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Strings in plane wave backgrounds reexamined

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String theory in an exact plane wave background is explored. The four-tachyon scattering amplitude is constructed. The spectrum of states found from the poles in the factorization turns out to be equivalent to that of the theory in flat space-time. The massless vertex operator is obtained from the residue of the first order pole. It exhibits nontrivial modifications with respect to the flat space case.

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I. INTRODUCTION

As a fundamental generalization of Einstein gravity and Yang-Mills theory, the concepts behind string theory remain quite mysterious. Although nonperturbative effects are being slowly discovered, at the moment we only have a perturbative formulation of a yet unknown full quantum theory. In the present framework, gravity appears at different levels. The string spectrum in flat Minkowski space-time contains gravitons whose interactions determine the classical backgrounds consistent with the string dynamics. The appearance of a nontrivial curved geometry as an infinite genus effect was realized by Amati and Klimcik [1]. They showed the coincidence between the summation of string loop diagrams of high energy graviton scattering in flat space-time performed in [2] with the S matrix relating in and out free excitations of a string in a shock wave background metric. The interplay between the different hierarchies assumed by gravity in these schemes is still unclear.

In the absence of an adequate understanding of the string picture, a classification of the specific symmetries is a useful tool in the search for the definite theory. The possible geometries compatible with compactification to four dimensions provide information about these symmetries (for instance, the classification of acceptable string vacua has made manifest the existence of mirror symmetry) and allow identification of phenomenologically viable models. A deeper insight has been gained by formulating the 10- or 26-dimensional theory in background fields. Duality symmetry, first discovered in toroidal compactification [3], was generalized to any nonlinear σ model with an isometry in [4] and is considered a potential string answer to the occurrence of space-time singularities and the problem of the cosmological constant in general relativity [5]. Target space duality provides novel features in string theory as compared to point particle theory. The impossibility of performing experiments permitting one to distinguish the geometry in which the string is living, a nonsingular Planckian region in string cosmology [6], and a plausible mechanism for the selection of four macroscopic space-time dimensions, suggested in [7], are among the most distinctive characteristics.

Many interesting string backgrounds have been identified starting from the Wess-Zumino-Witten (WZW) model. The requirement that the theory be conformal invariant at the quantum level amounts to the vanishing of the β functions of the couplings of the nonlinear σ model. WZW models are exactly solvable conformal field theory (CFT) and thus exact solutions to these equations. Gauging a one-dimensional subgroup of $SL(2,R)$ Witten [8] found an exact two-dimensional (2D) black hole. The 3D black hole background was recently obtained in Ref. [5]. The discrete symmetries referred to above allow one to identify mathematically equivalent vacua with radically different geometries. Even though not conclusive, the existence of candidates for dual 2D and 3D black hole backgrounds without curvature singularities [9–12] as well as the equivalence between string propagation on $SL(2)/U(1)$ with appropriate twists [13] (corresponding to a geometrical background with the same singularity structure as 4D black holes), and on a nonsingular circle of radius 1, are indications of the possible ways in which string theory could cure the singularities of Einstein's gravity. Even when the physical and geometrical meanings of these stringy features are not completely understood, they undoubtedly open new paths in the comprehension of the string theory of gravity.

The computation of scattering amplitudes is the main physical task in string theory which is basically an S-matrix theory. Particles of various masses and spins are exchanged in the different channels of a scattering process. They appear as poles in the square momentum when the points where some of the external vertices are attached coincide. However, when the theory is formulated in background fields other divergences could appear in physical operators [14–16] and in the partition function [17]. Furthermore, the spectrum of string states can be modified as compared with the flat space theory [17]. The interpretation of these features is still unclear and probably has to await the emergence of a nonperturbative formulation but is essential to properly understand the predictions of string theory for the structure of space-time at the Planck scale. Indeed, the perturbative expansion might be performed around the incorrect vacuum.

Thus, our aim in this paper is to further explore the consequences of string theory in nontrivial backgrounds. We consider the bosonic string in exact plane wave metrics. These metrics are solutions to all orders of the conformal invariance conditions of the bosonic string theory (the β functions of the nonlinear σ model) in an α' expansion and even nonperturbatively [18]. String theory in gravitational waves has been extensively discussed in the literature [1, 18, 14]. A monochromatic plane wave background was recently constructed by Nappi and Witten [19] from an ungauged WZW model based on a central extension of the 2D Poincaré algebra. The conformal field theory description of such algebra and its cosets was considered in Ref. [20]. Plane waves are interesting because the existence of a covariantly constant null vector leads to a definition of frequency which is conserved, and therefore there is no analogue of the particle creation mechanism of semiclassical theories in curved space-times. This is not true in a general time-dependent background where "string creation" should be considered in the context of second quantization.

The paper is organized as follows. In Sec. II we introduce the notation and review the previous relevant literature. We present new exact results for the string mass operator in certain shapes of the gravitational wave. In Sec. III we construct the scattering amplitude of tachyons in this background. Physical amplitudes provide further evidence on the structure of the theory, the spectrum of states, and its symmetries in nontrivial backgrounds. They are thus other elements to be considered in this search for the string theory of gravity. The spectrum of states obtained from the poles of the amplitude upon factorization is shown to be equivalent to that of the flat space-time theory. From Taylor expanding this residue the vertex operators of the corresponding exchanged states can be extracted. Indeed, the residue corresponds to the scattering amplitude of the intermediate higher mass states with the remaining tachyons. The factorization and Taylor expansion are performed in Sec. IV where the massless vertex operator is obtained. Final conclusions and a discussion of the results are contained in Sec. V.

Similar issues have been recently addressed in Ref. [20] from the current algebra giving rise to certain gravitational waves different from those considered here. We comment on the relation of these results with ours in Sec. V.

