\mathcal{D}\boldsymbol{\zeta}\,\Psi_f^*[\boldsymbol{\pi}(\mathbf{x},+T)]\Psi_i[\boldsymbol{\pi}(\mathbf{x},-T)] \times \exp\left(i\int d^4x\,\zeta^i\dot{\pi}^i - \mathcal{H}(\boldsymbol{\pi},\boldsymbol{\zeta}) + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\pi}\right) \ . \qquad (2.8)$$

$T_{fi}[\mathcal{J}]$ is the generating function for n-point Green's functions in the theory, which are extracted in the usual way by functionally differentiating Eq. (2.8) n times with respect to the external source $\mathcal{J}$. In the current context, the advantages of the phase-space FPI are twofold. First, it is the natural framework in which the FPI makes contact with Hamiltonian quantum mechanics as per Adkins, Nappi, and Witten [1]. Second, it has the technical merit that so long as one is careful to make a *canonical* change of variables to the collective coordinate basis [26], then the induced Jacobians cancel identically between field space and momentum space, as verified below. This is because of the volume-preserving property of canonical transformations. Anyone familar with the related topic of perturbation theory in instanton backgrounds [37], where the phase-space FPI is not helpful, and where the unavoidable, uncanceled, Jacobians are best incorporated into the Feynman rules by means of discrete ghosts, will appreciate this simplification.

We assume that the static Euler-Lagrange equation

$$0 = \frac{\delta M_s[\boldsymbol{\pi}]}{\delta \pi^a}, \quad M_s[\boldsymbol{\pi}] = \int d^3\mathbf{x}\, V(\boldsymbol{\pi}, \partial_i \boldsymbol{\pi}) \tag{2.9}$$

admits a hedgehog Skyrmion solution $U_{\rm cl} = \exp[iF(r)\hat{\mathbf{r}} \cdot \boldsymbol{\tau}]$. Equivalently,

$$\pi^i_{\rm cl}(\mathbf{x}) = \frac{f_\pi}{2} F(r)\hat{\mathbf{r}}^i \quad \text{or} \quad \pi^i_{\rm cl}(\mathbf{x}) = \frac{f_\pi}{2} \sin F(r)\hat{\mathbf{r}}^i \tag{2.10}$$

depending on which of the two parametrizations, (2.2) or (2.3), is chosen. Isospin then generates an SU(2) family of static solutions $D^{(1)}_{ij}(A)\pi^j_{\rm cl}$. Far away from the center of the Skyrmion, for any well-behaved model, $\boldsymbol{\pi}_{\rm cl}$ must be annihilated by the static Klein-Gordon operator $\nabla^2 - m_\pi^2$, so that

$$F(r) \approx \sin F(r) \underset{r\to\infty}{\sim} B\left(\frac{m_\pi}{r} + \frac{1}{r^2}\right) e^{-m_\pi r} . \tag{2.11}$$

For any particular choice of Skyrmion Lagrangian, the numerical value of the constant B is obtained by solving the nonlinear equation for $F(r)$. Fortunately, up to chiral corrections,[9] there is a model-independent interpretation of B in terms of the pion-nucleon coupling constant [1],

$$B = \frac{3g_{\pi NN}}{4\pi f_\pi M_N} , \tag{2.12}$$

that we will exploit later on.

From now on, we restrict the FPI (2.8) to configurations that live in the baryon-number-unity sector of the theory. In order to model physical processes involving both baryons and mesons, we must allow for fluctuations away from the SU(2) family of Skyrmions, although still within this topological sector. But unless care is taken, the resulting perturbation theory in the fluctuating field will be plagued by infrared singularities, due to the Skyrmion's zero modes. Specifically, the small fluctuations operator cannot be inverted, so the propagator is not well defined. The cure is well known [26]: one orthogonalizes the fluctuating fields to these zero modes by means of Faddeev-Popov constraints. To minimize clutter, in this paper we will ignore the three translational and focus exclusively on the three rotational zero modes $h^{(k)}_m = \epsilon_{mkl}\pi^l_{\rm cl}$, with $k = 1$, 2, or 3. For each value of this index, the orthogonalization condition reads

$$0 = \mathcal{O}^{(k)}_\pi[A;\boldsymbol{\pi}] = \int d^3\mathbf{x} h^{(k)}_m [D^{(1)}_{nm}(A)\pi^n(\mathbf{x},t) - \pi^m_{\rm cl}(\mathbf{x})] . \tag{2.13}$$

A word on notation: we will frequently abbreviate the quantity in square brackets as $\delta\pi^m$, which is the fluctuating field in the body-fixed frame of the rotating Skyrmion. Likewise, we will denote the body-fixed total field as $\pi^m_{\rm tot}$, defined via

$$\pi^n = D^{(1)}_{nm}(A)(\pi^m_{\rm cl} + \delta\pi^m) = D^{(1)}_{nm}(A)\pi^m_{\rm tot} . \tag{2.14}$$

The additional incorporation of the translational modes, while straightforward in principle, serves ultimately just to Lorentz contract the Skyrmion [26], which does not affect the decay widths to leading order in $1/N_c$. Nevertheless, for a more accurate numerical comparison to experiment, and because it is obvious how to do so, we will reinsert Skyrmion recoil "by hand," in the form of a Lorentz-dilated Skyrmion mass, in Sec. VI below.

Formally, the three constraints (2.13) are implemented by inserting the Faddeev-Popov factor of unity into the FPI:

$$1 = \int \mathcal{D}A(t) \det J^{ij}_\pi \prod_{k=1,2,3} \delta(\mathcal{O}^{(k)}_\pi[A;\boldsymbol{\pi}]) . \tag{2.15}$$

The definition of the Jacobian matrix J_π depends on how one specifies the three coordinates a_1, a_2, a_3 needed to parametrize SU(2). We will postpone making this choice explicit for as long as possible, in which case we have, quite generally,

$$J^{ij}_\pi = \frac{d\mathcal{O}^{(j)}_\pi}{da^i} = A^i_{nm} \int d^3\mathbf{x}\, h^{(j)}_m \pi^n_{\rm tot} , \tag{2.16}$$

where

$$A^i_{nm} = \frac{dD^{(1)}_{lm}(A)}{da^i} D^{(1)}_{ln}(A) = i\epsilon_{mnl}{\rm Tr}\left(\tau_l A^\dagger \frac{dA}{da^i}\right) . \tag{2.17}$$

Observe that *nothing* in the above expressions requires that the three constraint functions $\mathbf{h}^{(m)}$ be, as we originally took them to be, the Skyrmion's rotational zero modes. They merely need to have nonzero overlap with the zero modes, for the purpose of removing the infrared singularities from the perturbative expansion. With this caveat, from now on we will think of the $\mathbf{h}^{(m)}$ as *arbitrary* functions, and will verify explicitly that our final physical result, which cannot depend on the division between $\boldsymbol{\pi}_{\rm cl}$ and $\delta\boldsymbol{\pi}$, as stressed in Sec. I, is indeed independent of the $\mathbf{h}^{(m)}$. To emphasize further this "gauge freedom" we choose three other constraint functions $\mathbf{f}^{(m)}$ for the momentum sector, subject only to the technical requirement that the overlap matrix

[9] For the particular choice of Lagrangian (2.1), these corrections are $O(m_\pi/ef_\pi)$, which $\sim 1/N_c$ by Eq. (2.4).

$$\Lambda_{ij} = \int d^3\mathbf{x}\, \mathbf{f}^{(i)} \cdot \mathbf{h}^{(j)} \tag{2.18}$$

be invertible. We denote by P^n the momenta conjugate to the a^n. The three momentum constraints are then

$$0 = \mathcal{O}_\zeta^{(k)}[A,\mathbf{P};\boldsymbol{\pi},\boldsymbol{\zeta}] = \Lambda_{kl}^{-1} \int d^3\mathbf{x}\, f_i^{(l)} [D_{mi}^{(1)}(A)\zeta^m - \zeta_{\rm cl}^i] \,, \tag{2.19}$$

where the numerical prefactor Λ^{-1} is inserted for later convenience, and the "classical momentum" $\boldsymbol{\zeta}_{\rm cl}(x;a_i,P_i)$ will be specified in a moment. The corresponding Faddeev-Popov factor of unity reads

$$1 = \int \mathcal{D}\mathbf{P}(t) \det J_\zeta^{ij} \prod_{k=1,2,3} \delta(\mathcal{O}_\zeta^{(k)}[A,\mathbf{P};\boldsymbol{\pi},\boldsymbol{\zeta}]) \,,$$

$$J_\zeta^{ij} = \frac{\partial \mathcal{O}_\zeta^{(j)}}{\partial P^i} = -\Lambda_{jl}^{-1} \int d^3\mathbf{x}\, \mathbf{f}^{(l)} \cdot \frac{\partial \boldsymbol{\zeta}_{\rm cl}}{\partial P^i} \,. \tag{2.20}$$

We will refer to the quantity in square brackets in Eq. (2.19) as the body-fixed "fluctuating momentum" $\delta\zeta^i$, and will likewise define the body-fixed total momentum $\zeta_{\rm tot}^i$ analogously to Eq. (2.14).

The Faddeev-Popov insertions (2.15) and (2.20) effect a *change of variables* in the phase-space path integral (2.8), from the original lab-frame coordinates $\{\boldsymbol{\pi}(x),\boldsymbol{\zeta}(x)\}$ to the far more useful set $\{\mathbf{a}(t),\mathbf{P}(t)\} \oplus \{\delta\boldsymbol{\pi}(x),\delta\boldsymbol{\zeta}(x)\}$ in which the SU(2) collective coordinates have been explicitly broken out, and the remaining fluctuating degrees of freedom are expressed in the rotating (body-fixed) frame.[10] While Eq. (2.20) is an identity for any choice of $\boldsymbol{\zeta}_{\rm cl}$, it is particularly convenient to choose $\boldsymbol{\zeta}_{\rm cl}$ in such a way that this change of variables is *canonical*, meaning that the form of the Legendre term in the action is preserved:

$$\int d^3\mathbf{x}\, \boldsymbol{\zeta} \cdot \frac{d}{dt}\boldsymbol{\pi} = P^n \dot{a}^n + \int d^3\mathbf{x}\, \delta\boldsymbol{\zeta} \cdot \frac{d}{dt}\delta\boldsymbol{\pi} \,. \tag{2.21}$$

Paralleling the U(1) derivation (A8)–(A11), one easily verifies that so long as $\boldsymbol{\zeta}_{\rm cl}$ is a linear combination of the three constraints $\mathbf{h}^{(k)}$, and $\boldsymbol{\zeta}_{\rm tot}$ satisfies

$$P^n = A_{jk}^n \int d^3\mathbf{x} \zeta_{\rm tot}^j \pi_{\rm tot}^k \,, \tag{2.22}$$

then the criterion (2.21) will be met. These two conditions are uniquely satisfied by the choice

$$\boldsymbol{\zeta}_{\rm cl} = -\mathbf{h}^{(k)} \cdot (J_\pi)_{km}^{-1} \left(P^m - A_{cb}^m \int d^3\mathbf{x}\, \delta\zeta^c \pi_{\rm tot}^b \right) \,. \tag{2.23}$$

As a bonus, the prefactor Λ^{-1} in Eq. (2.19) then trivially ensures that $J_\zeta^T = J_\pi^{-1}$ so that the two Faddeev-Popov determinants cancel precisely in the phase-space FPI, as promised.

