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Quantum tunneling with global charge

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We investigate quantum tunneling in the theory of a complex scalar field with a global U(1) symmetry when the charge density of the initial configuration does not vanish. We discuss the possible final configurations and set up the Euclidean path integral formalism to find the bubble nucleation and to study the bubble evolution. For the stationary path, or the bounce solution, in the Euclidean time, the phase variable becomes pure imaginary so that the charge density remains real. We apply this formalism to examples when the initial charge density is small. While the phase transition considered here occurs in zero temperature, the bubble dynamics is richly complicated, involving conserved charge, the sound wave, and the supersonic bubble wall.

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I. INTRODUCTION

Recently, there has been some interest in the first order phase transition involving nonzero global charge. The finite temperature effective potential and the phase structure in this type of phase transition have been extensively studied [1]. However, how the phase transition proceeds has not been discussed in detail. Here we investigate the phase transition in a model which involves nonzero charge and a nontrivial bubble wall dynamics. This model is the theory of a complex scalar field with a global Abelian symmetry. Even in the zero temperature phase transition, this model has a rather rich dynamics in the phase transition, depending on the initial configuration and the potential energy. Since there are only two field degrees of freedom in this model, this model can be rather easily approached analytically and numerically. We hope that our toy model illuminates some aspects of the phase transition involving global charge and that some of the insight gained would be applicable to the QCD phase transition and the electroweak phase transition.

The general formalism of the Euclidean phase integral involving nonzero charge in our model was developed sometime ago [2]. While this formalism has been applied to wormhole physics, there has been no direct attempt to apply it to the first order phase transition. We extend this formalism to the first order phase transition at zero temperature, following the standard formalism [3]. One interesting aspect of our formalism is that the stationary path of the angle variable of the complex scalar field becomes pure imaginary. From this Euclidean path integral, one can find the bounce solution and calculate the bubble nucleation rate. In addition, one can gain some insight into the bubble evolution.

The initial configuration we are interested in here is a homogeneous configuration which is classically stable but not quantum mechanically. The charge density of the initial configuration is nonzero and uniform. The final configuration after the phase transition, it turns out, could be more complicated than the configuration of the lowest potential energy. In our model, there could be an attractive force between charges and charges clump together forming Q balls [4]. Thus the final configuration could be inhomogeneous with Q balls floating in the symmetric phase.

The Euclidean path integral allows us to calculate the imaginary part of the energy for the metastable initial configuration by the semiclassical method. The contribution to the path integral is dominated by bounce solutions. Contrasted with the usual case [3], the initial charge density now breaks the O(4) symmetry to the O(3) symmetry. In this paper we focus on the case of small initial charge density, where the bounce solution is close to the O(4) bounce solution of zero charge. When the charge density is small, we can look at the perturbative correction to the O(4) symmetric solution. The current conservation equation in this background turns out to be a boundary value problem in the classical electrodynamics and can be solved in the thin wall limit. This leads to some insight into the current flow in the bounce solution and the deformation of the O(4) symmetric bounce solution. This in turn leads to an understanding of the bubble evolution via the analytic continuation.

When there is nonzero charge density and the initial configuration is metastable, there is always the sound wave of the speed less than the speed of light. When the bubble of a "true vacuum" is nucleated, it will expand. The bubble wall speed could reach the sound speed of the initial configuration in finite time, becoming supersonic. In addition, some bubbles speed up to the speed of light in finite time, which implies some sort of a new instability.

The plan of this paper is as follows. In Sec. II we introduce the theory of a complex scalar field. We study the stability condition of the possible initial configurations and discuss the final configurations we expect after the phase transition. In Sec. III we study the Euclidean

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path integral. We offer a formal discussion of the bubble nucleation and evolution. In Sec. IV we examine in detail various examples when the initial charge density is small and present the qualitative pictures of the bubble nucleation and evolution. In Sec. V we conclude with some remarks and questions.

II. MODEL

We study the theory of a complex scalar field $\phi = fe^{i\theta}/\sqrt{2}$ with a global U(1) symmetry. The Lagrangian is given by

$$\mathcal{L} = |\partial_\mu\phi|^2 - U(\sqrt{2}\phi) = \tfrac{1}{2}(\partial_\mu f)^2 + \tfrac{1}{2}f^2(\partial_\mu\theta)^2 - U(f). \quad (2.1)$$

The global symmetry arising from a constant shift of θ leads to the conserved current

$$J_\mu = -i(\phi^*\partial_\mu\phi - \partial_\mu\phi^*\phi) = f^2\partial_\mu\theta. \quad (2.2)$$

The total charge is $Q = \int d^3x\, f^2\dot{\theta}$. While there could be global strings in this theory, they seem not to play any essential role in our discussion and will be neglected here.

In this paper we are interested in the quantum evolution of a metastable initial configuration with nonzero charge density. To start, we should have a proper description of possible initial configurations. While some of them could be inhomogeneous and localized, here we will focus on the static homogeneous initial configurations $(f, \theta = \omega t)$ with a given charge density $\rho = f^2\dot{\theta} = f^2\omega$. The energy density for these configurations is given by

$$e(\rho, f) = \frac{\rho^2}{2f^2} + U(f). \quad (2.3)$$

The first term in the right-hand side of Eq. (2.3) is the centrifugal term due to the conserved charge. An initial configuration would lie at the local minimum of $e(\rho, f)$, to be stable under homogeneous fluctuations, and satisfies

$$\frac{\partial e}{\partial f} = -\frac{\rho^2}{f^3} + U'(f) = 0 \quad (2.4)$$

and $\partial^2 e/\partial f^2 > 0$ or

$$3\frac{\rho^2}{f^4} + U'' > 0. \quad (2.5)$$

We denote a local minimum by f_0 and $\omega_0 = \rho/f_0^2$. Figure 1 shows an example of $e(\rho, f)$ and $U(f)$ for nonzero ρ.