II. STRINGS IN PLANE WAVE BACKGROUNDS

In order to introduce the notation and to be self-contained, this section reviews previous relevant literature and at the end we present new results.

The so-called plane-fronted waves in D dimensions are given by

$$ds^2 = -dUdV + dX^a dX_a + F(U, X^a)dU^2, \tag{1}$$

with $U = T - X^D$, $V = T + X^D$, and X^a, $a = 1, 2, ..., D-2$, transverse coordinates.

Gravitational plane waves are a particular case of (1) in which F is quadratic in X^a, i.e., $F(U, X^a) = W_{ab}(U)X^a X^b$, and the only nonvanishing component of the Ricci tensor is $R_{UU} = -W^a_a$. In the so-called exact plane waves, W_{ab} is a traceless matrix and thus the metric is Ricci flat. For later convenience we restrict attention to a function W_{ab} nonvanishing only in a finite range of U, i.e., $W_{ab}(U) \neq 0$ for $0 \leq U \leq U_0$. In this way asymptotic "in" and "out" string states can be defined in the flat space-time regions $U < 0$ and $U > U_0$.

The action of the bosonic string moving in a plane fronted wave background is

$$S = \frac{1}{2}\int d^2\sigma[-\partial_\alpha U \partial^\alpha V + \partial_\alpha X^a \partial^\alpha X_a + F(U, X^a)\partial_\alpha U \partial^\alpha U] \tag{2}$$

(with $\alpha' = 1/2\pi$).

The classical equations of motion for the transverse fields X^a are

$$\partial_\alpha \partial^\alpha X_a + \frac{1}{2}\partial_a F P^2 = 0, \tag{3}$$

where the light cone gauge $U = P\tau$ was chosen. Transverse coordinates solving (3) automatically satisfy the constraint equations $T_{ab} = 0$.

Without loss of generality we assume the background space to be a tensor product of a four-dimensional plane wave with a $c = 22$ conformal field theory. In the case of exact plane waves with $W_{xx}(U) = -W_{yy}(U) = W_0 =$ const for $0 \leq U \leq U_0$, decomposing the two transverse coordinates into modes as

$$X(\sigma, \tau) = \sum_n X_n(\tau)e^{in\sigma}, \tag{4}$$

$$Y(\sigma, \tau) = \sum_n Y_n(\tau)e^{in\sigma}, \tag{5}$$

Eq. (3) decouples and reads, for each coordinate,

$$\ddot{X}_n + n^2 X_n - W_0 P^2 X_n = 0, \tag{6}$$

$$\ddot{Y}_n + n^2 Y_n + W_0 P^2 Y_n = 0. \tag{7}$$

In terms of right and left oscillators, each mode can be written as

$$X_n = \frac{i}{2\sqrt{n}}\left(a^x_n u_n - \tilde{a}^{x\dagger}_n \tilde{u}_n\right), \tag{8}$$

where $u_n(\tilde{u}_n)$ are solutions of Eq. (6) for $U < 0$, i.e., "in" modes of the form $e^{-in\tau}(e^{in\tau})$. Similarly, "out" modes $v_n(\tilde{v}_n)$ can be used for $U > U_0$ with the corresponding $b^x_n, \tilde{b}^x_n$ oscillators.

A linear Bogoliubov transformation relates "in" and "out" oscillators:

$$b^x_n = A_n a^x_n - B^*_n \tilde{a}^{x\dagger}_n, \tag{9}$$

$$\tilde{b}^x_n = A_n \tilde{a}^x_n - B^*_n a^{x\dagger}_n. \tag{10}$$

(The oscillators for Y_n are obtained from those for X_n changing $W_0 \to -W_0$.)

The expectation value of the "out" number operator of the nth right and left modes of a string that was initially in the ground state can be found from these transformations as

$$\langle 0_{\rm in}|N_n^{x\ \ {\rm out}}|0_{\rm in}\rangle = \langle 0_{\rm in}|b_n^{x\dagger} b_n^x|0_{\rm in}\rangle = |B_n^x|^2. \tag{11}$$

This can be used to compute the expectation value of the "out" mass-squared operator in the "in" region. Since

$$M_{\rm out}^2 = 4\sum_{n=1}^{\infty} n(b_n^{a\dagger} b_n^a + \tilde{b}_n^{a\dagger}\tilde{b}_n^a) - 8, \tag{12}$$

then

$$\langle M_{\rm out}^2\rangle = 4\sum_{n,a} n\langle N_n^a\rangle - 8. \tag{13}$$

The absence in string theory of singularity theorems such as those of Penrose and Hawking [21] led Horowitz and Steif [14] to suggest the convergence of this series as a criterion to decide whether a solution is singular or not in the sense of string theory. It was shown that a string trying to propagate through certain singular plane-fronted waves becomes infinitely excited. Shock wave metrics, i.e., $F(U) = \delta(U)f(X^a)$, were considered in [14, 22–24]. It was found that the string mass-squared operator diverges for certain functions $f(X^a)$ whereas it is finite in some particular cases. Other divergent profiles of the form $W(U)_{U\to 0} \sim |U|^{-\beta}$ were shown in [16] to lead to a divergent $\langle M^2\rangle$ when $1 \le \beta < 2$. The origin of the divergences in the mass operator was assigned in [16] to the infinite transverse size of the wave. However, as will be shown in the forthcoming Eqs. (15) and (20), the presence of a singular profile $W(U)$ is necessary in addition in order to get a divergent $\langle M^2\rangle$. Another example of singular string state was found in [15] by considering an antisymmetric tensor field background, i.e., adding a discontinuous axion field. These conclusions neglect back reaction effects which could drastically modify them.