Thanks to the factorized form of the Legendre term (2.21), the FPI (2.8) can be recast as a *quantum mechanical* sum over phase-space histories:

$$T_{fi}[\mathcal{J}] = \int \mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{P}(t)\Psi_f^*[A(+T)]\Psi_i[A(-T)] \times \exp\left(i\int_{-T}^{T} dt\, P^n \dot{a}^n \right) \exp(iS_{\rm eff}[A,\mathbf{P};\mathcal{J}]) \,. \tag{2.24}$$

Here we have anticipated the fact that to leading order in $1/N_c$, the Skyrmion wave functions will be functions of the collective coordinates only, with no dependence on the fluctuating degrees of freedom. For a given quantum mechanical path, the effective action $S_{\rm eff}[A,\mathbf{P};\mathcal{J}]$ is, in turn, expressible as a constrained FPI over the body-fixed fields:

$$\exp(iS_{\rm eff}[A,\mathbf{P};\mathcal{J}]) = \int \mathcal{D}[\delta\boldsymbol{\pi}(x)]\mathcal{D}[\delta\boldsymbol{\zeta}(x)] \prod_{k=1,2,3} \delta\left(\int d^3\mathbf{x}\, \mathbf{h}^{(k)} \cdot \delta\boldsymbol{\pi} \right) \delta\left(\int d^3\mathbf{x}\, \mathbf{f}^{(k)} \cdot \delta\boldsymbol{\zeta} \right) \times \exp\left(i\int d^4x\, \delta\zeta^i \delta\dot{\pi}^i - \frac{1}{2}\zeta_{\rm tot}^i \cdot g_{ij}^{-1}(\boldsymbol{\pi}_{\rm tot}) \cdot \zeta_{\rm tot}^j - V(\boldsymbol{\pi}_{\rm tot}) + \mathcal{J}^n \cdot D_{nm}^{(1)}(A)\pi_{\rm tot}^m \right) \,, \tag{2.25}$$

and it is to the steepest-descent evaluation of this expression that we now turn our attention.

III. SADDLE-POINT EVALUATION OF THE EFFECTIVE ACTION

Sufficient to leading-order in $1/N_c$, our plan is to evaluate the inner FPI (2.25) using saddle-point methods, the

[10] Generically one expects additional $O(\hbar^2)$ terms in the effective action after this change of variables [34], but these should not affect our leading-order result. Alex Kovner has suggested to us that the preservation of chiral symmetry at the quantum level might forbid such terms, as presumably they give a Yukawa-type falloff to $F(r)$, violating Goldstone's theorem when $m_\pi = 0$.

goal being[11] Eq. (1.1). In order to do so, one exponentiates the δ functions in the usual way, and extremizes the resulting effective Lagrangian

$$L_{\rm eff} = -M_s[\boldsymbol{\pi}_{\rm tot}] + \int d^3\mathbf{x}\{\delta\zeta^i\delta\dot{\pi}^i - \tfrac{1}{2}\zeta^i_{\rm tot}\cdot g^{-1}_{ij}(\boldsymbol{\pi}_{\rm tot}) \cdot\zeta^j_{\rm tot} + \alpha^{(k)}(t)\mathbf{h}^{(k)}\cdot\delta\boldsymbol{\pi} + \beta^{(k)}(t)\mathbf{f}^{(k)}\cdot\delta\boldsymbol{\zeta}\} . \quad (3.1)$$

The Lagrange multipliers $\alpha^{(k)}$ and $\beta^{(k)}$ implement the constraints (2.13) and (2.19), respectively. For simplicity, we are neglecting the back reaction of the external source $\mathcal{J}(x)$ on the saddle point. This is acceptable, since the effect of nonzero $\mathcal{J}$ can be reintroduced to any desired order in $\hbar/N_c$ using standard graphical methods. For one-pion processes such as $\Delta \to N\pi$ the simplest such graph is the one-loop "lollipop" (which is not forbidden by G parity as the cubic $\delta\boldsymbol{\pi}$ vertex is nonvanishing in the Skyrmion background). Due to the loop, this is a $1/N_c$ correction, and can be ignored.[12]

We look for stationary solutions to Eq. (3.1) that are time independent in the rotating (body-fixed) frame of the Skyrmion, $\delta\dot{\pi}^i = 0$, so the Legendre term $\int\delta\zeta^i\delta\dot{\pi}^i$ can be set to zero. Calculating from Eqs. (2.23) and (2.16) that

$$\frac{\delta\zeta^i_{\rm tot}(x)}{\delta\delta\pi^b(y)} \equiv \frac{\delta\zeta^i_{\rm cl}(x)}{\delta\pi^b_{\rm tot}(y)} = h^{(k)}_i(x)(J_\pi)^{-1}_{km}A^m_{cb}\zeta^c_{\rm tot}(y) , \quad (3.2)$$

one writes down the opaque (but soon to be simplified) intermediate expression

$$\begin{aligned} 0 &= \frac{\delta L_{\rm eff}}{\delta\delta\pi^b(y)} \\ &= -\frac{\delta M_s[\boldsymbol{\pi}_{\rm tot}]}{\delta\pi^b_{\rm tot}(y)} - (J_\pi)^{-1}_{km}A^m_{cb}\zeta^c_{\rm tot}(y)\int d^3\mathbf{x}\, h^{(k)}_i g^{-1}_{ij}\zeta^j_{\rm tot} \\ &\quad + \frac{1}{2}\int d^3\mathbf{x}\,\zeta^i_{\rm tot}g^{-1}_{ik}\frac{\delta g_{kl}(x)}{\delta\pi^b_{\rm tot}(y)}g^{-1}_{lj}\zeta^j_{\rm tot} + \alpha^{(k)}(t)h^{(k)}_b(y) . \end{aligned} \quad (3.3)$$

Recasting Eq. (3.3) in understandable form requires that we eliminate all $\boldsymbol{\zeta}_{\rm tot}$ dependence in favor of $\boldsymbol{\pi}_{\rm tot}$. To do so, we once again stationarize $L_{\rm eff}$, this time with respect to the fluctuating momentum:

$$\begin{aligned} 0 &= \frac{\delta L_{\rm eff}}{\delta\delta\zeta^j} \\ &= -\zeta^i_{\rm tot}g^{-1}_{ij} - (J_\pi)^{-1}_{kn}A^n_{ji}\pi^i_{\rm tot}\int d^3\mathbf{x}\,\zeta^a_{\rm tot}g^{-1}_{ab}h^{(k)}_b \\ &\quad + \beta^{(k)}(t)f^{(k)}_j . \end{aligned} \quad (3.4)$$

This equation is easily solved for $\boldsymbol{\zeta}_{\rm tot}$, giving[13]

$$\begin{aligned} \zeta^k_{\rm tot} &= g_{kj}A^n_{ji}\pi^i_{\rm tot}\tilde{\mathcal{I}}^{-1}_{mn}P^m , \\ \tilde{\mathcal{I}}_{mn}[\boldsymbol{\pi}_{\rm tot}] &= \int d^3\mathbf{x}(A^m_{ij}\pi^i_{\rm tot})g_{jk}(\boldsymbol{\pi}_{\rm tot})(A^n_{lk}\pi^l_{\rm tot}) . \end{aligned} \quad (3.5)$$

Inserting Eq. (3.5) into Eq. (3.3) and neglecting the $\alpha^{(k)}$ term for the moment, we derive the pleasingly compact variational equation

$$0 = \frac{\delta}{\delta\pi^b_{\rm tot}}\{M_s[\boldsymbol{\pi}_{\rm tot}] + \tfrac{1}{2}P^m\tilde{\mathcal{I}}^{-1}_{mn}P^n\} . \quad (3.6)$$

We can do even better, by reexpressing the second term using the Skyrmion's true moment of inertia,

$$\mathcal{I}_{mn}[\boldsymbol{\pi}_{\rm tot}] = \int d^3\mathbf{x}(\epsilon_{ijm}\pi^i_{\rm tot})g_{jk}(\pi_{\rm tot})(\epsilon_{lkn}\pi^l_{\rm tot}) , \quad (3.7)$$

which, unlike $\mathcal{I}_{mn}$, is independent of the collective coordinates. From Eq. (2.17), we obtain $\frac{1}{2}P^k\Omega^{-1}_{km}\mathcal{I}^{-1}_{mn}\Omega^{-1}_{ln}P^l$, where $\Omega_{km} = -i\,{\rm Tr}(\tau_k A^\dagger dA/da^m)$. The reader is then invited to choose his favorite parametrization of SU(2) (ours can be found in Sec. V and Appendix B) and verify that $P^k\Omega^{-1}_{km} = J^m$, the angular momentum of the Skyrmion. Thus Eq. (3.6) finally becomes

$$0 = \frac{\delta}{\delta\pi^b_{\rm tot}}\{M_s[\boldsymbol{\pi}_{\rm tot}] + \tfrac{1}{2}J^m\mathcal{I}^{-1}_{mn}J^n\} , \quad (3.8)$$

subject still to the constraints $\int\mathbf{h}^{(k)}\cdot\delta\boldsymbol{\pi} = 0$.

The results of this section are captured in a nutshell by the expression

$$\begin{aligned} T_{fi}[\mathcal{J}] \cong \int &\mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{P}(t)\Psi^*_f[A(+T)]\Psi_i[A(-T)] \\ &\times\exp\left(i\int_{-T}^{T}dt\,P^k\dot{a}^k - H_{\rm rot}\right) \\ &\times\exp\left(i\int d^4x\,\mathcal{J}^a(x)D^{(1)}_{ai}(A(t))\pi^i_{\rm tot}(\mathbf{x};\mathbf{J})\right) , \end{aligned} \quad (3.9)$$

where $\mathbf{J}$ is the Skyrmion's angular momentum, and $H_{\rm rot}$ is the rotationally enhanced energy read off from Eq. (3.8),

$$H_{\rm rot} = M_s + \tfrac{1}{2}J^m\mathcal{I}^{-1}_{mn}J^n , \quad (3.10)$$

evaluated on the (constrained) saddle-point solution $\boldsymbol{\pi}_{\rm tot}(\mathbf{x};\mathbf{J})$ that minimizes $H_{\rm rot}$, that is to say, on the rotationally improved Skyrmion.

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[11] Appendix B might well substitute for this rather technical section on a first reading.

[12] A warning: this conclusion is special to one-pion events. For two-pion processes such as $\pi N \to \pi N$, the back reaction of $\mathcal{J}$ contributes at *leading* order, and must be taken into account; see Sec. VII.

[13] To obtain this result, multiply Eq. (3.4) through by $h^{(k)}_j$ and integrate to obtain $\beta^{(k)}(t) \equiv 0$, and then, starting once again from Eq. (3.4), multiply through by $g_{jk}\pi^l_{\rm tot}A^m_{lk}$ and integrate, using Eq. (2.22) to solve for $\int\zeta^a_{\rm tot}g^{-1}_{ab}h^{(k)}_b$. Observe that the saddle-point value of $\boldsymbol{\zeta}_{\rm tot}$, in contrast with $\boldsymbol{\zeta}_{\rm cl}$, is "gauge invariant," that is, independent of the constraint functions.