A metastable initial configuration should be stable, even under small inhomogeneous fluctuations, $f_0 + \delta f$ and $\theta = \omega_0 t + \delta\theta$. From the classical field equation, we get the linear equation for the fluctuations:

$$-\delta\ddot{f} + \partial_i^2\delta f + \omega_0^2\delta f + 2\omega_0 f_0\delta\dot{\theta} - U''(f_0)\delta f = 0,$$
$$-2\omega_0 f_0\delta\dot{f} - f_0^2\delta\ddot{\theta} + f_0^2\partial_i^2\delta\theta = 0. \quad (2.6)$$

With $\delta f, \delta\theta \sim e^{i(\alpha t - \mathbf{k}\cdot\mathbf{r})}$, Eq. (2.6) yields a dispersion relation

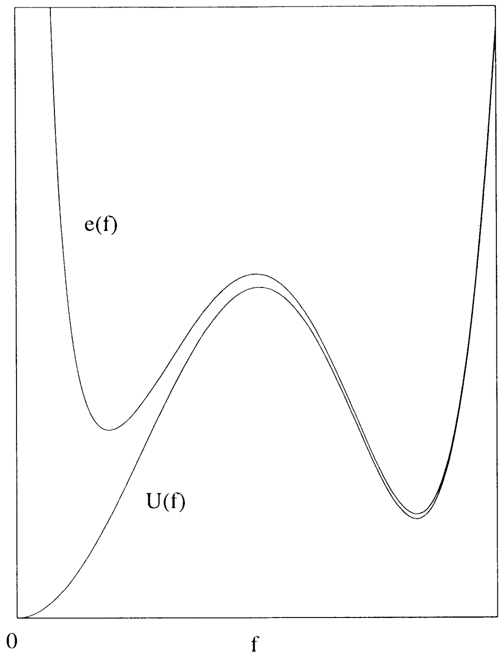

FIG. 1. Plot of $e(\rho, f)$ and $U(f)$ for a typical case.

$$[\alpha^2 - \mathbf{k}^2][\alpha^2 - \mathbf{k}^2 + \omega_0^2 - U''(f_0)] - 4\alpha^2\omega_0^2 = 0 \quad (2.7)$$

or

$$\alpha^2 = \mathbf{k}^2 + \frac{3\omega_0^2 + U''}{2} \pm \left\{\left(\frac{3\omega_0^2 + U''}{2}\right)^2 + 4\omega_0^2\mathbf{k}^2\right\}^{1/2}. \quad (2.8)$$

One can see that there are one massive mode and one massless mode. The massless mode can be interpreted as the sound wave and satisfies the dispersion relation $\alpha^2 = v_s^2\mathbf{k}^2$ for the long wavelength or for small $\mathbf{k}$, where the sound speed is given by

$$v_s^2 = \frac{f_0^4 U''(f_0) - \rho^2}{f_0^4 U''(f_0) + 3\rho^2}. \quad (2.9)$$

Thus the configuration is stable under small fluctuations only if

$$f_0^4 U''(f_0) - \rho^2 > 0, \quad (2.10)$$

which is a stronger condition than Eq. (2.5).

When the charge density is high, the centrifugal term would be balanced by the highest order term in the potential, say, $U \approx \epsilon f^n/n$. From Eqs. (2.4) and (2.9) we see that the sound speed in this limit is given as

$$v_s = \left(\frac{n-2}{n+2}\right)^{1/2}. \quad (2.11)$$

It is interesting to note that the potentials of $n = 4$ in four dimensions, $n = 6$ in three dimensions, and $n = \infty$

in two dimensions are renormalizable interactions and that the corresponding sound speeds $1/\sqrt{3}$, $1/\sqrt{2}$, and 1 are those of hot relativistic gases in the corresponding dimensions.

The stability condition (2.10) can be examined more concretely in the potential

$$U(f) = \frac{m^2}{2}f^2 + \frac{g}{4}f^4 + \frac{\lambda}{6}f^6\frac{\kappa}{8}f^8. \quad (2.12)$$

(Here we are not concerned about the renormalizability of the theory. We are interested in the general characteristics of the tunneling when global charge is involved.) Assume that the charge density is very small compared with other scales. When $m^2 > 0$, the condition (2.4) can be satisfied for f_0 near the symmetric phase:

$$f_0 = \sqrt{\rho/m} + O(\rho^2). \quad (2.13)$$

The stability condition (2.10) becomes

$$f_0^4 U''(f_0) - \rho^2 = 3g\left(\frac{\rho}{m}\right)^3 [1 + O(\rho)] > 0, \quad (2.14)$$

which is satisfied only if $g > 0$. The configuration is thus stable only if there is a short range repulsion due to the self-interaction. The velocity of the sound in the case $g > 0$ is given as

$$v_s^2 = \frac{3g\rho}{4m^3}[1 + O(\rho)], \quad (2.15)$$

which is much smaller than unity when the charge density is small.

In the case where the potential (2.12) takes a local minima at $f = v$, $U \approx \frac{1}{2}m'^2(f - v)^2$, with $m'^2 = U''(v)$. The condition (2.4) implies $\rho^2 \approx m'^2 v^3(f_0 - v)$. The stability condition (2.10) becomes

$$m'^2 v^4 + 3\rho^2 + v\rho^2 U'''(v)/m'^2 + O(\rho^3) > 0, \quad (2.16)$$

which is automatically satisfied when the charge density is small. The sound velocity becomes

$$v_s^2 = 1 - \frac{4\rho^2}{m'^2 v^4} + O(\rho^4). \quad (2.17)$$

This shows that the massless Goldstone boson becomes the sound wave as the nonzero charge density is introduced in the broken phase.

We have examined the stability condition on the possible initial configurations. If an initial configuration is metastable, it will evolve quantum mechanically so that the potential energy is converted to the kinetic energy of bubbles and the radiation energy. The important question is how we know whether an initial configuration is metastable quantum mechanically. We would say a configuration is not stable quantum mechanically if we can find a field configuration of a lower energy and the same quantum number so that there is no superselection rule preventing the transition between these two configurations.

In the case of the theory of a real scalar field, the answer comes immediately from the potential energy density [3]. The potential energy density for the final configuration would be the lowest. A metastable initial configuration can decay via nucleations of bubbles whose interior is in the true vacuum. We know well how this phase transition proceeds.