Since a careful analysis is needed for divergent profiles [23, 24] we consider functions $W(U)$ parametrized in such a way that the limit $W(U) \to \delta(U)$ can be unambigously taken. The Bogoliubov coefficient B_n can be exactly computed for the profile considered above [namely, $W(U) = W_0$ for $0 \le U \le U_0$ and $W(U) = 0$ otherwise]. It turns out to be

$$|B_n^x|^2 = \left(\frac{1}{2}\frac{P^2 W_0}{nn_-}\sin(n_- U_0)\right)^2, \tag{14}$$

with $n_\pm = \sqrt{n^2 \pm W_0 P^2}$. Note that the WKB approximation performed in [15] cannot be used in this case. Therefore

$$\langle M^2\rangle = 4\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{1}{2}\frac{P^2 W_0}{n_-}\sin(n_- U_0)\right)^2 + \left(\frac{1}{2}\frac{P^2 W_0}{n_+}\sin(n_+ U_0)\right)^2\right] - 8. \tag{15}$$

Each of the oscillatory modes of the string gets excited as it passes through the gravitational wave but the expectation value of the mass-squared operator remains finite; i.e., the sum in (15) is convergent for W_0 finite. Some interesting limits can be taken. When $U_0 \to 0$, $\langle M^2\rangle \to -8$, which is the tachyon mass, the state of the string before the gravitational wave reached it. If W_0 increases while U_0 decreases ($W_0 \to \infty, U_0 \to 0$), keeping $W_0 U_0 = 1$, then the profile $W(U)$ tends to a δ function $[W(U) \to \delta(U)]$ and the expectation value of the number operator yields

$$\langle N_n^{\rm out}\rangle \to \frac{1}{2}\left(\frac{P^2}{n}\right)^2. \tag{16}$$

Therefore the mass-squared operator $\langle M_{\rm out}^2\rangle = 2P^4\sum_{n=1}^{\infty}\frac{1}{n} - 8$ diverges and the string gets infinitely excited. In this case it is necessary to consider back reaction effects before extracting any definite conclusions.

Let us now consider a more complicated (continuum) profile $W(U) = \frac{W_0}{\cosh^2(\frac{\alpha U}{P})}$. In this case the classical equations of motion (3) read

$$\ddot{X}_n - \frac{W_0 P^2}{\cosh^2(\alpha\tau)}X_n = -n^2 X_n, \tag{17}$$

and this can be solved exactly in terms of a hypergeometric function $F(a, b, c; z)$:

$$X_n(\tau) = \left(1-\xi^2\right)^{-\frac{in}{2\alpha}} F\left(-\frac{in}{\alpha} - s, -\frac{in}{\alpha} + s + 1, -\frac{in}{\alpha} + 1; \frac{1-\xi}{2}\right), \tag{18}$$

where $\xi = \tanh(\alpha\tau)$ and $s = \frac{1}{2}\left(-1 + \sqrt{1 - \frac{4W_0 P^2}{\alpha^2}}\right)$.

The asymptotic expansion of (18) for $\tau \to \infty$ is

$$X_n(\tau) \to \frac{\Gamma(\frac{in}{\alpha})\Gamma(1-\frac{in}{\alpha})}{\Gamma(-s)\Gamma(1+s)}e^{in\tau} + \frac{\Gamma(-\frac{in}{\alpha})\Gamma(1-\frac{in}{\alpha})}{\Gamma(-\frac{in}{\alpha}-s)\Gamma(-\frac{in}{\alpha}+s+1)}e^{-in\tau}. \tag{19}$$

The Bogoliubov coefficient B_n can be easily read from the expression above since it is the coefficient of the negative frequency solution at late times $e^{in\tau}$, i.e.,

$$|B_n^x|^2 = \left|\frac{\Gamma(\frac{in}{\alpha})\Gamma(1-\frac{in}{\alpha})}{\Gamma(-s)\Gamma(1+s)}\right|^2 = \left(\frac{\cos\left(\frac{\pi}{2}\sqrt{1-\frac{4W_0P^2}{\alpha^2}}\right)}{\sinh(\frac{\pi n}{\alpha})}\right)^2, \tag{20}$$

and $|B_n^y|$ is the same expression changing $W_0 \to -W_0$. In this case the sum in $\langle M^2 \rangle$ is always convergent for W_0 and α finite. Taking $W_0 \to \infty, \alpha \to \infty$ while keeping $W_0 = \frac{\alpha}{2}$, then $W(U) \to \delta(U)$ and we recover the result of Eq. (16). This provides an example of a singularity in the sense of string theory as defined in [14, 15] whereas space-time is nonsingular in the sense of general relativity (the geodesics are complete though discontinous). If the same limits are taken but now keeping $\frac{4W_0P^2}{\alpha^2} = 1$, the expectation values of both the number and square mass operators become divergent. In this case there is an infinite discontinuity in the geodesics. Notice that this is a quantum effect. Classically the masses of the states remain finite since the oscillators are decoupled [Eqs. (6), (7)].

As advanced above Eqs. (15) and (20) show that a finite $\langle M^2 \rangle$ is obtained even in the presence of an infinite transverse size of the wave as long as its height is finite. These results indicate that there is no conclusive evidence to assign a unique origin and/or definition to string singularities. In any case, since the essential issue is the resolution given by string theory, as a candidate to provide a full theory of quantum gravity, to the problem of the singularities inherent in general relativity, the specific symmetries should be further explored. With this goal, in the next section we consider the scattering of tachyons in plane wave metrics.

III. TACHYON SCATTERING AND STRING SPECTRUM

We now consider the scattering of tachyons in this plane wave metrics.