$T_{fi}[\mathcal{J}]$ is a generating functional for (lab-frame) Green's functions. In particular, we can extract as a functional derivative the leading-order amplitude for one-pion emission at the space-time point $(\tilde{t},\tilde{\mathbf{x}})$ between Skyrmion states Ψ_i and Ψ_f sharp in the collective coordinates:

$$\langle A(+T)|\pi_a(\tilde{t},\tilde{\mathbf{x}})|A(-T)\rangle = -i\frac{\delta}{\delta\mathcal{J}^a(\tilde{x})}T_{fi}[\mathcal{J}]\,|_{\mathcal{J}\equiv 0}\ . \tag{3.11}$$

Of course, the physical "in" and "out" Skyrmions we are really interested in are not sharp in A, but rather, sharp in spin-isospin quantum numbers. In Sec. VI we will review the simple rules [1] for projecting out nucleons, Δ's, etc., from $|A(\pm T)\rangle$. But already it is clear that Eq. (3.11) entails evaluating $D^{(1)}_{ai}(A(\tilde{t}))\pi^i_{\rm tot}(\tilde{\mathbf{x}};\mathbf{J})$ between quantum states. We must therefore be prepared to answer two questions. First, what does the rotationally improved Skyrmion $\boldsymbol{\pi}_{\rm tot}$ look like? And second, since $\mathbf{J}$ is eventually promoted by the phase-space FPI to a Hamiltonian operator [1] $\hat{\mathbf{J}}$, and since $\hat{\mathbf{J}}$ does not commute with the operator $\hat{A}$, how is the ordering ambiguity in the product $D^{(1)}_{ai}(\hat{A})\pi^i_{\rm tot}(\mathbf{x};\hat{\mathbf{J}})$ resolved?[14] These questions are answered, respectively, in the two sections to follow.

IV. ASYMPTOTICS OF THE ROTATIONALLY IMPROVED SKYRMION

It would appear that generating a picture of $\pi^i_{\rm tot}(\mathbf{x};\mathbf{J})$ is a complicated computational problem, as the rotational kinetic term in Eq. (3.8) breaks the equivalence between spatial rotations and isorotations (see Appendix C). Thus the hedgehog symmetry of the solution is spoiled, and a purely radial equation for the Skyrmion profile is no longer available. But since we intend to focus on the pole location in momentum space, we only need such a picture at large distances, and here "pure thought" suffices.

It is helpful to recall some salient results from the U(1) model. From Eqs. (A17) and (A18), plus the fact that in all reasonable massive models the metric $g_{il}(\boldsymbol{\phi}_{\rm tot}) \to \delta_{il}$ exponentially fast at large distances, we observe that the isorotational kinetic term $P^2/2\mathcal{I}$ asymptotically contributes a negative mass-squared term $-P^2/\mathcal{I}^2$ to the Euler-Lagrange equation:

$$\frac{\delta}{\delta\phi^{\rm tot}_i}\frac{P^2}{2\mathcal{I}} = -\frac{P^2}{2\mathcal{I}^2}\frac{\delta\mathcal{I}}{\delta\phi^{\rm tot}_i} = -\frac{P^2}{\mathcal{I}^2}\epsilon_{ji}g_{jl}\epsilon_{lk}\phi^{\rm tot}_k \underset{r\to\infty}{\sim} -\frac{P^2}{\mathcal{I}^2}\phi^{\rm tot}_i\ . \tag{4.1}$$

The implications of the mass shift were reviewed in Sec. I.

[14] Henceforth we put hats on quantities to denote quantum operators.

The situation for SU(2) is more complicated, because the moment of inertia $\mathcal{I}$ is now a tensor, and requires a little more care. We note

$$\frac{\delta}{\delta\pi^b_{\rm tot}}\frac{1}{2}J^m\mathcal{I}^{-1}_{mn}J^n = -\frac{1}{2}J^m\mathcal{I}^{-1}_{mk}\frac{\delta\mathcal{I}_{kl}}{\delta\pi^b_{\rm tot}}\mathcal{I}^{-1}_{ln}J^n = -\tilde{J}^k\epsilon_{ijk}\pi^i_{\rm tot}g_{jn}\epsilon_{bnl}\tilde{J}^l\ , \tag{4.2}$$

adopting the shorthand $\tilde{J}^k = J^m\cdot\mathcal{I}^{-1}_{mk}[\boldsymbol{\pi}_{\rm tot}]$. Far away from the center of the Skyrmion, we again have $g_{jn}(\boldsymbol{\pi}_{\rm tot}) \to \delta_{jn}$ so that

$$\frac{\delta}{\delta\pi^b_{\rm tot}}\frac{1}{2}J^m\mathcal{I}^{-1}_{mn}J^n = \mathcal{M}_{bc}\pi^c,\quad \mathcal{M}_{bc} = -\tilde{\mathbf{J}}^2\delta_{bc} + \tilde{J}^b\tilde{J}^c\ . \tag{4.3}$$

The mass matrix $\mathcal{M}$ is diagonalized by inspection: one nullvector proportional to $\tilde{\mathbf{J}}$ itself, and two eigenstates with eigenvalue $-\tilde{\mathbf{J}}^2$ spanning the plane perpendicular to $\tilde{\mathbf{J}}$, for which $\tilde{\mathbf{J}}\times\hat{\mathbf{r}}$ and $\tilde{\mathbf{J}}\times\tilde{\mathbf{J}}\times\hat{\mathbf{r}}$ are a convenient basis. Accordingly, let us decompose $\boldsymbol{\pi}_{\rm tot} = f_1\tilde{\mathbf{J}} + f_2\tilde{\mathbf{J}}\times\hat{\mathbf{r}} + f_3\tilde{\mathbf{J}}\times\tilde{\mathbf{J}}\times\hat{\mathbf{r}}$, where the f_i are *a priori* general functions of the invariants $\tilde{\mathbf{J}}^2, \hat{\mathbf{r}}\cdot\tilde{\mathbf{J}}$, and r. As we have just noted, these are constrained by the requirement that at large distances, $f_1\tilde{\mathbf{J}}$ must be annihilated by $-\nabla^2 + m_\pi^2$ [coming from the first term on the right-hand side of Eq. (3.8)], while the f_2 and f_3 terms are annihilated by $-\nabla^2 + (m_\pi^2 - \tilde{\mathbf{J}}^2)$. A second constraint on the f_i is the requirement that $\boldsymbol{\pi}_{\rm tot}(\mathbf{x};\mathbf{J})$ smoothly approach the hedgehog configuration $\boldsymbol{\pi}_{\rm cl}(\mathbf{x})$ in the limit in which the classical vector $\tilde{\mathbf{J}} \to \mathbf{0}$. While a hedgehog is not itself a mass eigenstate of $\mathcal{M}$, it can be formed from a superposition of the f_1 and f_3 terms, since $\hat{\mathbf{r}} = ((\tilde{\mathbf{J}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{J}} - \tilde{\mathbf{J}}\times\tilde{\mathbf{J}}\times\hat{\mathbf{r}})/\tilde{\mathbf{J}}^2$ for any $\tilde{\mathbf{J}}$.

Collecting the various thoughts contained in the previous paragraph, and remembering Eqs. (2.10)–(2.12), we write down the following asymptotic expression for the rotationally improved Skyrmion:

$$\boldsymbol{\pi}_{\rm tot}(\mathbf{x};\mathbf{J}) \underset{r\to\infty}{\sim} \frac{3g_{\pi NN}}{8\pi M_N\tilde{\mathbf{J}}^2}\left\{\left(\frac{m_\pi}{r}+\frac{1}{r^2}\right)e^{-m_\pi r}(\tilde{\mathbf{J}}\cdot\hat{\mathbf{r}})\tilde{\mathbf{J}} - \left(\frac{(m_\pi^2-\tilde{\mathbf{J}}^2)^{1/2}}{r}+\frac{1}{r^2}\right) \times e^{-(m_\pi^2-\tilde{\mathbf{J}}^2)^{1/2}r}\tilde{\mathbf{J}}\times\tilde{\mathbf{J}}\times\hat{\mathbf{r}}\right\} + O(\tilde{\mathbf{J}})\ . \tag{4.4}$$

The $O(\tilde{\mathbf{J}})$ term includes the entire contribution of the f_2 term, as well as higher order pieces from the f_1 and f_3 terms. We reiterate that despite the overall $1/\tilde{\mathbf{J}}^2$, this expression has a perfectly smooth limit, the hedgehog $\boldsymbol{\pi}_{\rm cl}$, as $\tilde{\mathbf{J}} \to 0$.

In the above discussion, we have neglected the Faddeev-Popov field constraints (2.13) that remain as subsidiary conditions on Eq. (3.8), and, one might fear, modify $\boldsymbol{\pi}_{\rm tot}(\mathbf{x},\mathbf{J})$ in some complicated way. Fortunately, they merely result in a rigid spatial rotation of the Skyrmion (4.4) through a small angle that vanishes in the large-N_c limit, and hence they have no effect on the leading-order widths. These statements are proved in Appendix C.

The $1/N_c$ expansion allows a second simplification of Eq. (4.4), namely, the approximation of $\mathcal{I}_{mk}[\boldsymbol{\pi}_{\rm tot}]$ by $\mathcal{I}_{mk}[\boldsymbol{\pi}_{\rm cl}]$ in the definition[15] of $\tilde{\mathbf{J}}$. Since the moment of inertia tensor evaluated on a hedgehog configuration collapses to a scalar, $\mathcal{I}_{mk}[\boldsymbol{\pi}_{\rm cl}] \equiv \mathcal{I} \cdot \delta_{mk}$, we simply set $\tilde{\mathbf{J}} \cong \mathbf{J}/\mathcal{I}$ in Eq. (4.4). Furthermore $\mathcal{I} \propto f_\pi^2 \sim N_c$, which implies that the $O(\tilde{\mathbf{J}})$ contributions to $\boldsymbol{\pi}_{\rm tot}$ are also irrelevant. Implementing these large-N_c simplifications, and Fourier transforming the rotationally improved Skyrmion (4.4) to momentum space for later use, one therefore has

$$\boldsymbol{\pi}_{\rm tot}(\mathbf{q};\mathbf{J}) = -\frac{3ig_{\pi NN}}{2M_N\mathbf{J}^2}\left\{\frac{(\mathbf{J}\cdot\mathbf{q})\mathbf{J}}{\mathbf{q}^2+m_\pi^2} - \frac{\mathbf{J}\times\mathbf{J}\times\mathbf{q}}{\mathbf{q}^2+m_\pi^2-\mathbf{J}^2/\mathcal{I}^2}\right\} + (1/N_c \text{ corrections}) + (\text{nonpole terms}) \ . \tag{4.5}$$

Strictly speaking, Eqs. (4.4) and (4.5) were derived assuming $m_\pi^2 - \mathbf{J}^2/\mathcal{I}^2 > 0$, so that both exponentials are real and decaying. Later, when we calculate the on-shell decay amplitudes, we will need to extract the LSZ residue at $m_\pi^2 - \mathbf{J}^2/\mathcal{I}^2 = -\mathbf{q}^2$. It is possible that the rotationally improved Skyrmion itself has bizarre properties when this difference goes negative [27–31], such as a divergent mass at a subleading order in $1/N_c$; in fact, there is probably no solution at all in this regime to the defining equation (3.8). While interesting to contemplate, and potentially useful to understand for other applications, for our present purposes these pathologies are *irrelevant*: Eq. (4.5) will be used to construct a Green's function with Feynman boundary conditions, and like all such Green's functions, it is amenable to analytic continuation.[16] In sum, our procedure is: fix $m_\pi^2 > 0$ as per Eq. (2.4); then, since both m_π^2 and $\mathbf{J}^2/\mathcal{I}^2 \sim N_c^{-2}$, there is a finite N_c-independent radius of convergence in $\mathbf{J}^2$ where the analysis leading to Eq. (4.5) is justified; and finally, analytically continue to the kinematic regime of interest with the help of the usual Feynman prescription $m_\pi^2 \to m_\pi^2 - i\epsilon$.