For the theory of a complex scalar field, the story is more complicated as there are two degrees of freedom which work together or against each other. If we regularize the system in a large finite box, we expect that the possible final configuration has the minimum energy for a given total charge. The excessive energy of the initial configuration would be channeled into elementary excited modes in the final configuration through the radiation and bubble collisions. In general, it is not easy to find such a final configuration. A possible final configuration could be inhomogeneous. In this paper, we concentrate on the cases where the potential $U(f)$ has a local minimum at $f = 0$ and the phase transition occurs between the $f \approx 0$ and $f \neq 0$ phases. Figure 2 shows such potentials. In addition, we assume that the initial charge density is small compared with other scales in the problem.

In these limits, we can analyze the problem in a somewhat satisfactory way. As far as the f field is concerned, this field wants to settle at the ground state of $e(\rho, f)$ in Eq. (2.3). However, we will see that the charge will move so that the energy per unit charge, $e(\rho, f)/\rho$, takes the lowest value. Since we are concerned with the phase transition between the symmetric and asymmetric phases, we choose $U(f = 0) = 0$. It turns out that there are three different cases of the phase transitions to consider: (A) The ground state of $U(f)$ is an asymmetric phase where $U(f) < 0$, and the initial configuration is in the symmet-

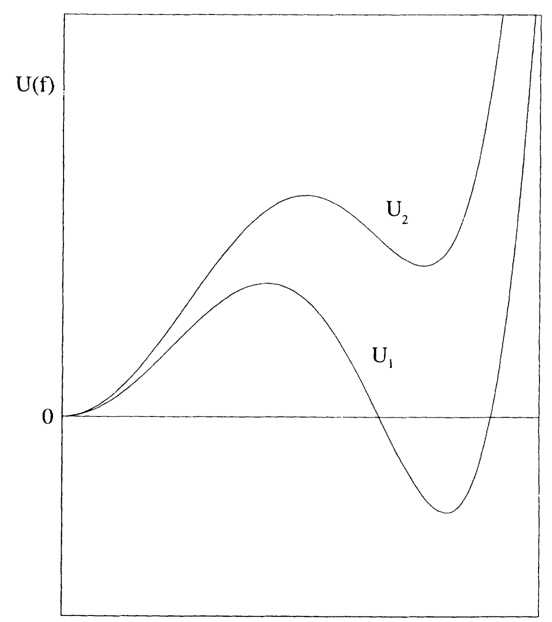

FIG. 2. Plot of two potentials $U_1(f)$ and $U_2(f)$.

ric phase; (B) the potential $U(f)$ has a local minima at an asymmetric phase where $U(f) > 0$, and the initial configuration is in the asymmetric phase; (C) the potential $U(f)$ is identical to case (B), but the initial configuration is in the symmetric phase. In Fig. 2, $U_1(f)$ corresponds to case (A) and $U_2(f)$ to cases (B) and (C).

Let us first study the cases (B) and (C) where the symmetric phase is the ground state of $U(f)$. To find out the final configuration, let us recall that in these cases there could be Q balls in the theory for an appropriate potential. Let us here recapitulate the Q-ball physics briefly [4]. In the symmetric phase, charged particles have mass m and so the ratio e/ρ would be m. For a given large charge Q, the lowest energy state, however, does not need to be made of a collection of these charged particles at rest, whose energy is $m|Q|$. Rather, that could be a Q ball if there is an enough attraction between the charged particles.

To find the condition on the potential for Q balls to exist, let us examine a homogeneous configuration whose energy per unit charge is lowest. We first minimize e/ρ with respect to ρ:

$$\frac{\partial(e/\rho)}{\partial\rho} = \frac{1}{2f^2} - \frac{1}{\rho^2}U(f) = 0. \tag{2.18}$$

The charge density is then fixed as a function of f:

$$\rho = 2f^2 U. \tag{2.19}$$

The energy per unit charge is given by

$$\frac{e}{\rho} = \left(\frac{2U}{f^2}\right)^{1/2}. \tag{2.20}$$

We have to minimize e/ρ in Eq. (2.20) with respect to the f field. Figure 3 shows $\sqrt{2U/f^2}$ for various potentials. When $\sqrt{2U/f^2}$ takes the local minimum value w_* at the nonzero $f = f_*$ field, such a configuration is called Q matter whose charge density is fixed to be $\rho_* = f_*^2\omega_*$. A Q ball is a sphere whose inside is made of Q matter and outside is just the symmetric phase. When the total charge is large, the size of a Q ball would be large, the surface energy will be negligible compared with the volume energy, and the energy per charge would be very close to ω_*. Thus, if ω_* is less than m, the Q balls are stable against decaying into charged particles. Such Q balls are possible with U_1, U_2 of Fig. 3.

For case (B) the initial configuration should be classically stable, which means $g > 0$ as shown in Eq. (2.15). U_2 in Fig. 3 represents such a potential.

When the stable Q balls are possible, the final configuration for cases (B) and (C) is inhomogeneous. There will be a region with Q matter and the rest as the symmetric phase of zero charge density. The domain walls separating two regions would slowly evolve to reduce the surface energy. The ratio of the volumes between two regions would vary depending on the initial charge density. If the initial charge density is small, Q balls will float in the symmetric vacua. If the initial charge density is large, the balls of the symmetric vacua will float in the Q

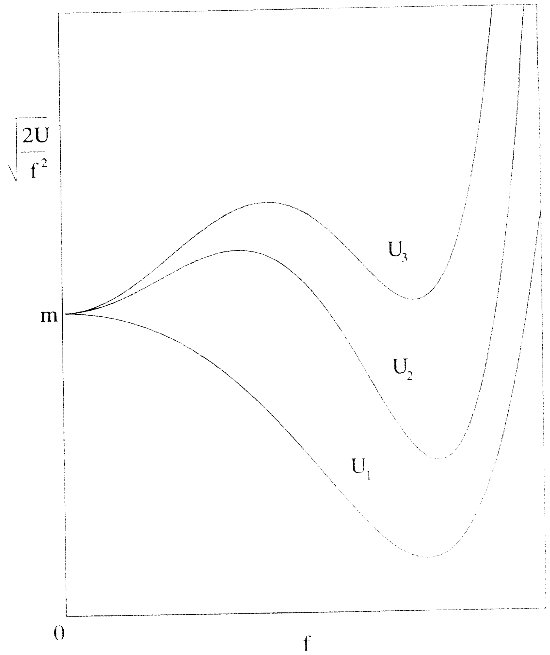

FIG. 3. Plot of $\sqrt{2U(f)/f^2}$ for three potentials U_1, U_2, U_3.

matter. In case (B) the phase transition would proceed with the nucleations of Q balls, and in case (C) it would proceed with the nucleations of bubbles with less charge density.