The set of coordinates used in the preceding section, i.e., (U, V, X^a), called harmonic coordinates, is physically convenient since a single chart can be used to cover the whole plane wave space-time and the curvature depends only on one component of the metric tensor. However, in order to fully display the symmetries of the geometry and for computational convenience, the so-called "group coordinates" are more suitable. In these coordinates the metric takes the form

$$ds^2 = -dudv + g_{ab}(u)dx^a dx^b. \tag{21}$$

Both sets of coordinates are related by

$$U = u,$$
$$V = v + \frac{1}{2}\dot{g}_{ab}(u)x^a x^b, \tag{22}$$
$$X^a = P^a_b(u)x^b,$$

where $g_{ab}(u) = P^c_a(u)P^c_b(u)$ and the matrix $P^a_b(u)$ is determined by

$$\ddot{P}^a_b = W_{ac}P^c_b, \tag{23}$$

which must be solved with the initial conditions

$$\dot{P}^c_a P^c_b - \dot{P}^c_b P^c_a = 0. \tag{24}$$

A possible solution is

$$P^1_1(u) = p_1(u) = e^{\sqrt{W_0}u},$$
$$P^2_2(u) = p_2(u) = e^{i\sqrt{W_0}u}, \tag{25}$$
$$P^1_2(u) = P^2_1(u) = 0.$$

The constraint (24) is automatically satisfied by the exact plane waves. Note that with this choice the metric and coordinates are complex and thus unphysical. Therefore we will consider physical operators those expressed in harmonic coordinates.

In order to compute scattering amplitudes of the lowest energy states in this metric, the vertex operator responsible for the emission of a tachyon is needed. As is well known these vertices must be conformal operators of anomalous dimension 2 and the conditions they must satisfy in an arbitrary curved background were given by Callan and Gan [25]. To first order in an α' expansion, the tachyon vertex $\mathcal{V}_T$ must be a solution of a Klein-Gordon equation

$$\alpha' \Delta \mathcal{V}_T^{\mathrm{PW}}(k, u, v, x^a) = k^2 \mathcal{V}_T^{\mathrm{PW}}(k, u, v, x^a), \tag{26}$$

where we take Δ as the Laplacian in the plane wave (PW) metric (21) with $k^2 = -m^2 = 8$ the mass of the tachyon.

Garriga and Verdaguer [26] found a solution that in this metric reads

$$\mathcal{V}_T^{\mathrm{PW}}(k, u, v, x^a) =: \exp\left[i\left(k_a x^a - k_- v - \frac{1}{4k_-}\int_0^u du(g^{ab}k_a k_b + m^2) + \frac{i}{2}\sqrt{W_0}(1+i)u\right)\right] :, \tag{27}$$

where k_a, k_- are the separation constants and play the role of components of the momentum of the tachyon [27]. It reduces to the usual vertex operator in flat space when the limit $W_0 \to 0$ is taken:

$$\mathcal{V}_T^{\mathrm{PW}} \to: \exp\left[i\left(k_a x^a - k_- v - k_+ u\right)\right] : \ . \tag{28}$$

It is useful to define a shifted momentum in the direction of u as

$$\tilde{k}_+(u) = \frac{1}{4k_-}\left[g^{ab}(u)k_a k_b + m^2\right] - \frac{i}{2}\sqrt{W_0}(1+i). \tag{29}$$

The four-tachyon scattering amplitude on the sphere is defined through

$$\mathcal{A}_{4T} = \int \prod_{i=1}^4 d^2 z_i \int \mathcal{D}u\mathcal{D}v\mathcal{D}x^a |J| \prod_{i=1}^4 \mathcal{V}_T^{\mathrm{PW}}(k_i, z_i) \times e^{iS[u,v,x]}, \tag{30}$$

where the vertex $\mathcal{V}_T^{\rm PW}(k_i, z_i)$ is placed at the point z_i of the complex plane; $|J|$ is the determinant of the Jacobian of the transformation between the harmonic and group coordinates (it is included because the physical vertices are assumed to be those in harmonic coordinates, but we shall see that it does not contribute to the amplitude due to momentum conservation), and $S[u,v,x]$ is the action expressed in group coordinates:

$$S[u,v,x^a] = \frac{1}{2}\int d^2\sigma\left[-\partial_\alpha u\partial^\alpha v + g_{ab}(u)\partial_\alpha x^a \partial^\alpha x^b\right]. \tag{31}$$

It is possible to simplify the calculations by making a Lorentz transformation so that all $k_-^{(i)}$ vanish, except two of them. We then consider

$$k_-^{(i)} = 0, \quad i = 2,3,$$
$$k_-^{(i)} \neq 0, \quad i = 1,4.$$

Of course this transformation is not a symmetry of the metric. However, this loss of generality is irrelevant for our purposes, as we shall see. With this choice the vertices $\mathcal{V}_2^{\rm PW}$ and $\mathcal{V}_3^{\rm PW}$ have the same form as the Minkowski space vertices (with $k_- = 0$), because Eq. (26) only depends on the transverse coordinates. This independence of v means that these two tachyons travel in the same direction as the gravitational wave and never collide with it:

$$\mathcal{V}_{2,3} = e^{-ik_+u + ik_a x^a}. \tag{32}$$

Putting everything together in the functional integral, the amplitude $\mathcal{A}_{4T}$ in Eq. (30) reads

$$\mathcal{A}_{4T} = \int \mathcal{D}u\mathcal{D}v\mathcal{D}x^a\, |J|\, \exp\left(i\left[\sum_{i=1}^{4} k_a^{(i)}x^a(z_i) - \sum_{i=1,4} k_-^{(i)}v(z_i) - \sum_{i=1}^{4}\int_0^u du\, \tilde{k}_+^{(i)}[u(z_i)] - \sum_{i=2,3} k_+^{(i)}u(z_i)\right]\right) e^{iS[u,v,x]}. \tag{33}$$