V. RESOLVING THE OPERATOR ORDERING AMBIGUITY

In the preceding section, the pole pieces of the rotationally improved Skyrmion were calculated with $\mathbf{J}$ treated as a c number. In order to promote $\mathbf{J}$ to a q-number $\hat{\mathbf{J}}$, one must settle the ordering question implicit in the expression $D_{ai}^{(1)}(\hat{A})\pi_{\rm tot}^i(\mathbf{x};\hat{\mathbf{J}})$, where $D^{(1)}$ is the rotation matrix from Eqs. (2.14) and (3.9) that relates the lab-fixed and body-fixed frames.

Such ordering ambiguities are not peculiar to the Skyrme model, or to our particular choice of formalism, but on the contrary appear to be unavoidable in soliton quantization. They can always be resolved by appealing to physics. In the kink model [26], the operator ordering is fixed in an elegant way, by demanding that the commutation relations obeyed by the generators of one-dimensional Lorentz transformations be preserved at the quantum level [39]. In our U(1) toy model, we simply needed to invoke conservation of energy [4]. In both these models, the physically relevant solution turned out to be Weyl ordering, and hence, equivalent to the midpoint discretization of the phase-space FPI [34,40]. Unfortunately, the concepts of Weyl ordering and midpoint discretization do not readily generalize to SU(2), which is a curved manifold (unlike these one-dimensional examples). Nevertheless the ordering ambiguities are easily resolved, as we now explain.

It is helpful to specify an explicit representation of SU(2), namely,

$$A = a_0 + i\mathbf{a}\cdot\boldsymbol{\tau}, \quad a_0^2 + \mathbf{a}^2 = 1 \ , \tag{5.1}$$

in which case

$$\begin{aligned} D_{ai}^{(1)}(A) &\equiv \tfrac{1}{2}\mathrm{Tr}\tau_a^\dagger A\tau_i A^\dagger \\ &= (a_0^2 - \mathbf{a}^2)\delta_{ai} + 2a_a a_i + 2\epsilon_{aik}a_0 a_k \ . \end{aligned} \tag{5.2}$$

In the $|A\rangle$ basis used by Adkins, Nappi, and Witten, the Skyrmion's mutually commuting spin and isospin operators $\hat{\mathbf{J}}$ and $\hat{\mathbf{I}}$ are represented as derivatives [1]:

$$\begin{aligned} \hat{J}^k &= \frac{i}{2}\left(a_k\frac{\partial}{\partial a_0} - a_0\frac{\partial}{\partial a_k} - \epsilon_{klm}a_l\frac{\partial}{\partial a_m}\right) \\ &= -\frac{1}{4}\mathrm{Tr}(\tau_k\hat{A}^\dagger\partial_A) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \hat{I}^k &= \frac{i}{2}\left(-a_k\frac{\partial}{\partial a_0} + a_0\frac{\partial}{\partial a_k} - \epsilon_{klm}a_l\frac{\partial}{\partial a_m}\right) \\ &= \frac{1}{4}\mathrm{Tr}(\hat{A}^\dagger\tau_k\partial_A) \ , \end{aligned} \tag{5.4}$$

where $\partial_A = \partial/\partial a_0 + i\tau_j\partial/\partial a_j$. Note that $\hat{\mathbf{J}}^2 \equiv \hat{\mathbf{I}}^2 = -\frac{1}{4}\partial_\mu\partial_\mu$. The ordering issue arises because the components of $\hat{\mathbf{J}}$ do not commute with one another, nor with[17] $D_{ai}^{(1)}(\hat{A})$. It is easily resolved by appealing to one fundamental property of the spectrum of a relativistic quantum field theory, as follows. Consider $\Delta \to N\pi$. As anticipated already in Sec. I, the correct mass-shell pole will be given by the second term on the right-hand side of Eq. (4.5). As further discussed in Sec. I, the spectral representation of quantum field theory rules out the possibility of an extra nearby isolated pole in the Green's function in a theory of pions alone. Therefore, the first term of Eq. (4.5), which contains such a nearby pole,

[15] While this hedgehog approximation is valid for $\mathcal{I}_{mn}$, it is invalid for the *variation* of $\mathcal{I}_{mn}$ at large r as we have seen, and would therefore miss entirely the interesting pole structure in momentum space.

[16] This is not unlike the instanton case [38]. There, too, the on-shell single-particle pole generated by the configuration can only be reached with an analytic continuation away from the region (Euclidean space) where the configuration itself is well defined.

[17] In fact, $\hat{\mathbf{J}}$, $\hat{\mathbf{I}}$, and $D_{ai}^{(1)}$ precisely generate the old SU(4) spin-flavor algebra [21,41].

cannot contribute to Δ decay, nor to any off-diagonal transition $J_{\rm out} = J_{\rm in} \pm 1$.

We assert that the ordering[18] $D^{(1)}_{ai}(\hat{A})\hat{J}^i(\hat{\mathbf{J}}\cdot\mathbf{q})$ for the numerator of the first term has precisely this required property. This follows instantly from the identity

$$D^{(1)}_{ai}(\hat{A})\hat{J}^i = -\hat{I}^a \ , \quad (5.5)$$

since by definition, $\hat{\mathbf{J}}$ and $\hat{\mathbf{I}}$ do not change the spin and/or isospin representation of the Skyrmion, but rather act like the usual SU(2) ladder operators within each representation. Furthermore, we can construct a second operator that is likewise proportional to $\hat{I}^a(\hat{\mathbf{J}}\cdot\mathbf{q})$ on any given representation, but that has the advantage of not containing $\hat{\mathbf{I}}$ or $\hat{\mathbf{J}}$ explicitly, namely, $\mathcal{P}_{\Delta J=0}D^{(1)}_{ai}(\hat{A})q^i$, where $\mathcal{P}_{\Delta J=0}$ is the projection operator that ensures that the Skyrmion representation is preserved. By the Wigner-Eckart theorem, these two operators must therefore be proportional to one another:

$$D^{(1)}_{ai}(\hat{A})\hat{J}^i(\hat{\mathbf{J}}\cdot\mathbf{q}) = c(\hat{\mathbf{J}}^2)\mathcal{P}_{\Delta J=0}D^{(1)}_{ai}(\hat{A})q^i \ . \quad (5.6)$$

The constant $c(\hat{\mathbf{J}}^2)$ is fixed by letting both sides act on a Skyrmion wave function [1,42],

$$\langle A|^{I=J}_{i_z s_z}\rangle = (2J+1)^{1/2}(-)^{J-s_z}D^{(J)}_{-s_z,i_z}(A^\dagger) \ , \quad (5.7)$$

and using the standard formula for the tensor product of two Wigner D matrices.[19] One quickly finds $c(\hat{\mathbf{J}}^2) = \hat{\mathbf{J}}^2$, neatly canceling the factor of $1/\hat{\mathbf{J}}^2$ in front of Eq. (4.5).

By the same logic, the second term in Eq. (4.5) cannot contribute to the diagonal transitions $J_{\rm in} = J_{\rm out}$. By rewriting

$$-\frac{1}{\hat{\mathbf{J}}^2}\hat{\mathbf{J}}\times\hat{\mathbf{J}}\times\mathbf{q} = \mathbf{q} - \frac{1}{\hat{\mathbf{J}}^2}\hat{\mathbf{J}}(\hat{\mathbf{J}}\cdot\mathbf{q}) \ , \quad (5.8)$$

we reduce this ordering problem to the one above, which implies

$$D^{(1)}_{ai}(\hat{A})\left(q^i - \frac{1}{\hat{\mathbf{J}}^2}\hat{J}^i(\hat{\mathbf{J}}\cdot\mathbf{q})\right) = (1-\mathcal{P}_{\Delta J=0})D^{(1)}_{ai}(\hat{A})q^i = \mathcal{P}_{\Delta J=1}D^{(1)}_{ai}(\hat{A})q^i \ . \quad (5.9)$$

As the notation suggests, $\mathcal{P}_{\Delta J=1}$ forces the Skyrmion spin to change by one unit, just as required for this term.

There is one further ordering issue to be resolved, this one not unique to the Skyrme model, but present also in the U(1) model [4], namely, the meaning of $\hat{\mathbf{J}}^2/\mathcal{I}^2$ in the *denominator* of the second term in Eq. (4.5). This is the quantity interpreted by LSZ as the squared pion energy ω_π^2. As stated earlier, the solution to this ordering question is dictated unambiguously by conservation of energy, which equates ω_π to the difference of Skyrmion energies:[20]

$$\omega_\pi = \left(M_s + \frac{J_{\rm in}(J_{\rm in}+1)}{2\mathcal{I}}\right) - \left(M_s + \frac{J_{\rm out}(J_{\rm out}+1)}{2\mathcal{I}}\right) \ . \quad (5.10)$$

For the two allowed cases $J_{\rm in} = J_{\rm out} \pm 1$, Eq. (5.10) gives

$$\omega_\pi^2 = \frac{1}{2\mathcal{I}^2}[J_{\rm in}(J_{\rm in}+1) + J_{\rm out}(J_{\rm out}+1)] \ , \quad (5.11)$$

that is to say, the average of $\hat{\mathbf{J}}^2/\mathcal{I}^2$ acting on the bra and on the ket (i.e., an anticommutator).

The various results of this section are assembled as follows:

$$\{D^{(1)}_{ai}(\hat{A})\pi^i_{\rm tot}(\mathbf{q};\hat{\mathbf{J}})\}_{\text{properly ordered}} = -\frac{3g_{\pi NN}}{2M_N}\left(\frac{i}{\mathbf{q}^2+m_\pi^2}\mathcal{P}_{\Delta J=0}D^{(1)}_{ai}(\hat{A})q^i + \frac{i}{\mathbf{q}^2+m_\pi^2-\partial/\partial\beta}e^{\beta\hat{\mathbf{J}}^2/2\mathcal{I}^2}\mathcal{P}_{\Delta J=1}D^{(1)}_{ai}(\hat{A})q^i e^{\beta\hat{\mathbf{J}}^2/2\mathcal{I}^2}\Big|_{\beta=0}\right) + (1/N_c \text{ corrections}) + (\text{nonpole terms}) \ . \quad (5.12)$$

The β differentiation is just a concise bookkeeping device for the nested anticommutators implied by Eq. (5.11), and ensures that $\partial/\partial\beta \to \omega_\pi^2$. The effective pointlike field-theoretic vertex equivalent to this soliton expression is the full-strength $O(N_c^{1/2})$ pseudovector pion-baryon coupling (1.3), as advertised, resolving the "Yukawa problem."