If Q balls are impossible because $\omega_* > m$ as in U_3 of Fig. 3, then the final configuration would be in the symmetric phase with nonzero charge density. If the initial charge density is larger than ρ_*, the final configuration could be more complicated and will not be discussed here.

To understand the condition (2.18) better, consider the pressure of a homogeneous configuration given by

$$p = \frac{\rho^2}{2f^2} - U(f). \tag{2.21}$$

We see that Eq. (2.18) is identical to the zero pressure condition. When Q balls can exist, we have argued that the final configuration could be inhomogeneous where Q balls float in the symmetric vacuum. The zero pressure condition means that there is no pressure difference between Q balls and the symmetric vacuum, leading to an equilibrium situation.

Having analyzed the last two cases where the symmetric phase is the ground state of $U(f)$, let us now consider case (A) where the asymmetric phase is the ground state of $U(f)$. For this case, both the f field and charge prefer the asymmetric phase because the energy per charge in the asymmetric phase would be negative compared with that in the symmetric phase. The final configuration would be in the asymmetric phase with uniform charge density. An interesting observation on this case has been made when the initial state is not classically stable because $g < 0$ in the potential (2.12) [5]. Note that the initial configuration of zero charge density is classically stable. When we introduce a small charge density, charges get concentrated in several regions and these regions would evolve classically into the true vacuum. For the initial configuration at the symmetric phase to be stable, there should be a repulsive force between charges ($g > 0$) at long distances.

We note that the energy per unit charge and the pressure are sensitive to the shift of the potential energy. In the above arguments, $U(0) = 0$ was used crucially because we are dealing with tunneling between the symmetric and asymmetric phases. If we attempted to understand the tunneling between two asymmetric phases, we would have needed another device to figure out the final state. We will try to investigate this example elsewhere.

III. QUANTUM TUNNELING

We have studied the general characteristics of the phase transition from a homogeneous initial configuration with nonzero charge density. We found out qualitatively what will be the final configuration after the quantum tunneling. We now want to approach this problem more analytically by using the Euclidean path integral formalism [3]. When a nonzero global charge is involved, the standard formalism should be extended to accommodate the nontrivial boundary term [2]. Here we summarize and expand the known results.

We start with the Euclidean generating functional

$$\langle F|e^{-HT}|I\rangle = \int [f\, df\, d\theta]\Psi_F^* e^{-S_E}\Psi_I, \tag{3.1}$$

where the Euclidean action is given by

$$S_E = \int d^4x \left\{ \tfrac{1}{2}(\partial_\mu f)^2 + \tfrac{1}{2}f^2(\partial_\mu\theta)^2 + U(f) \right\}. \tag{3.2}$$

The initial and final states $\Psi_{I,F}$ describe the configurations of the charge density, $\rho_{I,F}$, or

$$\Psi_{I,F}(\theta, f) \sim \exp\left\{ i\int d^3x\, \rho_{I,F}\theta \right\}. \tag{3.3}$$

In our case, the initial and final states are identical and describe the initial metastable configuration.

By calculating the imaginary correction to the energy of the metastable initial configuration, we can find out the tunneling rate or the bubble nucleation rate. When we sum over the multibounce contributions to the energy,

$$\begin{aligned} e^{-\mathcal{E}VT} &= \langle i|e^{-HT}|i\rangle \\ &= K_0 e^{-\mathcal{E}_0 VT}\left\{ 1 + VTKe^{-B} \right. \\ &\qquad \left. + \tfrac{1}{2}(VTKe^{-B})^2 + \cdots \right\} \\ &= \exp\{-(\mathcal{E}_0 - Ke^{-B})VT\}, \end{aligned} \tag{3.4}$$

where $\mathcal{E}_0$ is the energy density of the initial configuration, K_0 is the prefactor from the small fluctuations around the initial configuration, the B factor is the difference between the bounce action and the background action, $\mathcal{E}_0 VT$, and KK_0 is the prefactor arising from the small fluctuations around the bounce solution. There is a negative mode around the bounce solution which implies that the factor K is purely imaginary. The bubble nucleation rate per unit volume is then given by $2|K|e^{-B}$.

The path integral will be dominated by the stationary configurations of the path integral. In our case the boundary condition on the f field of such a stationary configuration is fixed to be the initial classical configuration. However, Eq. (3.3) implies that the boundary condition on the θ field is free. Thus the stationary configuration in the path integral (3.1) satisfies the Euler equation for the action $S_E + \Sigma$, where Σ is the boundary term,

$$\Sigma = -i\int d^3r \{\rho_F\theta(\mathbf{r}, \tau_F) - \rho_I\theta(\mathbf{r}, \tau_I)\}. \tag{3.5}$$

From the Euclidean equation from the combined action $S_E + \Sigma$, one can see that the stationary path of the angle variable θ should be purely imaginary:

$$\theta \equiv -i\eta. \tag{3.6}$$

The Euclidean field equations for the f, η fields are then

$$\begin{gathered} \partial_\mu^2 f + f(\partial_\mu\eta)^2 - U'(f) = 0, \\ \partial_\mu(f^2\partial_\mu\eta) = 0. \end{gathered} \tag{3.7}$$

The boundary condition on η becomes

$$f^2\frac{\partial\eta}{\partial\tau}(\mathbf{r}, \tau_{I,F}) = \rho_{I,F}(\mathbf{r}). \tag{3.8}$$

The f field should approach the time-independent f given by the initial configuration. We assume that there is no vortex in the initial configuration. Then the classically stable configuration for a given charge should satisfy

$$\partial_i^2 f + \frac{\rho^2}{f^3} - U'(f) = 0, \tag{3.9}$$

since $\partial_\tau f = 0$ and $\partial_i\eta$ at the boundary. In our case, the initial configuration is homogeneous in space and so Eq. (3.9) becomes identical to Eq. (2.4).

