Finite black hole entropy and string theory

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An accelerating observer sees a thermal bath of radiation at the Hawking temperature which is proportional to the acceleration. Also, in string theory there is a Hagedorn temperature beyond which one cannot go without an infinite amount of energy. Several authors have shown that in the context of Hawking radiation a limiting temperature for string theory leads to a limiting acceleration, which for a black hole implies a minimum distance from the horizon for an observer to remain stationary. We argue that this effectively introduces a cutoff in Rindler space or the Schwarzschild geometry inside of which accelerations would exceed this maximum value. Furthermore, this natural cutoff in turn allows one to define a finite entropy for Rindler space or a black hole as all divergences were occurring on the horizon. In all cases if a particular relationship exists between Newton's constant and the string tension then the entropy of the string modes agrees with the Bekenstein-Hawking formula.

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I. INTRODUCTION

Usually entropy in physics can be described both in a thermodynamic sense and in terms of a counting of states. However, although black hole entropy has long been formulated in a thermodynamic sense involving the Hawking temperature, only recently has it been possible to approach black hole entropy as a counting of states. In [1] 't Hooft calculated the number of particle states surrounding a black hole in a "brick wall model," where particles are not allowed to be closer than a certain cutoff distance to the horizon. He found a contribution to the entropy proportional to the area of the horizon but divergent as the cutoff distance was taken to zero. A different approach was taken by Bombelli *et al.* [2] and Srednicki [3], who traced over particle states inside the sphere of the horizon and also found a divergent entropy proportional to the area. Callan and Wilczek [4] and Kabat and Strassler [5] showed that the brick wall model of 't Hooft and the geometric model of [2,3] were in fact equivalent. The divergence in the entropy arises because of an infinite number of states which appear on the horizon itself and occurs whenever the brick wall cutoff is removed. Susskind and Uglum [6] calculated the density of states of the Schwarzschild geometry in the limit of infinite mass, which was essentially equivalent to Rindler space, the spacetime seen by an accelerating observer. The canonical particle entropy was again divergent, but they identified a contribution in string theory consisting of open strings with the ends attached to the horizon that, owing to the different ultraviolet properties of string theory, could in principle yield a finite black hole entropy [7–9].

In this paper we pursue an alternate route to a finite entropy in superstring theory. We show that the brick wall cutoff used by 't Hooft to obtain a finite entropy has a natural interpretation in terms of a string theory's maximum acceleration. This is fundamentally a string theory phenomenon related to the existence of a Hagedorn temperature [10,11], the limiting temperature in string theory. Essentially an accelerating observer sees thermal radiation at the Hawking temperature $T = a/2\pi$ with a the acceleration [12]. In string theory, however, there is a limiting temperature, the Hagedorn temperature; this suggests that for Hawking radiation there is a limiting Hawking temperature and maximum acceleration. Sakai [13] has studied this phenomenon by calculating the thermal response function and vacuum stress in Rindler space and finds a limiting Hawking temperature related to the Hagedorn temperature by $T_{\text{Hawking-Max}} = T_{\text{Hagedorn}}/\pi$. Parentani and Potting [14] find the same relation in terms of a thermal Green's function approach (see, in addition, Bowick and Giddings [15]). This limiting acceleration also applies to stationary observers outside a black hole and yields minimum distance from the horizon for an observer to remain stationary. In this paper we find that the structure of Rindler space or the Schwarzschild geometry must be suitably altered close to the horizon, where accelerations would exceed the maximum value and infinite vacuum stress would be present. We then argue that this effectively introduces a cutoff in Rindler space or the Schwarzschild geometry and yields a finite entropy in string theory.

II. MAXIMUM ACCELERATION IN STRING THEORY

The essence of the Hawking effect is that an observer at constant acceleration a will feel the existence of a heat bath at temperature $T = a/2\pi$. It is also well known that in string theory there exists a maximum temperature T_{Hagedorn} above which the string partition function diverges. Essentially this is because the number of string states of given mass is $\rho(m) = m_0^{n_s - 1} m^{-n_s} \exp(bm)$ and the partition function involves multiplying by $\exp(-\beta m)$

and summing over m, where $\beta = 1/T$. Thus the critical temperature is given by $T_{\rm Hagedorn} = 1/b$. One might expect that because the Hawking radiation is thermal, string theories possess a limiting Hawking temperature corresponding to $1/b$. Indeed Sakai [13] and also Parentani and Potting [14] showed that there is maximum Hawking temperature; however, the limiting value turns out to be $T_{\rm Hawking\text{-}Max} = 1/b\pi$ so that the limiting acceleration is simply $a_{\rm max} = 2/b$. For the mass degeneracy $\rho(m) = m_0^{n_s-1}m^{-n_s}\exp(bm)$, the quantities n_s and b are determined by the spectrum of various string theories [16–18]. One has for n noncompact dimensions $n_s = \frac{n+1}{2}$ for open strings, $n_s = n$ for closed strings, b is given by $4\pi\sqrt{\alpha'}$ for bosonic strings, $2\sqrt{2}\pi\sqrt{\alpha'}$ for superstrings, and $(2+\sqrt{2})\pi\sqrt{\alpha'}$ for heterotic superstrings while m_0 is of the order $1/\sqrt{\alpha'}$.