VI. THE SKYRMION DECAY AMPLITUDE

The numerical calculation of the decay widths of the $I = J$ baryons now follow in short order. For thematic consistency, rather than working from this point forward with the effective field theory directly, we will complete the calculation in the same way we started it: as a FPI dominated in saddle-point approximation by the rotationally improved Skyrmion. But the real reason we stick

[18] N.B.: There exist other orderings which, though they appear distinct, give the same final result.

[19] We normalize the volume of SU(2) to unity.

[20] It is tempting to conjecture that for this case, the unique ordering for $\hat{\mathbf{J}}^2/\mathcal{I}^2$ specified by conservation of energy can also be arrived at in a completely different way, by demanding that the chiral algebra close at the quantum level, analogous to Ref. [39]. Note also, the lack of final-state Skyrmion recoil in this expression for the energy difference is just a harmless by-product of our decision at the outset, valid to leading order in $1/N_c$, to ignore the translational modes.

with the Skyrmion approach is that it is *much easier.* In the perturbative Fock space of the pion, the Skyrmion decays we are considering look like $0 \to 1$ transitions in a nontrivial background field, i.e., the Skyrmion itself, and hence the widths are integrals simply over one-body phase space. In contrast, the traditional relativistic effective Lagrangian approach to Δ decay [43–45], not to mention the higher spin baryons [45], involves the construction of Rarita-Schwinger spinors, subsidiary spin projection conditions, and other complications.

Resuming our train of thought with Eq. (3.11), and recalling the definition (5.7) of the Skyrmion wave functions, we write the cumbersome but conceptually simple expression

$$\begin{aligned} {}_{\rm out}\langle{}^{I'=J'}_{i'_z s'_z}|\pi^a(\omega,\mathbf{q})|{}^{I=J}_{i_z s_z}\rangle_{\rm in} = &\int d\tilde{t}\, e^{i\omega\tilde{t}} \int_{SU(2)} dA(T)\langle{}^{I'=J'}_{i'_z s'_z}|A(T)\rangle \int_{SU(2)} dA(\tilde{t}) \\ &\times \int_{SU(2)} dA(-T)\langle A(-T)|{}^{I=J}_{i_z s_z}\rangle \left[\int_{\tilde{t}}^{T} \mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{P}(t)\exp\left(i\int_{\tilde{t}}^{T} dt(P^k\dot{a}^k - H_{\rm rot})\right)\right] \\ &\times\langle A(\tilde{t})|\{D^{(1)}_{ai}(\hat{A}(\tilde{t}))\pi^i_{\rm tot}(\mathbf{q};\hat{\mathbf{J}})\}_{\rm properly\ ordered}|A(\tilde{t})\rangle \\ &\times\left[\int_{-T}^{\tilde{t}} \mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{P}(t)\exp\left(i\int_{-T}^{\tilde{t}} dt(P^k\dot{a}^k - H_{\rm rot})\right)\right] . \end{aligned} \tag{6.1}$$

Note that the FPI has been divided into two time intervals on either side of the one-point insertion, $-T < t < \tilde{t}$ and $\tilde{t} < t < T$. A technical point: in the path integration over each of these intervals, the quantum mechanical field $A(\tilde{t})$ formally enters, wrongly, as a fixed boundary condition, and so to lift this unphysical restriction we need an additional explicit integration over $A(\tilde{t})$, as we have indicated in Eq. (6.1). The reason for splitting up the FPI in this way is that in each time segment the Skyrmion propagates *freely* on the SU(2) manifold. Therefore, one can exploit the well-known sum-over-states expression for the propagator for a free particle moving on the SU(2) group manifold, derived in a classic paper by Schulman [42]:

$$\begin{aligned} &\int_{t_1}^{t_2} \mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{P}(t)\exp\left(i\int_{t_1}^{t_2} dt(P^k\dot{a}^k - H_{\rm rot})\right) \\ &\quad = \sum_{J=\frac{1}{2},\frac{3}{2},\ldots}\ \sum_{i_z,s_z=-J}^{J} \langle A(t_2)|{}^{I=J}_{i_z s_z}\rangle e^{i(t_2-t_1)M_J}\langle{}^{I=J}_{i_z s_z}|A(t_1)\rangle , \end{aligned} \tag{6.2}$$

where

$$M_J = M_s + J(J+1)/2\mathcal{I} . \tag{6.3}$$

The boundary conditions are that $A(t_1)$ and $A(t_2)$ are held fixed. The fact that only the diagonal component of $\mathcal{I}_{mn}$ appears follows from the use of the hedgehog wave functions (5.7), and is justified in large N_c.

Inserting Eq. (6.2) into Eq. (6.1) and performing the three independent SU(2) integrations using standard identities, one extracts the much simpler expression for the one-point function, free of collective coordinates, independent of the choice of Lagrangian (2.5) and of constraints (2.13) and (2.19), and valid for both $\Delta J = 0$ and $\Delta J = 1$:

$$\begin{aligned} &{}_{\rm out}\langle{}^{I'=J'}_{i'_z s'_z}|\pi^a(\omega,\mathbf{q})|{}^{I=J}_{i_z s_z}\rangle_{\rm in} \\ &\quad = -\frac{3g_{\pi NN}}{2M_N}\frac{iq^i}{\mathbf{q}^2+m_\pi^2-\omega^2}e^{iTM_{J'}}e^{iTM_J}(-)^{J+J'} \\ &\qquad \times\langle Ji_z|1J'ai'_z\rangle\langle J's'_z|1Jis_z\rangle 2\pi\delta(M_J - M_{J'} - \omega) \\ &\qquad +(1/N_c\ \text{corrections}) + (\text{nonpole terms}) . \end{aligned} \tag{6.4}$$

Next, one amputates the pion leg by multiplying by the inverse pion propagator $-i(q^2 - m_\pi^2)$ and going on mass shell, killing all nonpole contributions. The amputation of the nonrelativistic "in" and "out" baryon legs simply means erasing the two exponential factors, which represent free Skyrmion propagation. The conventional definition of the T matrix also requires that we cross out the energy-conserving δ function, leaving the product of Clebsch-Gordan coefficients:

$$\frac{3g_{\pi NN}q^i}{2M_N}(-)^{J+J'}\langle Ji_z|1J'ai'_z\rangle\langle J's'_z|1Jis_z\rangle . \tag{6.5}$$

We now set $J' = J - 1$, integrate the square of this amplitude over the one-body relativistic phase-space of the pion, and sum over final states, to obtain the total Skyrmion decay width:

$$\Gamma_{J\to J-1} = \left(\frac{3g_{\pi NN}}{2M_N}\right)^2 \sum_{a,i,i'_z,s'_z} (\langle Ji_z|1,J-1,ai'_z\rangle\langle J-1,s'_z|1Jis_z\rangle)^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{(q^i)^2}{2\sqrt{\mathbf{q}^2+m_\pi^2}}2\pi\delta(M_J - E_{\rm out}(\mathbf{q})) . \tag{6.6}$$

Spherical symmetry permits $(q^i)^2 \to \frac{1}{3}\mathbf{q}^2$ inside the integral, in which case the Clebsch-Gordan sum decouples, and gives $(2J-1)/(2J+1)$. The numerical value of the integral depends sensitively on how the final-state energy $E_{\rm out}$ is defined. With the naive choice $E_{\rm out} = M_{J'} + \sqrt{\mathbf{q}^2 + m_\pi^2}$ as in Eq. (6.4), we obtain

$$\Gamma_{J\to J-1} = \frac{3g_{\pi NN}^2 \bar{q}^3}{8\pi M_N^2}\frac{2J-1}{2J+1}\,, \qquad (6.7)$$

where $\bar{q}$ denotes the value of $|\mathbf{q}|$ which satisfies the δ function. Alternatively, we can already anticipate an obvious consequence of quantizing the Skyrmion's translational as well as rotational zero modes, namely, the Lorentz dilation of the Skyrmion mass [26]. This suggests the better choice[21] $E_{\rm out} = \sqrt{\mathbf{q}^2 + M_{J'}^2} + \sqrt{\mathbf{q}^2 + m_\pi^2}$, in which case the expression (6.7) is multiplied by an extra factor of $\sqrt{\mathbf{q}^2 + M_{J'}^2}/M_J = 1 + O(N_c^{-2})$. For $J = \frac{3}{2}$ these recoil-corrected formulas give $\bar{q} = 227$ MeV in which case[22] $\Gamma_{\Delta\to N\pi} = 114$ MeV, against an experimental width of 120 ± 5 MeV, while the cruder estimate (6.7) gives $\bar{q} = 258$ MeV and $\Gamma_{\Delta\to N\pi} = 212$ MeV. In both cases we have used the experimental values for all the parameters, eliminating M_s and $\mathcal{I}$ from Eq. (6.3) in favor of M_N and M_Δ as per Ref. [1] (one of several reasonable prescriptions).

The recoil-corrected expressions also yield $\Gamma_{5/2\to3/2} = 803$ MeV, $\Gamma_{7/2\to5/2} = 2643$ MeV, and $\Gamma_{9/2\to7/2} = 6437$ MeV, the masses of these large-N_c baryons being 1720, 2404, and 3284 MeV, respectively. Extrapolating to large J, $\Gamma_{J\to J-1} \sim J^3$ while the masses grow only like J^2. So these higher-spin "large-N_c artifacts," often considered a failing of the Skyrme model in particular, and of large-N_c phenomenology in general, are so broad that they effectively drop out of the particle spectrum[23] for physical values of the parameters, and *pose no problem whatsoever.*

VII. APPLICATION TO πN SCATTERING AND SOME CONCLUDING THOUGHTS

By grounding the Skyrme model in the FPI, systematizing the $1/N_c$ expansion, and paying careful attention to the analytic properties of the rotationally improved Skyrmion, we have taken a significant step towards showing how the Skyrmion bootstraps itself into an effective relativistic quantum field theory with explicit pointlike fields for the nucleon, Δ, etc. In particular, we have confirmed using soliton quantization [26] the effective large-N_c meson-baryon coupling (1.3), originally put forward by Adkins, Nappi, and Witten with no explicit input from or connection to Skyrmion physics, as these authors acknowledge (see Sec. 5 of Ref. [1]). By so doing, we have solved completely the so-called "Yukawa problem," namely, the emergence of Eq. (1.3) directly from Skyrmion quantization. (Previous major progress towards the solution may be found in Refs. [18–20].) Our approach has the advantages of focusing on the Yukawa coupling directly (rather than extracting it as the "square root" of a πN scattering amplitude or NN potential) and being manifestly "gauge" (i.e., constraint) independent. That the model-independent width of the Δ works out well, and the problematic higher spin baryons are too broad to be seen, adds credibility to the large-N_c program. Extensions of our methods to the case of three flavors, and to the study of pion photoproduction from a Skyrmion, are currently in progress.