The solution of Eq. (3.7) is the so-called bounce for quantum tunneling. When the initial charge density vanishes, we know that O(4) symmetric solution of Eq. (3.7) has been shown to exist by the undershoot-overshoot method [3]. In our case, the boundary condition reduces this O(4) symmetry to the O(3) symmetry because the charge density selects a preferred time direction. Thus we are interested here in the bounce which is O(3) symmetric invariant under the spatial rotation. This makes Eq. (3.7) a partial differential equation, whose solution is much harder to find. We can either use some analytic tools or numerical analysis to find the bounce solution. In the next section, we use the perturbation method to get an approximate bounce solution when the initial charge density is very small.

Once we find the O(3) bounce solution f_b, η_b, we can calculate its action $S_E + \sigma$ from Eqs. (3.2) and (3.5). By using Eqs. (3.7) and (3.8), we can see that the combined action becomes

$$S_E + \sigma = \int d^4x \{ \tfrac{1}{2}(\partial_l f_b)^2 + \tfrac{1}{2} f_b^2 (\partial_l \eta_b)^2 + U(f_b) \}. \tag{3.10}$$

The bubble nucleation per unit volume is then given by Ke^{-B} where

$$B = (S_E + \sigma)(\text{bounce}) - (S_E + \sigma)(\text{background}). \tag{3.11}$$

While the action for the bounce $S_E + \sigma$ could be infinite, the difference B between that of the bounce and that of the initial configuration should be finite when one expects a finite tunneling rate. While we will not attempt to calculate the effect of the fluctuations around the bounce solution, we note that the field fluctuations $\delta f, \delta\theta$ should be kept real in the functional integral. This is exactly what happens in a Gaussian integral:

$$\int dx\, e^{-x^2 + ipx}.$$

To find out the escape point or the bubble configuration at the nucleation moment, we use the time translation and reflection symmetries of the action under $\tau \to -\tau$ and $\eta \to -\eta$ of $S_E + \sigma$ to choose the origin to be the center of the bounce so that

$$\frac{\partial f_b}{\partial \tau}(\mathbf{r}, \tau = 0) = 0, \qquad \eta_b(\mathbf{r}, \tau = 0) = 0. \tag{3.12}$$

As in our problem the charge density remains real in Minkowski and Euclidean times, it is natural to identify the initial charge density of the bubble to be given by that of the bounce. The initial bubble configuration is then

$$\begin{aligned} f(\mathbf{r}, t = 0) &= f_b(\mathbf{r}, \tau = 0), \\ \theta(\mathbf{r}, t = 0) &= 0, \\ \frac{\partial f}{\partial t}(\mathbf{r}, t = 0) &= 0, \\ f^2 \frac{\partial \theta}{\partial t}(\mathbf{r}, t = 0) &= f_b^2 \frac{\partial \eta_b}{\partial \tau}(\mathbf{r}, \tau = 0). \end{aligned} \tag{3.13}$$

In usual quantum tunneling, momenta are imaginary and coordinates are real under the potential barrier, and we find the escape point in the coordinate space. In our case, the charge or momentum density and the angle variable have changed their role. Under the barrier the angle variable is imaginary and the charge density is real, and we find the escape point in the charge density.

Once we know the initial bubble configuration (3.13), we can solve the field equation in Minkowski time to find out how a given bubble evolves. The bounce solution (f_b, η_b) can also by analytically continued to a solution in Minkowski time:

$$f(\mathbf{r}, t) = f_b(\mathbf{r}, it), \qquad \theta(\mathbf{r}, t) = -i\eta_b(\mathbf{r}, it). \tag{3.14}$$

A further insight about bubble nucleation with a global charge can be gained by using the dual formulation [2]. It is well known that Goldstone bosons can be described by an antisymmetric tensor field $B_{\mu\nu}$. In Minkowski time, the dual Lagrangian is given by

$$\mathcal{L}_{DM} = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{12 f^2} H^2_{\mu\nu\rho} - U(f), \tag{3.15}$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. The field strength of the antisymmetric tensor is related to the original current by

$$f^2 \partial^\mu \theta = \tfrac{1}{6} \epsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho}. \tag{3.16}$$

The uniform initial charge density becomes the condition of the uniform "magnetic" field H_{123}. In Euclidean time, there will be no boundary term arising from the wave function $\Psi(f, B_{ij})$. The Euclidean Lagrangian becomes

$$\mathcal{L}_{DE} = \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{12 f^2} H^2_{\mu\nu\rho} + U(f). \tag{3.17}$$

The bounce equation becomes

$$\partial_\mu^2 f + \frac{1}{6 f^3} H^2_{\mu\nu\rho} - U'(f) = 0, \qquad \partial^\mu \left(\frac{1}{f^2} H_{\mu\nu\rho} \right) = 0. \tag{3.18}$$

We can relate the Euclidean fields either through the Euclidean time dual transformation or by comparing the current. The relation between the antisymmetric tensor field and the angle variable in Euclidean time is given by

$$f^2 \partial^\mu \eta = \tfrac{1}{6} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}. \tag{3.19}$$

Thus the bounce solution in terms of the $B_{\mu\nu}$ field would be real and there would be no contribution to the bounce action from the boundary.

IV. EXAMPLES

We are now in position to examine in more detail the three cases of the phase transition which we have discussed in Sec. II. To be more specific, we choose the potential to be given by Eq. (2.12). Case (A) has the initial configuration in the symmetric phase. The potential energy has the absolute minimum at the asymmetric phase. For cases (B) and (C) the potential energy $U(f)$ has the absolute minimum at the symmetric phase and the local minimum at the asymmetric phase. The initial configuration of case (B) is at the asymmetric phase, and the initial configuration of case (C) is at the symmetric phase. In cases (A) and (B) the tunneling would proceed even when there is no initial charge density because the initial configurations are at the metastable points of $U(f)$. Introducing a small amount of charge density would not change much of the original bubble nucleation. Thus we would expand perturbatively the bounce solution by the initial charge density and see how the zeroth order O(4) symmetric solution deforms. These are the cases we will examine closely in this section. For case (C) we do not have the zeroth order bounce solution because the quantum tunneling occurs solely due to the charge density. However, we can still get some insight into this case as we will see later.