The Jacobian $J(\frac{\partial X^\mu}{\partial x^\nu})$ can be easily evaluated because it depends only on u; then

$$\det J = \prod_{z,\bar{z}} p_1(u)p_2(u) = \exp\left(\sqrt{W_0}(1+i)\int d^2z\, u(z)\right). \tag{34}$$

Now integrating the action by parts and collecting all the pieces under the same integral symbol in the exponential by introducing the delta functions $\delta^2(z - z_i)$, $\mathcal{A}_{4T}$ can be expressed as

$$\mathcal{A}_{4T} = \int \prod_{i=1}^{4} d^2 z_i \int \mathcal{D}u\mathcal{D}v\mathcal{D}x^a e^{\tilde{S}[u,v,x^a,k_a^{(i)},z_i]}, \tag{35}$$

where

$$\tilde{S} = \int d^2z\left[v\left(\frac{1}{2}\partial_\alpha\partial^\alpha u - \sum_{i=1,4} k_-^{(i)}\delta^2(z-z_i)\right) - \sum_{i=2,3} k_+^{(i)}u(z_i)\delta^2(z-z_i)\right.$$
$$\left. + \int_0^{u(z)} du\left(\sqrt{W_0}(1+i) - \sum_{i=1,4}\tilde{k}_+^{(i)}(u)\delta^2(z-z_i)\right) + x^a\left(-\frac{1}{2}\partial_\alpha g_{ab}\partial^\alpha\right)x^b + \sum_{i=1}^{4} k_a^{(i)}x^a\delta^2(z-z_i)\right]. \tag{36}$$

Because of the linear dependence, the integral on v can be evaluated, giving the δ function

$$\delta\left[\frac{1}{2}\partial_\alpha\partial^\alpha u - \sum_{i=1,4} k_-^{(i)}\delta^2(z-z_i)\right], \tag{37}$$

where $\frac{1}{2}\partial_\alpha\partial^\alpha = 2\partial_z\partial_{\bar{z}}$. Integrating u therefore simply implies replacing it by

$$\bar{u}(z) = \sum_{i=1,4} k_-^{(i)} \ln|z - z_i| \tag{38}$$

and inserting the factor $\det^{-1}(\partial_\alpha\partial^\alpha)$, irrelevant for our calculations.

We are then left with

$$\mathcal{A}_{4T} \sim \int \prod_{i=1}^{4} d^2 z_i \, \exp\left(\left(\int d^2 z \int_0^{\bar{u}(z)} du \left[- \sum_{i=1,4} \tilde{k}_+^{(i)} \delta^2(z - z_i) - \sum_{i=2,3} k_+^{(i)} \delta^2 (z - z_i) \right] \right) \right.$$
$$\times \int \mathcal{D}x^a \exp \left(\left(\int d^2 z \left[x^a \left(-\frac{1}{2} \partial_\alpha g_{ab} \partial^\alpha \right) x^b + \sum_{i=i}^{4} k_a^{(i)} x^a(z) \delta^2(z - z_i) \right] \right) \right). \tag{39}$$

Note that the contribution from the Jacobian vanishes due to momentum conservation. Finally, one can integrate over x^a since the action is quadratic in those fields. This yields, for the amplitude,

$$\mathcal{A}_{4T} \sim \int \prod_{i=1}^{4} d^2 z_i \, \exp \left[-k_+^{(2)} \bar{u}(z_2) - k_+^{(3)} \bar{u}(z_3) - \int_0^{\bar{u}(z_1)} du \, \tilde{k}_+^{(1)} - \int_0^{\bar{u}(z_4)} du \, \tilde{k}_+^{(4)} \right]$$
$$\times \exp \left[\frac{1}{4} \int d^2 z \, d^2 z' \sum_{i=1}^{4} k_a^{(i)} \delta^2(z - z_i) A^{ab}(z, z') \sum_{j=i}^{4} k_b^{(j)} \delta^2(z' - z_j) \right], \tag{40}$$

where $A^{ab}(z,z')$ is the Green function of the quadratic operator $-\frac{1}{2}\partial_\alpha g_{ab} \partial^\alpha$: i.e.,

$$-\frac{1}{2} \partial_\alpha g_{ab}[\bar{u}(z)] \partial^\alpha A^{ab}(z, z') = \delta^2(z, z'). \tag{41}$$

Using expression (21) for $\bar{u}(z)$, we may write

$$\mathcal{A}_{4T} \sim \int \prod_{i=1}^{4} d^2 z_i \, |z_2 - z_1|^{-k_+^{(2)} k_-^{(1)}} |z_2 - z_4|^{-k_+^{(2)} k_-^{(4)}} |z_3 - z_1|^{-k_+^{(3)} k_-^{(1)}} |z_3 - z_4|^{-k_+^{(3)} k_-^{(4)}}$$
$$\times \exp \left[- \int_0^{\bar{u}(z_1)} du \, \tilde{k}_+^{(1)} - \int_0^{\bar{u}(z_4)} du \, \tilde{k}_+^{(4)} \right] \exp \left[\frac{1}{2} \sum_{i<j} k_a^{(i)} k_b^{(j)} A^{ab}(z_i, z_j) \right]. \tag{42}$$

Since the vertices are normal ordered, self-contractions are not considered. Note that the shift in $\tilde{k}_+^{(1)}$ and $\tilde{k}_+^{(4)}$ can be dropped in this case because of momentum conservation in k_- (i.e., $[\bar{u}(z_1) + \bar{u}(z_4)] = 0$); then we can replace $\int_0^{\bar{u}} du \, \tilde{k}_+^{(i)} \to \int_0^{\bar{u}} du \, k_+^{(i)}$. Therefore nontrivial interactions take place in the transverse space.