While we have focused narrowly on single-pion poles, there is obviously more to be learned from the analytic properties of the rotationally improved Skyrmion. For example, look again at the asymptotic behavior (2.11) of the Skyrmion profile $F(r)$, obtained simply by linearizing the defining equation for F. The next-leading terms, which we dropped, are $O(F^3)$, and can be treated in Born approximation. Their iterated contribution to the Skyrmion falls off as $e^{-\mu r}$, where the Källen-Lehmann spectral parameter μ assumes a continuum of values $\geq 3m_\pi$. In the language of the corresponding field theory, these are precisely the two-loop vertex corrections to the bare coupling (1.3) that are responsible for the three-pion cut in the Green's function $\mathcal{G}(q_\pi)$ (the two-pion cut being forbidden by G parity). Interestingly, the two-loop $O(F^3)$ level is also where the two alternative definitions of the pion field, Eqs. (2.2) and (2.3), diverge from one another—and might actually define distinct quantum theories [34].

Beyond diagrammatics, a profound consequence of analyticity is crossing symmetry. We cannot even speculate how crossing and large N_c are reconciled, since the kinematic regimes can be so far removed from one another; for instance, $\Delta \to N\pi$ with $q_\pi^2 \sim N_c^{-2}$ versus the virtual transition $\pi \to \bar{N}\Delta$ with $q_\pi^2 \sim N_c^2$, the latter being a so-called "forbidden process" (see Sec. 8.3 of Ref. [35]) suppressed beyond any finite order in $1/N_c$. Whether an obstruction to such an "ambitious" analytic continuation actually exists, or whether on the contrary it is completely legitimate, is an open research topic of some importance, and for which Skyrmions might prove useful.

On a more down-to-earth level, the pion-baryon vertex

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[21] Because N_c appears implicitly in both the kinematics of the theory and in the parameters themselves (unlike, say, α in QED), it is impossible in practice to be a "purist" in the $1/N_c$ expansion, refusing to mix orders. Nor is this even desirable in principle (a view shared by most workers in the field), as it would break up Lorentz invariants. Those who would object nonetheless to our use of the mixed-order expression $\sqrt{\mathbf{q}^2 + M_{J'}^2}$ would also need to explain how, from the experimental values, one might separate the "leading" contribution to (say) $g_{\pi NN}$ from the "subleading" pieces. In our view, using the recoil-corrected Skyrmion mass-energy is truer to the spirit of equivalence to relativistic field theory that is the main theme of this paper.

[22] As already noted, a comparable width of the Δ was first quoted by Adkins, Nappi, and Witten [1], and similar expressions reappear regularly in the Skyrme-model literature.

[23] Alternatively, an interesting, purely *group-theoretic* means of eliminating the $I = J \geq \frac{5}{2}$ baryons from the spectrum, while preserving unitarity, may be found in Ref. [46].

is easily assembled into more complicated diagrams, most notably $\pi N \to \pi N$, which one might write somewhat schematically as

$$\langle \Psi_f | D^{(1)}_{ai} \pi^i_{\rm tot}(x;\hat{\mathbf{J}}) D^{(1)}_{bj} \pi^j_{\rm tot}(y;\hat{\mathbf{J}}) | \Psi_i \rangle \ . \quad (7.1)$$

The source of this contribution is obvious: it comes from hitting the FPI (3.9) with $[\delta/\delta\mathcal{J}^a(x)][\delta/\delta\mathcal{J}^b(y)]$ and pulling down two disconnected copies of the rotationally improved Skyrmion. This contribution to πN scattering has been studied in Refs. [18–20]. The graphs in the corresponding effective field theory are the "Compton diagrams" where one pion is absorbed and another emitted directly from the baryon line.

However, there is another contribution to pion-Skyrmion scattering that has also been studied extensively in the literature [19,32,33], which one might abbreviate as $\langle \delta\pi^a(x)\delta\pi^b(y)\rangle$. This is the two-point function for the fluctuating field $\delta\boldsymbol{\pi}$, propagating in the classical background of the Skyrmion, and it contributes at the same order, N_c^0, as Eq. (7.1). Yet we argued in Sec. I that it is dangerous, and contrary to the semiclassical nature of large N_c, to split up the total pion field in this way, into classical and fluctuating pieces. Is there a way of generating this important contribution directly from the rotationally improved Skyrmion, at *zeroth* order in the semiclassical expansion?

The answer, naturally, is yes. The propagator contribution arises automatically when the first functional derivative pulls down the Skyrmion, and the second acts on the *very same* Skyrmion:

$$\frac{\delta}{\delta\mathcal{J}^b(y)} D^{(1)}_{ai} \pi^i_{\rm tot}(x;\hat{\mathbf{J}};[\mathcal{J}]) \mid_{\mathcal{J}\equiv\mathbf{0}} \ . \quad (7.2)$$

As the notation suggests, the rotationally improved Skyrmion is itself a functional of the external source $\mathcal{J}(y)$. In the case of Skyrmion decay, we were cavalier in Eq. (3.1) about the back reaction of $\mathcal{J}$ on the Skyrmion, arguing that it is a one-loop, hence $1/N_c$, correction. But for πN scattering this back reaction is critical. Indeed, Taylor-expanding $D^{(1)}_{ai}\pi^i_{\rm tot}(x;\hat{\mathbf{J}};[\mathcal{J}])$ about $\mathcal{J} = \mathbf{0}$ we find a linear term in $\mathcal{J}$, which is precisely the convolution $\int dx' \langle \delta\pi^a(x)\delta\pi^b(x')\rangle \mathcal{J}^b(x')$ of the propagator in the Skyrmion background with the source itself. This is, of course, a tree diagram, hence leading order, and cannot be ignored. The associated diagrams in the effective field theory are the "exchange-type graphs" in which the pion and baryon lines exchange an arbitrary number of quanta. Thus, both contributions, (7.1) and (7.2), can be viewed in an elegant, unified, semiclassical way—in terms of the rotationally improved Skyrmion.

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APPENDIX A: U(1) REDUX

In this appendix we review some of the results of the U(1) model discussed in Ref. [4], generalized to include a field-dependent metric $g_{ij}(\boldsymbol{\phi})$ in the kinetic energy term, so that the analogy to the Skyrme model is closer. The incorporation of the constraints is also more general than in Ref. [4]. This appendix should be read in tandem with Secs. II and III which it parallels closely, in a setting in which the algebra is more transparent.

Our starting point is the (1+1)-dimensional Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}_i g_{ij}(\boldsymbol{\phi})\dot{\phi}_j - V(\boldsymbol{\phi}, \partial_x\boldsymbol{\phi}) \ . \quad (A1)$$

Here $\boldsymbol{\phi}$ is a real scalar doublet, and $\mathcal{L}$ is presumed invariant under the U(1) transformation

$$\boldsymbol{\phi} \to M(\theta)\cdot\boldsymbol{\phi}, \quad M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ . \quad (A2)$$

We assume that the Euler-Lagrange equation admits a static solution $\boldsymbol{\phi}^{\rm cl}$ and hence a family of U(1) solutions swept out by[24] $M(\theta)\cdot\boldsymbol{\phi}^{\rm cl}$. As in Ref. [26], we constrain the fluctuations away from these solutions by imposing the condition

$$0 = \mathcal{O}_\phi[\theta;\boldsymbol{\phi}] = \int dx\, h_k(M_{lk}\phi_l - \phi^{\rm cl}_k) \ . \quad (A3)$$

The expression in parentheses is the body-fixed fluctuating field $\delta\boldsymbol{\phi}$. The constraint function h_k need not be equal to the soliton's U(1) zero mode $\epsilon_{kj}\phi^{\rm cl}_j$; as long as they have nonzero overlap, the constraint (A3) will have the desired effect of removing the infrared singularities from the perturbative expansion.

Formally, the constraint (A3) is implemented by inserting into the path integral the Faddeev-Popov factor of unity

$$1 = \int \mathcal{D}\theta(t) J_\theta \delta(\mathcal{O}_\phi[\theta;\boldsymbol{\phi}]) \ ,$$
$$J_\theta = \frac{\partial\mathcal{O}_\phi}{\partial\theta} = -\int dx\, h_j \epsilon_{jk} \phi^{\rm tot}_k \ . \quad (A4)$$

Here the body-fixed field $\boldsymbol{\phi}^{\rm tot}$ is defined by

$$\phi^l = M_{ln}(\phi^{\rm cl}_n + \delta\phi_n) = M_{ln}\phi^{\rm tot}_n \ , \quad (A5)$$

and we have used the fact that $M_{ln} dM_{lk}/d\theta = \epsilon_{nk}$.

Since we intend to use a phase-space FPI, with path integration over both the canonical fields and their conjugate momenta, we introduce the canonical momentum $\zeta_i = g_{ij}\dot{\phi}_j$ in terms of which $\mathcal{H} = \frac{1}{2}\zeta_i g^{-1}_{ij}\zeta_j + V$. We also introduce a quantum mechanical momentum P conjugate to the U(1) collective coordinate θ. The phase-space approach [26] requires that the momenta be constrained in analogy to Eq. (A3); thus,

[24]There needs to be at least one additional singlet field in order for this to be possible [48], but this technical point is irrelevant to our general treatment.

$$0=\mathcal{O}_\zeta[\theta,P;\boldsymbol{\phi},\boldsymbol{\zeta}]=\Lambda^{-1}\int dx\, f_j(M_{lj}\zeta_l-\zeta_j^{\rm cl})\ . \quad \text{(A6)}$$

Here Λ is a normalization constant that we will pick conveniently later, and the new constraint $\mathbf{f}$ can be chosen independently of $\mathbf{h}$ (so long as they have nonzero overlap). The "classical momentum" $\boldsymbol{\zeta}^{\rm cl}$ is a configuration that we select to ensure that the constraints (A3) and (A6) define a canonical transformation of the path integral variables, from the old variables $\{\boldsymbol{\phi},\boldsymbol{\zeta}\}$ to the new variables $\{\theta,P\}\oplus\{\delta\boldsymbol{\phi},\delta\boldsymbol{\zeta}\}$ in which the U(1) collective coordinates have been explicitly separated out. Here the body-fixed "fluctuating momentum" $\delta\boldsymbol{\zeta}$ as well as the body-fixed "total momentum" $\boldsymbol{\zeta}^{\rm tot}$ are defined as

$$\zeta^l=M_{ln}(\zeta_n^{\rm cl}+\delta\zeta_n)=M_{ln}\zeta_n^{\rm tot} \quad \text{(A7)}$$

in analogy with Eq. (A5). A necessary and sufficient condition for such a canonical transformation is that the form of the Legendre term $\int\zeta_k\dot\phi_k$ be preserved; thus,

$$\int dx\,\zeta_k\dot\phi_k=P\dot\theta+\int dx\,\delta\zeta_k\delta\dot\phi_k\ . \quad \text{(A8)}$$

It is an easy matter to see how the condition (A8) fixes $\boldsymbol{\zeta}^{\rm cl}$ for us, giving

$$\boldsymbol{\zeta}^{\rm cl}=-\mathbf{h}\cdot J_\theta^{-1}\left(P-\int dx\delta\zeta_k\epsilon_{kl}\phi_l^{\rm tot}\right)\ . \quad \text{(A9)}$$

To verify this claim, expand the left-hand side of (A8) as

$$\begin{aligned}\int dx\,\zeta_k\dot\phi_k&=\int dx\,M_{kj}(\zeta_j^{\rm cl}+\delta\zeta_j)\frac{d}{dt}[M_{ki}(\phi_i^{\rm cl}+\delta\phi_i)]\\&=\dot\theta\int dx\,\zeta_k^{\rm tot}\epsilon_{kl}\phi_l^{\rm tot}+\int dx\,\delta\zeta_k\delta\dot\phi_k\\&\quad+\left[\frac{d}{dt}\int dx\,\zeta_k^{\rm cl}\delta\phi_k-\int dx\,\dot\zeta_k^{\rm cl}\delta\phi_k\right]\ . \quad \text{(A10)}\end{aligned}$$

Comparing the right-hand sides of Eqs. (A8) and (A10), we see that we must have

$$P=\int dx\,\zeta_k^{\rm tot}\epsilon_{kl}\phi_l^{\rm tot}\ , \quad \text{(A11)}$$

while in addition the expression in square brackets in (A10) must vanish. Thanks to the constraint (A3), this latter condition is automatically satisfied if we pick $\boldsymbol{\zeta}^{\rm cl}$ proportional to $\mathbf{h}$, whence Eq. (A9) follows immediately from the additional requirement (A11).