Let us consider first the cases where the initial configuration is unstable even without any charge density. The symmetric phase of the potential U_1 and the asymmetric phase of the potential U_2 in Fig. 2 are such initial configurations. When there is no initial charge density, the bounce solution can be obtained by the O(4) symmetric ansatz $\tilde{f}(s \equiv \sqrt{\mathbf{r}^2+\tau^2})$ [3]. Let us assume that the thin wall approximation works. We call that inside the wall $f = f_i$ and outside the wall $f = f_e$. The wall radius a can be determined as follows. Suppose that the potential energy difference $\Delta U = U(f_e) - U(f_i) > 0$ between the f_e and f_i phases is small. Then the bubble radius will be large and we can approximate the bubble wall as a domain wall separating two phases. This wall satisfies the equation, $\partial_x^2 f + U'(f) = 0$, neglecting the potential energy difference. Define the tension of the wall to be the action density per unit three-volume, $T = \int dx[(df/dx)^2/2 + U(f)]$. The gain of the action due to this true vacuum bubble of the radius a is then

$$S(a) = 2\pi^2 T a^3 - \frac{\pi^2}{2}\Delta U a^4. \quad (4.1)$$

At $a = 3T/\Delta U$, $S(a)$ takes the maximum value $27\pi^2 T^4/2(\Delta U)^3$, which is the B factor in the bubble nucleation rate.

We ask what happens to this O(4) symmetric thin wall bounce solution if we introduce a small initial charge density. From Eq. (3.7), we see that the equation of the angle variable is first order in the charge density and the f field equation has a second order correction to the bounce equation of the zero charge density. Thus we can solve Eq. (3.7) by a perturbative expansion around this O(4) symmetric background. The phase variable will be first order in the charge density and satisfies the current conservation

$$\partial_l(\tilde{f}^2(s)\partial_l\eta) = 0, \quad (4.2)$$

with the boundary condition $f_e^2\eta(\mathbf{r},\tau=\pm\infty) = \rho_0$ with the initial charge density ρ_0. The above equation can be interpreted as a boundary problem of a dielectric media in four space dimensions. The electric field is $\partial_\mu\eta$ with the potential η, and the electric displacement is $J_\mu = f^2\partial_\mu\eta$ with the dielectric constant f^2. Equation (4.2) implies that the boundary condition at the wall that the normal component of J_μ and the tangential component of $\partial_\mu\eta$ should be continuous. The boundary condition at infinity is that there is a constant external electric displacement field $J_\tau = \rho_0$. For a given O(4) symmetric configuration described before, it is trivial to find the potential $\eta_{e,i}$ outside and inside the thin wall:

$$\eta_e = \frac{\rho_0}{f_e^2}\left(1 + \frac{f_e^2 - f_i^2}{3f_e^2 + f_i^2}\frac{a^4}{s^4}\right)\tau,$$
$$\eta_i = \frac{\rho_0}{f_e^2}\left(\frac{4f_e^2}{3f_e^2 + f_i^2}\right)\tau. \quad (4.3)$$

The charge density J_τ at $\tau = 0$ would be the charge profile of the bubble at the moment of nucleation as shown in Eq. (3.13). Let us now examine the implications of this solution (4.3) in various cases. The correction to the f field would be second order and will be considered in each case.

A. Case (A): from the symmetric phase to the asymmetric phase

Let us first consider the case when the asymmetric phase is the ground state and the initial configuration is near the symmetric phase. Thus $f_e \sim 0 \ll f_i \sim v$. Since there is no initial charge density, the previous argument would imply that $f_e = 0$. When we introduce a uniform charge density in the initial configuration, the initial value of f would be given by Eq. (2.13) with $\theta = mt$, invalidating our assumption $\tilde{f} = \vartheta(1)$. Here let us assume simply that f_e is nonzero even when there is no charge, say, due to a small bump in the potential at $f = 0$. This would not change the physics of the tunneling under consideration much and allows us to use Eq. (4.3).

The global current J_μ around the O(4) symmetric bounce solution can be obtained from Eq. (4.3). Outside the thin wall ($\zeta > a$), $J_\mu = f_e^2\partial_\mu\eta_e$, and inside the thin wall ($\zeta < a$), $J_\mu = f_2^i\partial_\mu\eta_i$. The energy per charge inside the wall is small, and so the charge is attracted to the interior region, making the charge density inside the bubble higher than that outside. The charge density profile of the bubble at the moment of nucleation would be given by J_τ at $\tau = 0$:

$$J^0_{\text{exterior}} = \left(1 - \frac{a^4}{r^4}\right)\rho_0,$$
$$J^0_{\text{interior}} = 4\rho_0, \quad (4.4)$$

when $f_i \gg f_e$. The charge density inside the bubble is 4 times larger than the initial charge density. From Eq. (4.4), we can see that $\int d^3x(J^0 - \rho_0) = 0$, implying that the charge inside the bubble came from the region near the bubble wall. Figure 4 shows this global current on the O(4) symmetric solution background.

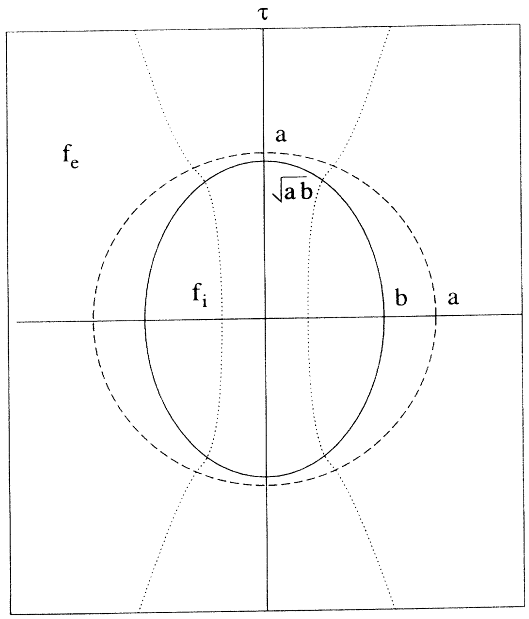

FIG. 4. Bounce solution for case (A). The dashed circle is the wall of the O(4) symmetric bounce solution. The dotted lines indicate the charge flow. The solid ellipse is the wall of the deformed bounce.