The next step is to compute the Green function (41) writing the operator $-\frac{1}{2}\partial_\alpha g_{ab} \partial^\alpha$ in the form

$$\Delta(z) = -\frac{1}{2} \partial_\alpha g_{ab} \partial^\alpha = -p^2(\Delta_0 + \delta\Delta), \tag{43}$$

where

$$\Delta_0 = \frac{1}{2} \partial_\alpha \partial^\alpha = 2\partial_z \bar{\partial}_{\bar{z}}, \tag{44}$$

$$\delta\Delta = \partial_z \ln p^2 \bar{\partial}_{\bar{z}} + \bar{\partial}_{\bar{z}} \ln p^2 \partial_z, \tag{45}$$

and p is either $p_1(\bar{u})$ or $p_2(\bar{u})$. Note that $\delta\Delta$ is proportional to $\sqrt{W_0}$, and so if we consider $\delta\Delta$ as a perturbation to the Minkowskian Green function Δ_0, formally,

$$\Delta(z) = -p^2 \Delta_0 \left[1 + \int d^2 \omega \Delta_0^{-1}(z - \omega) \delta\Delta(\omega) \right], \tag{46}$$

$$\Delta_0^{-1}(z - \omega) = -2 \ln |z - \omega|. \tag{47}$$

Then, to first order in $\sqrt{W_0}$,

$$\Delta^{-1}(z, z') \approx -\Delta_0^{-1}(z - z') p^{-2}(z') + p^{-2}(z') \int d^2 \omega \Delta_0^{-1}(z - \omega) \delta\Delta(\omega) \Delta_0^{-1}(\omega - z'). \tag{48}$$

Therefore $A^{ab}(z,z') \approx \Delta^{-1}(z,z')$, with $p^2 = p_1^2(\bar{u})$ for $a = b = 1$ and $p^2 = p_2^2(\bar{u})$ for $a = b = 2$. Of course in the limit $W_0 \to 0$ we recover the Minkowskian Green function.

IV. FACTORIZATION AND GRAVITON VERTEX OPERATOR

We now discuss the factorization of this amplitude ($\mathcal{A}_{4T}$) when two of the external vertices are placed at the same point. In this way the mass spectrum of the theory can be found from the physical poles corresponding to the particles interchanged in the process. The residues give rise to the scattering amplitude of the intermediate state with the remaining tachyons. From them, applying the procedure introduced in Ref. [28], the vertex operators of the exchanged states can be read.

An N-tachyon amplitude $\mathcal{A}_N(k^{(1)}, ..., k^{(N)})$, factorizes when r vertices collide to the same point, i.e., taking the limit $z_i \to z_r$ for $i = 1, ..., r-1$, as

$$\mathcal{A}_N \longrightarrow \frac{1}{\frac{1}{4}K^2-2}\mathcal{A}_{r+1}(k^{(1)},...,k^{(r)},-K)\mathcal{A}_{N-r+1}(K,k^{(r+1)},...,k^{(N)}), \tag{49}$$

where $K_\mu = \sum_{i=1}^{r} k_\mu^{(i)}$ is the momentum of the intermediate state, and the pole occurs at the physical mass $m^2 = -8$; i.e., the particle exchanged is a tachyon. Taylor expanding the residue, new poles corresponding to higher mass intermediate particles are found.

In fact, given a function $\Phi(\epsilon,\bar{\epsilon})$, regular for $\epsilon,\bar{\epsilon}\to 0$, a Taylor expansion leads to

$$\int d^2\epsilon|\epsilon|^{-2\nu}\Phi(\epsilon,\bar{\epsilon}) = \int d^2\epsilon|\epsilon|^{-2\nu}\sum_{l,m}\frac{\epsilon^l\bar{\epsilon}^m}{l!m!}\partial^l\bar{\partial}^m\Phi|_0 \;\; \epsilon^l\;\bar{\epsilon}^m = \sum_n \frac{\Lambda^{-2\nu+2n+2}}{-2\nu+2n+2}\frac{1}{(n!)^2}(\partial\bar{\partial})^n\Phi|_0, \tag{50}$$

where angular integration in polar coordinates implies $l=m=n$ and Λ is an arbitrary cutoff, irrelevant for $\nu\to n+1$.

Taking the limit $z_1\to z_2, (\epsilon = z_1 - z_2)$ from the amplitude (42), the momenta of the colliding tachyons are $k_-^{(1)}\neq 0$ and $k_-^{(2)} = 0$ so that the momentum of the intermediate state K_μ is general, i.e., $k_-^{(1)} + k_-^{(2)} \neq 0$. It is possible to isolate the divergent part of the amplitude (replacing z_1 by $z_2+\epsilon$), as

$$\begin{aligned}\mathcal{A}_{4T} \sim \prod_{i=2}^{4}\int d^2 z_i \int d^2\epsilon|\epsilon|^{\frac{1}{2}k^{(2)}\cdot k^{(1)}}|z_{23}+\epsilon|^{-k_+^{(3)}\cdot k_-^{(1)}}|z_{24}|^{-k_+^{(2)}\cdot k_-^{(4)}}|z_{34}|^{-k_+^{(3)}\cdot k_-^{(4)}}\\ \times\exp\left(-\int_0^{\bar{u}(z_2+\epsilon;\epsilon)} du\;\tilde{k}_+^{(1)}(u) - \int_0^{\bar{u}(z_4;\epsilon)} du\;\tilde{k}_+^{(4)}\right)\\ \times\exp\left(\frac{1}{2}\sum_{j>2}k_a^{(1)}k_b^{(j)}A^{ab}(z_2+\epsilon;z_j;\epsilon) + \frac{1}{2}\sum_{2<i<j}k_a^{(i)}k_b^{(j)}A^{ab}(z_i,z_j;\epsilon)\right),\end{aligned} \tag{51}$$

and therefore identify $\nu = -\frac{1}{4}k^{(1)}\cdot k^{(2)}$. Integrating ϵ after Taylor expanding the regular part, the poles at $\nu = n+1$ correspond to $K^2 = (k^{(1)}+k^{(2)})^2 = -8(n-1)$; i.e., the interaction with the gravitational wave does not change the mass spectrum of the theory with respect to the Minkowskian case. This need not be true in the presence of a nontrivial dilaton background. As shown in Ref. [17] in this case the graviton acquires a (tachyonic) mass proportional to the derivative of the dilaton.