The same choice of $\boldsymbol{\zeta}^{\rm cl}$ that makes the Legendre term (A8) work out elegantly has a second nice property, as follows. We insert into the path integral the Faddeev-Popov factor of unity for the momentum sector:

$$\begin{aligned}1&=\int\mathcal{D}P(t)J_P\delta(\mathcal{O}_\zeta[\theta,P;\boldsymbol{\phi},\boldsymbol{\zeta}])\ ,\\ J_P&=\frac{\partial\mathcal{O}_\zeta}{\partial P}=\Lambda^{-1}J_\theta^{-1}\int dx\,\mathbf{h}\cdot\mathbf{f}\ . \quad \text{(A12)}\end{aligned}$$

Therefore, if the normalization constant Λ is equated to $\int\mathbf{h}\cdot\mathbf{f}$, the two Faddeev-Popov Jacobians cancel identically: $J_\theta J_P=1$.

This reflects the volume-preserving character of canonical transformations.

We are now ready to discuss the saddle-point evaluation of the action. The Lagrangian reads

$$\begin{aligned}L_{\rm eff}&=\int dx\{\zeta_k\dot\phi_k-\mathcal{H}\}+(\text{exponentiated constraints})\\&=P\dot\theta+\int dx\{\delta\zeta_k\delta\dot\phi_k-\tfrac12\zeta_i^{\rm tot}g_{ij}^{-1}(\boldsymbol{\phi}^{\rm tot})\zeta_j^{\rm tot}\\&\quad-V(\boldsymbol{\phi}^{\rm tot},\partial_x\boldsymbol{\phi}^{\rm tot})+\alpha(t)\mathbf{h}\cdot\delta\boldsymbol{\phi}+\beta(t)\mathbf{f}\cdot\delta\boldsymbol{\zeta}\}\ .\end{aligned} \quad \text{(A13)}$$

Here α and β are Lagrange multipliers implementing the constraints (A3) and (A6), respectively. We look for stationary solutions that are time independent in the rotating frame of the soliton, $\delta\dot\phi_k=0$, so that the Legendre term $\int\delta\zeta_k\delta\dot\phi_k$ can be set to zero. Calculating from Eqs. (A9) and (A4) that

$$\frac{\delta\zeta_j^{\rm tot}(x)}{\delta\delta\phi_i(y)}\equiv\frac{\delta\zeta_j^{\rm cl}(x)}{\delta\phi_i^{\rm tot}(y)}=h_j(x)J_\theta^{-1}\epsilon_{li}\zeta_l^{\rm tot}(y)\ , \quad \text{(A14)}$$

one has

$$\begin{aligned}0&=\frac{\delta L_{\rm eff}}{\delta\delta\phi_i(y)}\\&=-J_\theta^{-1}\zeta_l^{\rm tot}(y)\epsilon_{li}\int dx\,h_jg_{jn}^{-1}\zeta_n^{\rm tot}\\&\quad+\int dx\left\{\frac12\zeta_n^{\rm tot}g_{nk}^{-1}\frac{\delta g_{kl}(x)}{\delta\phi_i^{\rm tot}(y)}g_{lj}^{-1}\zeta_j^{\rm tot}-\frac{\delta V(x)}{\delta\phi_i^{\rm tot}(y)}\right\}\\&\quad+\alpha(t)h_i(y)\ .\end{aligned} \quad \text{(A15)}$$

We would like to eliminate the $\boldsymbol{\zeta}^{\rm tot}$ dependence of this expression, recasting it purely in terms of $\boldsymbol{\phi}^{\rm tot}$. To do so, we stationarize $L_{\rm eff}$ with respect to the fluctuating momentum:

$$\begin{aligned}0&=\frac{\delta L_{\rm eff}}{\delta\delta\zeta_i}\\&=-\zeta_j^{\rm tot}g_{ji}^{-1}-J_\theta^{-1}\epsilon_{ij}\phi_j^{\rm tot}\int dx\,h_ag_{ab}^{-1}\zeta_b^{\rm tot}+\beta(t)f_i\ .\end{aligned} \quad \text{(A16)}$$

This equation is easily manipulated to give[25]

$$\begin{aligned}\zeta_k^{\rm tot}&=g_{kj}\epsilon_{jl}\phi_l^{\rm tot}\mathcal{I}^{-1}P\ ,\\ \mathcal{I}[\boldsymbol{\phi}^{\rm tot}]&=\int dx(\epsilon_{ij}\phi_j^{\rm tot})g_{il}(\boldsymbol{\phi}^{\rm tot})(\epsilon_{lk}\phi_k^{\rm tot})\ .\end{aligned} \quad \text{(A17)}$$

Substituting Eq. (A17) into (A15) yields, finally, the elegant expression

$$\begin{aligned}0&=\frac{\delta}{\delta\phi_i^{\rm tot}}\left\{M_s+\frac{P^2}{2\mathcal{I}[\boldsymbol{\phi}^{\rm tot}]}\right\}\ ,\\ M_s&=\int dx\,V\ ,\end{aligned} \quad \text{(A18)}$$

[25] To obtain this result, multiply Eq. (A16) through by h_i and integrate to find $\beta(t)\equiv0$, and then, returning to Eq. (A16), multiply through by $g_{ij}\epsilon_{jk}\phi_k^{\rm tot}$ and integrate, using the identity (A11) to solve for $\int h_ag_{ab}^{-1}\zeta_b^{\rm tot}$.

subject still to the field constraint $\int \mathbf{h}\cdot\delta\boldsymbol{\phi} = 0$, the (subleading) effect of which is discussed in Appendix C.

APPENDIX B: A HAND-WAVING JUSTIFICATION OF THE ROTATIONALLY ENHANCED EULER-LAGRANGE EQUATION

The purpose of this appendix is to give an heuristic justification of our use of Eq. (1.1) as an improved starting point. The reader seeking a more compelling derivation should work through Sec. III.

We start from the generalized Skyrme Lagrangian (2.5), and make the ansatz

$$\pi^i(x) \to D^{(1)}_{ik}[A(t)]\pi^k_{\rm tot}(\mathbf{x}) \,, \tag{B1}$$

where $\boldsymbol{\pi}_{\rm tot}$ is presumed to be time independent. Inserting this ansatz into the first term of Eq. (2.5) gives $8j^a\mathcal{I}_{ab}[\boldsymbol{\pi}_{\rm tot}]j^b$, where

$$j^a = -\frac{i}{4}{\rm Tr}\tau^a A^\dagger \dot{A} \tag{B2}$$

is the c-number analogue of the Skyrmion's spin operator (5.3), and $\mathcal{I}_{ab}$ is the moment of inertia tensor (3.7). In deriving this result we have exploited the fact that the metric transforms as a two-index tensor:

$$g_{ij}(D^{(1)}\cdot\boldsymbol{\pi}_{\rm tot}) = g_{ab}(\boldsymbol{\pi}_{\rm tot})D^{(1)}_{ia}D^{(1)}_{jb} \,. \tag{B3}$$

Inserting the ansatz (B1) into the second term (2.5) just gives $V(\boldsymbol{\pi}_{\rm tot})$ by isospin invariance. The sum of the two terms implies an action functional that can be inserted into an ordinary (*not* phase-space) FPI. A convenient choice of coordinates is the S^3-symmetric set a_μ given by Eq. (5.1), in terms of which the SU(2)-invariant path integration measure is proportional to the product over time slices of ordinary integrals:

$$\prod_{\substack{\rm time\\ \rm slices}} \int d^4a\,\delta(a_\mu a_\mu - 1)\exp\left(i\int dt(8j^a\mathcal{I}_{ab}j^b - M_s)\right) \,. \tag{B4}$$

When $\mathcal{I}_{ab}$ is diagonal the first term in the exponent collapses to $\mathcal{I}\,{\rm Tr}\dot{A}^\dagger\dot{A}$ which we recognize as the free SU(2) Lagrangian [42].

We will now show that our phase-space FPI construction, Eqs. (3.9) and (3.10), is formally equivalent to Eq. (B4). Following Ref. [1], we introduce four momenta p_μ conjugate to the a_μ, so that $p_\mu \leftrightarrow -i\partial/\partial a_\mu$, subject to the constraint $p_\mu a_\mu = 0$. The SU(2)-invariant momentum integration is then proportional to a product over time slices of $\int d^4p\,\delta(a_\mu p_\mu)$. The Legendre term can be rewritten as $\dot{a}_\mu p_\mu = 4\mathbf{J}\cdot\mathbf{j}$ where, as in Eq. (5.3),

$$J^a = -\frac{i}{4}{\rm Tr}\tau^a A^\dagger P, \quad P = p_0 + i\mathbf{p}\cdot\boldsymbol{\tau} \,. \tag{B5}$$

The phase-space FPI is then proportional to

$$\prod_{\substack{\rm time\\ \rm slices}} \int d^4a\,\delta(a_\mu a_\mu - 1)\int d^4p\delta(a_\mu p_\mu) \times\exp\left(i\int dt(4\mathbf{J}\cdot\mathbf{j} - M_s - \frac{1}{2}J^a\mathcal{I}^{-1}_{ab}J^b)\right) \,. \tag{B6}$$

Since $d^4p\delta(a_\mu p_\mu) \propto d^3\mathbf{J}$ we can perform the Gaussian $\mathbf{J}$ integrals and be left with Eq. (B4) precisely. This completes the heuristic justification of our phase-space starting point.