Let us now consider the effect of the charge flow on the f field. The f field equation (3.7) can be expanded around the O(4) symmetric solution as

$$\partial_\mu^2 \delta f - U''(f)\delta f = -\frac{J_\mu^2}{\tilde{f}^3}, \qquad (4.5)$$

where J_μ^2 can be obtained from Eq. (4.3) and $\tilde{f}$ would be given by the O(4) symmetric solution. Rather than try to solve this partial differential equation, let us approach the problem more qualitatively. When $f_e \ll f_i$, one can show that at the north and south poles $J_\mu^2 \sim 16\rho_0^2$ both inside and outside and that at the equator $J_\mu^2 \sim 0$ outside and $J_\mu^2 \sim 16\rho_0^2$ inside. The centrifugal term $J_\mu^2/2f^2$ would be then important at the poles, but not at the equator. Directly from the f equation (3.7) and the previous argument about the thin wall approximation, we see that the centrifugal term reduces the tension on the domain wall by reducing the effective potential energy barrier. Since the tensions at the two poles are lower while the tensor at the equator remains constant, the curvature at the poles would be larger than $1/a$ and that at the equator will remain $1/a$ where $a = 3T/\Delta U$ is the radius of the O(4) symmetric shell. (In addition, ΔU is increased at the pole and remains unchanged at the equator, amplifying the curvature change between the poles and equator.) Consequently, the O(4) symmetric wall would be shrunk at the equator. We take a liberty in choosing this wall configuration to be an ellipsoid:

$$\frac{r^2}{b^2} + \frac{\tau^2}{ab} = 1, \qquad (4.6)$$

where $b < a$ takes a complicated function of ρ_0. Figure 4 shows the deformed bounce wall.

Let us now think about the bubble nucleation and evolution. The bubble of a true vacuum will nucleate with a radius b and a higher charge density $4\rho_0$ and then expand. As argued in Eq. (3.14), we can analytically continue the bounce solution to find out how the bubble will evolve. From Eq. (4.6), we see that the bubble wall trajectory will be given by $R(t) = (\sqrt{b/a})\sqrt{t^2 + ab}$. Since $b < a$, the terminal velocity $v_t = \sqrt{b/a}$ is less than 1.

How do we understand the finite terminal velocity? Energy conservation implies that the change of the wall energy comes from the potential energy difference:

$$d\left(\frac{4\pi T(R)R^2}{\sqrt{1-\dot{R}^2}}\right) = d\left(\frac{4\pi}{3}\Delta U R^3\right). \qquad (4.7)$$

The tension of the bubble wall surface could depend on R. Integrating Eq. (4.7), we get

$$\dot{R} = \left(1 - \frac{3T(R)}{\Delta U R}\right)^{1/2}. \qquad (4.8)$$

When there is no charge density, T, ΔU are fixed and we see that the terminal velocity is the light speed. If the tension grows linearly with the radius for large R, $T(R) \sim \alpha R$, the terminal velocity $v_t = \sqrt{b/a}$ will be

$$v_t = \left(1 - \frac{3\alpha}{\Delta U}\right)^{1/2} < 1. \qquad (4.9)$$

The growth of tension or the energy density of wall per unit area can be understood by considering the phase variable θ. While the charge density $f_i^2\dot{\theta}$ inside the bubble is larger than the charge density $f_e^2\dot{\theta}$ outside, it is not large enough to keep the phase variable space independent since $f_i \gg f_e$. The phase increases by mt outside the bubble and more slowly inside the bubble, leading to an increase of its space gradient at the bubble wall. This is what we think is the source of the increasing tension or energy density at the bubble wall.

Let us now remind ourselves that the sound speed (2.15) in the symmetric phase is much smaller than the terminal speed (4.9) when the initial charge density is small. Thus the bubble wall forms a sort of supersonic front in the symmetric phase. However, our analysis is not accurate enough to compare the terminal speed and the sound speed (2.17) at the inside asymmetric phase. (When the initial charge density is large enough, there is a possibility that the terminal speed is less than the sound speed in the symmetric phase.) The thin wall approximation would fail eventually, because there is not enough charge lying outside the expanding bubble to keep the charge density inside the bubble to be 4 times larger than the initial value. The charge density profile around the expanding bubble should become more smoothly changing.

B. Case (B): from the asymmetric phase to the symmetric phase

Let us now consider the case where the initial state is the broken phase which has the higher potential energy than that of the symmetric phase. When there is no initial charge density, there will be an O(4) symmetric bounce solution, inside which the scalar field takes a value near the symmetric phase, $f_i \ll f_e$. Again, we ask what is the consequence of the small initial charge density. Equation (4.3) implies how the current flows around this bounce solution. The charge density at the moment of bubble nucleation would be given by

$$J^0_{\text{exterior}} = \left(1 + \frac{a^4}{3r^4}\right)\rho_0, \qquad J^0_{\text{interior}} \approx 0, \tag{4.10}$$

when $f_i \ll f_e$. The charge is excluded from the symmetric vacuum region. Figure 5 shows the charge flow around the O(4) symmetric bounce solution.

The effect of the charge flow on the f field is given by the centrifugal term $J_\mu^2/2f^2$ as in Eq. (4.5). We can calculate J_μ^2 for our bounce solution. Since $f_e \gg f_i$, Eq. (4.3) implies that at the north and south poles $J_\mu^2 \approx 0$ both inside and outside and that at the equator $J_\mu^2 \approx 0$ inside and $J_\mu^2 \approx 16\rho_0^2/9$ outside. This raises the energy density of the false vacuum at the equator, lowering the barrier energy and increasing ΔU. This in turn lowers the tension of wall at the equator. The tension at the poles would remain unchanged and so the bounce solution would be elongated at the equator. Now for the sake of the argument, we again approximate the wall as an ellipsoid:

$$\frac{r^2}{ab} + \frac{\tau^2}{b^2} = 1, \tag{4.11}$$

where $b < a$ is a complicated function of ρ_0. The deformed bounce solution is shown in Fig. 5.