We now turn to analyze the residues. For $n=0$ we find the tachyon pole $[k^{(1)}\cdot k^{(2)} = -4$ or equivalently $(k^{(1)}+k^{(2)})^2 = -m^2 = 8]$ and the residue corresponds to the product of two three-tachyon amplitudes [one of them is already divided by the (infinite) volume of the conformal group, leaving only a constant as a result].

For $n=1$, i.e., the massless pole $K^2=0$, the derivatives $\partial_\epsilon\bar{\partial}_\epsilon\Phi(\epsilon,\bar{\epsilon})$ lead to a residue corresponding to the scattering of a massless vertex with two tachyons. We find

$$\begin{aligned}\mathcal{A}_{GTT}^{\rm PW} = \Bigg|\frac{1}{2}\frac{k_+^{(3)}k_-^{(4)}}{z_{23}} + \frac{1}{2}\frac{\tilde{k}_+^{(1)}[\bar{u}(z_2)]\,k_-^{(4)}}{z_{24}} + \frac{1}{2}\frac{\tilde{k}_+^{(4)}[\bar{u}(z_4)]\,k_-^{(1)}}{z_{24}} + \frac{1}{2}\sum_{i=3}^{4}k_a^{(1)}k_b^{(i)}\left[\partial_{z_2}A^{ab}(z_i,z_2)+\partial_{z_2}A^{ab}(z_2,z_i)\right]\\ -\frac{1}{2}\sqrt{W_0}k_-^{(1)}k_a^{(3)}k_b^{(4)}\left[\frac{A^{ab}(z_3,z_4)}{z_{24}}+\frac{A^{ab}(z_4,z_3)}{z_{23}}\right]\Bigg|^2\mathcal{A}_{3T}(K,k^{(3)},k^{(4)}),\end{aligned} \tag{52}$$

where $\mathcal{A}_{3T}$ is the three-tachyon amplitude for states of momentum $K_\mu = k_\mu^{(1)}+k_\mu^{(2)}$, $k_\mu^{(3)}$, and $k_\mu^{(4)}$ $(z_{ij} = z_i - z_j)$.

Note that the shifts in $\tilde{k}_+^{(1)}$ and $\tilde{k}_+^{(4)}$ $[\tilde{k}_+ = k_+ - \frac{i}{2}\sqrt{W_0}(1+i)]$ decouple from the physical processes if momentum conservation $k_-^{(1)} + k_-^{(4)} = 0$ is used [recall Eq. (42)]. However, this is a consequence of the particular choice $k_-^{(2)} = k_-^{(3)} = 0$ we made, and in general this will not happen. The Minkowskian limit $\mathcal{A}_{GTT}^{(M)}$ is recovered when $W_0\to 0$.

Since the information on the plane wave background is contained in the contribution from the transverse coordinates and in the shift in k_+, when the transverse momentum of the particles vanishes, $\mathcal{A}_{GTT}^{\rm PW}$ reduces to $\mathcal{A}_{GTT}^{(M)}$ if conservation of k_- is used. This means that the particles collide normally to the gravitational wave and they do not "feel" the background. Nontrivial interactions take place in the transverse space. On the other hand if all the k_- components vanish, again $\mathcal{A}_{GTT}^{(M)}$ is recovered.

From this amplitude it is possible to obtain the massless vertex operator. A generic operator of naive conformal dimension 2 on a flat world sheet is of the form $\partial_\alpha X^\mu\partial^\alpha X^\nu F_{\mu\nu}(X)$. Translation invariance implies an exponential of the same form as the tachyon vertices. With these considerations we can read from Eq. (52) the operator responsible for the emission or absorption of a massless state as

$$\mathcal{V}_G^{\rm PW} = \tilde{\epsilon}_{\mu\nu}(k) : \partial X^\mu \bar{\partial} X^\nu \exp\left(-iK_- v + iK_i X^i - \frac{1}{4K_-}\int_0^u du \tilde{K}_+(u)\right) :, \tag{53}$$

with $X^\mu = (u, v, X^1, X^2)$, $\tilde{\epsilon}_{\mu\nu} = \tilde{\epsilon}_\mu \tilde{\epsilon}_\nu$, $\tilde{\epsilon}_\mu = \epsilon_\mu + \frac{i}{2}\sqrt{W_0}(1+i)\delta^u_\mu$, and $\epsilon_\mu = k^{(1)}_\mu$.

Indeed by computing the scattering amplitude of this vertex operator with two tachyons it can be checked that $\mathcal{A}^{\rm PW}_{GTT}$ in Eq. (52) is recovered. (Recall that $\mathcal{A}^{\rm PW}_{GTT}$ was obtained by factorizing $\mathcal{A}^{\rm PW}_{4T}$.) The polarization tensors obtained in this way are of course particular ones. They depend only on $k^{(1)}_\mu$ because of the particular vertices that were made to coincide and the way the limit $z_1 \to z_2$ was taken. However, once the conditions to be satisfied by the polarization tensors are imposed, namely, transversality $K^\mu \epsilon_{\mu\nu} = 0$, the form of the vertex operator is completely general. Notice that the polarization tensor can be decomposed into a traceless part (graviton) and a trace part (dilaton). The antisymmetric tensor cannot be produced in this way since it does not couple to tachyons.