APPENDIX C: EFFECT OF THE FIELD CONSTRAINTS ON THE ROTATIONALLY IMPROVED SKYRMION

In Sec. IV we examined the rotationally improved Skyrmion while ignoring the effect of the field constraints (2.13) that remain as subsidiary conditions on Eq. (3.8). We justified our cavalier approach with two claims, first, that the constraints can be implemented trivially by rigidly rotating the configuration (4.4) in real space, and second, that the angle of this rotation is vanishingly small in the large-N_c limit, so that the leading-order LSZ residues are unaffected. Let us prove these two statements.

It is easiest to start with the one-dimensional U(1) model as reviewed in Appendix A [4]. Let $\boldsymbol{\phi}^{\rm tot}(x;P)$ be a static solution to Eq. (A18); the configuration we have been calling $\boldsymbol{\phi}^{\rm cl}(x)$, analogous to the undistorted hedgehog in the Skyrme model, is then $\boldsymbol{\phi}^{\rm tot}(x;0)$. The relevant observation is that for any P, there is a U(1) manifold of degenerate static solutions, $M(\theta)\boldsymbol{\phi}^{\rm tot}(x;P)$. Since in this model there is only one field constraint, Eq. (A3), generically there will be one point, or at most a discrete set of points, on this U(1) manifold that satisfy the constraint. We now show, self-consistently, that we arrive at one such point by picking a particular relative angle $\theta(P)$ between $\boldsymbol{\phi}^{\rm tot}(x;P)$ and $\boldsymbol{\phi}^{\rm cl}(x)$, and that $\theta(P)$ is in fact *small.* [The term "relative angle" presupposes that for $\theta=0$, $\boldsymbol{\phi}^{\rm tot}(x;P)$ and $\boldsymbol{\phi}^{\rm cl}(x)$ are equivalently oriented in the internal space, for instance to point in the $\binom{1}{0}$ direction for $x\to\infty$.] The effect of a small isorotation can be Taylor-expanded:

$$\phi^{\rm tot}_k(x;P) \overset{\rm isorotation}{\longrightarrow} \phi^{\rm tot}_k(x;P) + \theta(P)\epsilon_{kl}\phi^{\rm tot}_l(x;P) + O(\theta(P)^2) \,. \tag{C1}$$

The constraint becomes

$$0 = \int dx\,h_k[\phi^{\rm tot}_k(x;P) + \theta(P)\epsilon_{kl}\phi^{\rm tot}_l(x;P) + O(\theta(P)^2) - \phi^{\rm cl}_k(x)] \tag{C2}$$

so that

$$\theta(P) \cong -\frac{\int dx\,\mathbf{h}\cdot[\boldsymbol{\phi}^{\rm tot}(x;P) - \boldsymbol{\phi}^{\rm cl}(x)]}{\int dx\,h_k\epsilon_{kl}\phi^{\rm tot}_l(x;P)} \,. \tag{C3}$$

To the extent that the rotational term $P^2/2\mathcal{I}$ in Eq. (A18) is a perturbative correction to M_s (as it is in the Skyrme model, where it is down by N_c^2), this ratio is obviously small, and our claim is established in a self-consistent manner.

The Skyrme model is just a little more complicated, because of the possibility of both spatial and isorotations in three dimensions; respectively, $\boldsymbol{\pi}(\mathbf{x}) \to \boldsymbol{\pi}(D^{(1)}\cdot\mathbf{x})$ versus $\boldsymbol{\pi}(\mathbf{x}) \to D^{(1)}\cdot\boldsymbol{\pi}(\mathbf{x})$. The effect of the perturbation $\frac{1}{2}J^m\mathcal{I}^{-1}_{mn}[\boldsymbol{\pi}_{\rm tot}]J^n$ is to break the SU(2) degeneracy of the rotationally improved Skyrmion in *iso*space down to U(1), namely, isorotations in the plane perpendicular to **J**. However, there is still a full SU(2) complement of degenerate configurations obtained by *spatial* rotations [a U(1) subgroup of which is redundant with the remaining isorotations]. Thus for any **J** there is a three-parameter manifold of degenerate rotationally improved Skyrmions, and since there are now three constraints, we once again expect a single solution point, or at most a discrete number of solutions. So in either model the constraints, being in 1-to-1 correspondence with the zero modes, have done their job, and eliminated the flat directions from the FPI. The resulting perturbation theory about the rotationally improved Skyrmion will be free of IR singularities to all orders in $1/N_c$, and no additional collective coordinates, nor extra isorotational kinetic terms in the action, can be justified. We leave it to the reader to write down the SU(2) analogues of Eqs. (C2)–(C3), which now involve matrix inverses, and are not particularly illuminating.

APPENDIX D: PION PSEUDOVECTOR COUPLING TO THE $I = J$ BARYONS

As stated in Sec. I, the pion-baryon vertex (1.3) is uniquely specified by the twin requirements of the chiral and large-N_c limits. The former implies the derivative coupling ("Adler's rule"). The latter augments the usual such coupling to the nucleon, namely,

$$(g_{\pi NN}/2M_N)\partial_\mu\pi^a\bar{N}\gamma^\mu\gamma^5\tau^a N\ , \tag{D1}$$

with coupling to the entire tower of $I = J$ baryons, in such a way that the following three requirements are satisfied.

(1) The coupling is invariant under isospin and angular momentum. The field $|A\rangle$ transforms as[26]

$$|A\rangle \stackrel{\rm isospin}{\longrightarrow} |U_I A\rangle \text{ and } |A\rangle \stackrel{\rm ang\ mom}{\longrightarrow} |AU_J^\dagger\rangle \tag{D2}$$

so that

$$\begin{aligned}\int_{\rm SU(2)} dA\, D^{(1)}_{ab}(A)|A\rangle\langle A| \to \int_{\rm SU(2)} dA\, D^{(1)}_{ab}(A)|U_I A U_J^\dagger\rangle\langle U_I A U_J^\dagger| \\ = \int_{\rm SU(2)} dA\, D^{(1)}_{ab}(U_I^\dagger A U_J)|A\rangle\langle A| \\ = D^{(1)}_{aa'}(U_I^\dagger)D^{(1)}_{bb'}(U_J^\dagger)\int_{\rm SU(2)} dA\, D^{(1)}_{a'b'}(A)|A\rangle\langle A|\ .\end{aligned} \tag{D3}$$

Here we have used the group invariance of the SU(2) measure, $d(U_I^\dagger A U_J) = dA$, and the reality of the rotation matrices. Similarly,

$$\partial_b\pi^a \to \partial_{b''}\pi^{a''}D^{(1)}_{a''a}(U_I)D^{(1)}_{b''b}(U_J)\ . \tag{D4}$$

Combining these last two equations confirms the invariance.

(2) Equation (1.3) includes the usual pion-nucleon interaction (D1). Expanding $|A\rangle\langle A|$ into baryon fields with good spin and isospoin using the wave function (5.7), and performing the resulting integral over three D matrices, gives

$$-\frac{3g_{\pi NN}}{2M_N}\sum_{a,b}\partial_b\pi^a\sum_{J,i_z,s_z}\sum_{J',i'_z,s'_z}(-1)^{J+J'}\langle J1i_za|J'i'_z\rangle\langle J'1s'_zb|Js_z\rangle|^{J'}_{i'_zs'_z}\rangle\langle^{J}_{i_zs_z}|\ . \tag{D5}$$

We now pick out the terms with $J = J' = 1/2$ in this expression. Isospin and angular momentum invariance can be made more manifest by rewriting this subset of terms as

$$(g_{\pi NN}/2M_N)\sum_{a,b}\sum_{i_z,s_z}\sum_{i'_z,s'_z}\tau^a_{i'_zi_z}\sigma^b_{s'_zs_z}\partial_b\pi^a|^{1/2}_{i'_zs'_z}\rangle\langle^{1/2}_{i_zs_z}| \tag{D6}$$

which we recognize as the nonrelativistic (or, equivalently, in the present context, large-N_c) limit of Eq. (D1).

(3) Equation (1.3) correctly implements the "$I_t = J_t$ rule" [49] and the "proportionality rule" [21,49,41] governing meson couplings to the higher-spin fields in the $I = J$ tower. A careful reading of Ref. [49] reveals these criteria will be

[26] We remind the reader of the compact notation of Eq. (1.3), whereby $\langle A|$ is shorthand for the superposition of *explicit* pointlike fields for the nucleon, Δ, and so forth up the $I = J$ tower of baryons, any one of which may be projected out using Eq. (5.7).

automatically satisfied due to the diagonality of Eq. (1.3) in the collective coordinate A. It is instructive nevertheless to see how this comes about explicitly. The bra and ket in Eq. (D5) can be written in terms of fields with good t-channel (exchange-channel) quantum numbers as

$$\left|{}^{J'}_{i'_z s'_z}\right\rangle\left\langle{}^{J}_{i_z s_z}\right| = \sum_{I_t, I_{tz}} \sum_{J_t, J_{tz}} (-1)^{J+i_z}(-1)^{J'+s'_z}\langle I_t I_{tz}|J'Ji'_z, -i_z\rangle\langle JJ's_z, -s'_z|J_tJ_{tz}\rangle \times \left|{}^{I_t;JJ'}_{I_{tz}}\right\rangle\left\langle{}^{J_t;JJ'}_{J_{tz}}\right| , \tag{D7}$$

where the phases in the above are the usual cost of turning bras into kets in SU(2) [50]: $|jm\rangle \leftrightarrow (-1)^{j+m}\langle j, -m|$. Plugging Eq. (D7) into Eq. (D5) and using Clebsch-Gordan orthogonality gives for the pion-baryon coupling:

$$-\frac{g_{\pi NN}}{2M_N} \sum_{I_{tz}, J_{tz}} \partial_{J_{tz}} \pi^{I_{tz}} \sum_{J,J'} (-1)^{J+J'}[(2J+1)(2J'+1)]^{1/2} \left|{}^{I_t=1;JJ'}_{I_{tz}}\right\rangle\left\langle{}^{J_t=1;JJ'}_{J_{tz}}\right| . \tag{D8}$$

This expression correctly embodies the two aforementioned large-N_c selection rules: the square-root proportionality factors relating the pion's couplings to the various baryon fields in the $I = J$ tower illustrate the proportionality rule, while the fact that the exchanged angular momentum $J_t = 1$ (i.e., P-wave pion emission) is equal to the isospin $I_t = 1$ of the pion is a specific example of the more general $I_t = J_t$ rule. This latter observation is not entirely "content-free," as one might initially suspect. True, for the special case $\Delta \to N\pi$, or for the specific off-shell coupling $N \to N\pi$, the fact that the pion is emitted in a P-wave follows trivially from parity and angular momentum conservation. But for the higher members of the $I = J$ tower of baryons there is no obvious conservation law forbidding, or even suppressing, F-wave hard pion emission when the off-shell virtuality of the pion is order $q_\pi^2 \sim N_c^0$. The fact that P-wave emission/absorption nevertheless continues to dominate in this kinematic regime is a specific dynamical prediction of large N_c, already incorporated into the effective field-theoretic coupling (1.3), and thanks to the equivalence exhibited in this paper, also embodied by the Skyrme model. Unfortunately, as we have also shown that these higher-spin states do not exist as particles, this particular piece of phenomenology is somewhat moot.

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