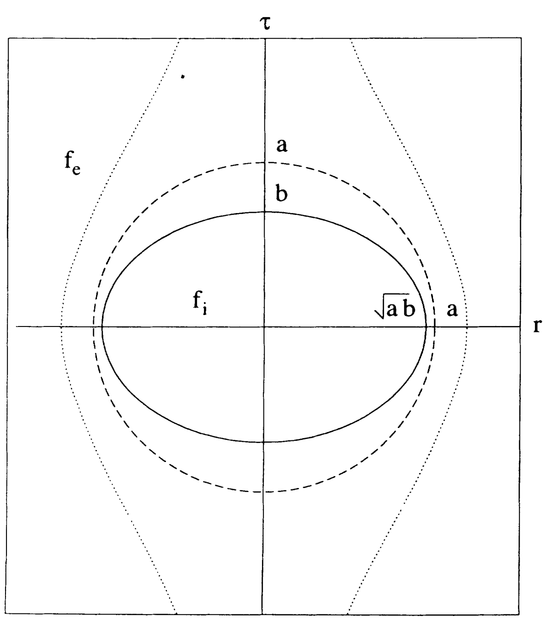

FIG. 5. Bounce solution for case (B). The dashed circle is the wall of the O(4) symmetric bounce solution. The dotted lines indicate the charge flow. The solid ellipse is the wall of the deformed bounce.

In Minkowski time, the bubble wall trajectory is given by $R(t) = \sqrt{a/b}\sqrt{t^2 + b^2}$, with the terminal velocity $v_t = \sqrt{a/b} > 1$, which clearly violates the causality. Something should happen before the wall speed becomes the light speed. When there is a tachyonic mode, we say there is an unstable or exponentially growing mode. There are many possibilities. Since the charge is pushed out from the bubble and is accumulated at the wall, the wall could stop expanding. Or the thin wall approximation could break down before the wall reaches the light speed.

In case (B) the charge is pushed way from the bubbles and accumulated at the initial asymmetric phase. At the end of the phase transition, we would be left with the islands of the original phase with a high charge density, which are exactly Q balls floating in the symmetric phase.

C. Case (C): forming Q balls by quantum tunneling

If the minimum of $\sqrt{2U/f^2}$ is lower than the mass of the charged particles in the symmetric phase, the charge tends to clump into Q balls. Suppose the initial configuration lies at the symmetric phase with a very small charge density and is stable under local fluctuations. Since the minimum of the potential is chosen to be the symmetric phase, the initial configuration would be stable if there is no charge density. After a small uniform charge density is introduced, the initial configuration, however, becomes unstable under the quantum mechanical tunneling transition to form Q balls. Since we start from the minimum of the potential, we do not have the bounce solution at zero charge density. However, we can still gain some understanding of the general features of the bounce solution from what we learned in case (A).

First, the bubble at the moment of nucleation would be a Q ball of the minimum size, where the surface energy is as important as the volume energy. The charge density ρ_* inside the Q ball would be much larger than the initial charge density ρ_0. The minimum size of a Q ball could be rather large if ω_* is very close to m so that the energy gain by the charge is small and a lot of charge is needed to compensate the surface energy. We know that a large current will flow into the bounce wall from outside in this case because the interior charge density $\rho_i = f_i^2\omega_*^2$ is much larger than ρ_0. With a similar argument given for case (A), the current would be large inside and outside the wall at the poles and would be zero outside and large inside the equator. The energy density outside the bounce wall at the poles is larger than that

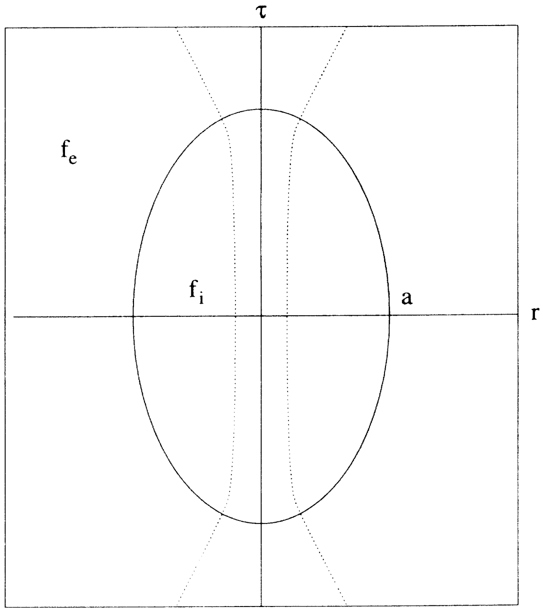

FIG. 6. Bounce solution for case (C). The dotted lines indicate the charge flow. The solid ellipse is the wall of the bounce for Q-ball nucleation.

outside the bounce wall at the equator. The barrier energy at the poles would in turn be lower than that at the equator, and so the wall tension at the poles will be lower than that at the equator. Thus the bounce solution in case (C) would also be elongated along the τ direction as in case (A). Figure 6 shows such a bounce solution for Q-ball nucleation.

Once a Q ball is nucleated, it will grow very slowly. The reason is that a Q ball can grow only when it absorbs the charge from outside and that there is not much charge around it because the formation itself has already diluted the initial charge density around its neighborhood. This is consistent with the picture that the bounce solution is elongated along the τ direction, which also implies a slow terminal velocity as argued after Eq. (4.9). The explicit nature of the Q-ball nucleation and expansion will however, require a better analysis and will not be attempted here.

V. DISCUSSION

We have studied the phase transitions in the theory of a complex scalar field with a global U(1) symmetry when there is nonzero initial charge density. We have discussed the metastability condition on the possible initial configurations and the possible inhomogeneous final configurations. We argued that there are many cases of the phase transitions to be studied in the theory. We have set the Euclidean formalism of the bubble nucleation when there is nonzero charge density. We applied our formalism to the case when the initial charge density is small and when the phase transitions involve the symmetric phase as the initial configuration or a part of the final configuration. Here we studied the characteristics of the bounce solutions and the bubble evolution. Our system is shown to have a rich variety of possible phase transitions and could be a good simple toy model of the phase transition involving charges, the supersonic bubble wall, and the sound wave.

However, there are still many loose ends and questions we have not attempted here. One of the interesting questions seems to concern the later development of the bubbles. Depending on the cases of the phase transitions, there is a possibilty of rich dynamics. Additional interesting questions to be explored are concerned with the phase transition between the asymmetric phases and the phase transition when the initial charge density is not small. Finally, we note that it is rather straightforward to extend our formalism in Sec. III to the case involving nonzero local gauge charges.

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