Callan and Gan [25] deduced the conditions that a massless vertex must satisfy in order to be an eigenoperator with eigenvalue 2 of the anomalous dimension matrix in a general σ-model background. They define a general operator of naive dimension 2 as

$$V = F_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + \alpha' \, {}^{(2)}R F(X), \tag{54}$$

where the second term is unavoidable when studying string theory on a world sheet of curvature ${}^{(2)}R$. Setting the dilaton to zero on a flat world sheet the equations to be satisfied by the massless vertex operators are

$$\begin{aligned} &-\nabla^2 F_{\mu\nu} - \nabla_\mu \nabla_\nu F^\lambda_\lambda + R^{\lambda\sigma}_\mu{}_\nu F_{\lambda\sigma} + 2\nabla_\mu \nabla^\lambda F_{\lambda\nu} + 2\nabla_\nu \nabla^\lambda F_{\lambda\mu} + \nabla_\mu \nabla_\nu F = 0, \\ &\frac{1}{4}\nabla^2 F^\lambda_\lambda - \frac{1}{4}\nabla^\lambda \nabla^\sigma F_{\lambda\sigma} + \frac{1}{4}R^{\lambda\sigma}F_{\lambda\sigma} - \nabla^2 F = 0. \end{aligned} \tag{55}$$

Using the Ricci flat metric (21) whose nonvanishing Christoffel symbols and Riemann tensor reduce to

$$\Gamma^{\,a}_{\,bu} = \frac{1}{2}g^{ac}\dot{g}_{cb} \;, \quad \Gamma^v_{ab} = \dot{g}_{ab}, \tag{56}$$

$$R^{\,a}_{\;ubu} = -\partial_u \Gamma^a_{bu} - \Gamma^a_{cu}\Gamma^c_{bu}, \tag{57}$$

these equations are satisfied by

$$F_{\mu\nu}(x) = \tilde{\epsilon}_{\mu\nu}\exp\left(-iK_- v + iK_a x^a - \frac{i}{4K_-}\int_0^u du \, \tilde{K}_+\right), \tag{58}$$

with $\tilde{\epsilon}_{\mu\nu}$ given by Eq. (53). This can be verified using that $K^2 = 0$ and the transversality condition $K_\mu \epsilon^{\mu\nu} = 0$. Notice that even though the shift in $\tilde{k}^{(1)}_+$ can be eliminated from the physical amplitude (52) using momentum conservation, it is unavoidable in order to satisfy Eqs. (55). The function $F(x)$ is a tachyonlike operator with momentum K_μ, and so $\nabla^2 F = K^2 F = 0$. Similarly $F_\mu{}^\mu = \tilde{\epsilon}_\mu{}^\mu = \left(k^{(1)}\right)^2 + 2ik_-\sqrt{W_0}(1+i) = -m^2 + 2ik_-\sqrt{W_0}(1+i)$, m being the tachyon mass. Then $\nabla^2 F_\mu{}^\mu = [-m^2 + 2ik_-\sqrt{W_0}(1+i)]K^2 F = 0$.

V. CONCLUSIONS

We considered string theory in plane wave backgrounds. Profiles of the gravitational wave where the Bogoliubov transformation can be exactly solved were found. The mass-squared operator remains convergent as long as the "height" of the wave is finite. The limit of a δ-like wave was unambigously taken, leading to a divergent mass-squared operator.

Tachyon scattering amplitudes were constructed and solved to first order in a perturbative expansion in $\sqrt{W_0}$. The information about the gravitational wave is contained both in the transverse coordinates and in the shift in $\tilde{k}_+$. Two particular situations can be distinguished: (a) If all $k^{(i)}_- = 0$, i.e., none of the tachyons collide with the wave, then the interaction reduces to the flat space case; (b) if all transverse momenta vanish, i.e., the particles collide normally to the wave, again the interaction reduces to the Minkowskian case. This is completely different to what happens to classical and quantum point particles which converge at the same point at the same time; i.e., the transverse distance among geodesics becomes zero and their z coordinates are the same for a normal collision [26]. A similar observation was made in Ref. [20] where the current algebra giving rise to this class of backgrounds was analyzed and two kinds of irreducible representations were found: those corresponding to particles having momentum in the transverse directions and those having only longitudinal momentum. However, the gravitational wave obtained from the central extension of the 2D Poincaré algebra [19] is different from the one we have considered here.

From the poles appearing in the factorization of this amplitude the mass spectrum of the theory was found to coincide with the flat space case. The vertex operator of the massless states was "read" from the residue of the massless pole corresponding to the graviton-tachyon scattering amplitude. The effect of the wave on the vertex is to shift the k_+ momentum component by a factor $\frac{i}{2}\sqrt{W_0}(1+i)$. However, this shift does not amount to a shift in the mass spectrum of strings in this background.

The vertices responsible for the emission or absorption of higher mass particles can be constructed in the same way from higher orders of the Taylor expansion. We believe that the difference with their flat space analogues will be contained in the shift in $\tilde{k}_+$ and in the exponential "tachyonic" part.

The procedure used here to obtain the higher mass vertex operators can be generalized to arbitrary Riemann surfaces [28]. The advantage of the formalism is mainly that normal ordering and self-contractions arise naturally from the physical amplitudes. Arbitrary regularizations and possible sources of Weyl anomalies are thus avoided.

Klimcik and Tseytlin [29] analyzed the action of duality transformations on plane wave metrics. They found that the corresponding dual σ-model target spaces belong to the same class of metrics. Duality may connect curved backgrounds with flat metrics, but in general nontrivial antisymmetric tensor and dilaton fields are necessary. It would be interesting to consider the action of these duality transformations on the vertex operators.

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