Before discussing the derivation of this maximum acceleration let us quickly review the structure of Rindler space [19], the spacetime as seen from an accelerated observer, in order to set notation useful later. Solving the differential equation

$$\frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\frac{dx}{dt}\right) = a \tag{2.1}$$

with $v = dx/dt$ and $x_\perp = 0$, we obtain the spacetime trajectory of a particle with proper acceleration a, initial position and velocity x_i, v_i at time t_i given by

$$x = x_i + \frac{1}{a}\left\{\left[1+\left(a(t-t_i)+\frac{v_i}{\sqrt{1-v_i^2}}\right)^2\right]^{1/2} - \left(1+\frac{v_i^2}{1-v_i^2}\right)^{1/2}\right\}. \tag{2.2}$$

Now if the relations $t_i = \frac{1}{a}\frac{v_i}{\sqrt{1-v_i^2}}$ and $x_i = \frac{1}{a}(1+\frac{v_i^2}{1-v_i^2})^{1/2}$ hold for the initial conditions, the spacetime trajectory simplifies dramatically to

$$x = \left(t^2+\frac{1}{a^2}\right)^{1/2}. \tag{2.3}$$

This is the trajectory of a Rindler observer. Note that the distance of closest approach to the origin $x_{\rm min} = 1/a$ is smaller for larger accelerations, whereas one might expect large accelerations to cause a turnaround further out. This happens because the initial conditions of a Rindler trajectory are such that highly accelerated observers start closer to the origin. The Rindler coordinates are defined by $x = s\cosh\tau, t = s\sinh\tau$, and cover the right-hand wedge of Minkowski space $x > |t|$. Then from (2.3) we have the relation

$$s = 1/a, \tag{2.4}$$

so that large accelerations correspond to small s close to the horizon $x = |t|$. In terms of Rindler coordinates the flat spacetime metric takes the form

$$d\ell^2 = -s^2d\tau^2 + ds^2 + dx_\perp dx_\perp.$$

As described by Susskind and Uglum, the Schwarzschild metric

$$d\ell^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega^2$$

of a very large black hole can also be described by the Rindler coordinates with the transformation $\tau = t/4GM, s = \sqrt{8GM(r-2GM)}$, and the area of the horizon taken to be $A = 4\pi(2GM)^2$.

The phenomenon of maximum acceleration in string theory has been discussed in several contexts. Sakai [13] studies the detector response functional for string theories in an accelerating frame and found that it deverged for $a > 2/b = a_{\rm max}$. Parentani and Potting [14] studied the Feynman propagator in Rindler space and found the same value for the limiting acceleration. Another approach regarding acceleration in string theory was taken in [20,21] where a mode instability for classical solutions of an extended object in Rindler space led to a critical acceleration $a_c = (\frac{3}{\pi(n-2)})^{1/3}\frac{1}{\sqrt{\alpha'}}$. Because the critical acceleration a_c is somewhat larger than the maximum acceleration $a_{\rm max} = 2/b$ of Sakai, we mainly work with $a_{\rm max}$, as these effects should occur first, however, similar conclusions can also be reached regarding a_c. A physically intuitive derivation of the maximum acceleration was also given by Sakai who studied the difference in the vacuum energy between Rindler space and Minkowski space, similar to the Casimir effect. One begins with the stress tensor for a massive scalar field:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + m^2\phi^2). \tag{2.5}$$

The vacuum stress [22,23] is then defined by $\bar{T}_{\mu\nu} = \langle 0_R|T_{\mu\nu}|0_R\rangle - \langle 0_M|T_{\mu\nu}|0_M\rangle$, where $|0_R\rangle$ and $|0_M\rangle$ represent the Rindler and Minkowski vacuum, respectively.

The vacuum stress can be computed by solving the eigenvalue problem in Rindler and Minkowski space and using these solutions to form the two-point function and then the stress tensor from

$$\bar{T}_{\mu\nu} = \lim_{x\to x'}\left(\partial_\mu\partial'_\nu - \frac{1}{2}g_{\mu\nu}(g^{\rho\sigma}\partial_\rho\partial'_\sigma + m^2)\right)\bar{G}(x,x'), \tag{2.6}$$

where

$$\bar{G}(x,x') = \langle 0_R|\phi(x)\phi(x')|0_R\rangle - \langle 0_M|\phi(x)\phi(x')|0_M\rangle.$$

The quantum field ϕ is written

$$\phi = \int dw \int \frac{d^{n-2}k}{(2\pi)^{(n-2)/2}}a_k u_k + \text{H.c.} \tag{2.7}$$

Here u_k are eigenfunctions in Rindler space and k is the conjugate momentum to $x_\perp$. The a_k annihilate the Rindler vacuum $a_k|0_R\rangle = 0$ and are related to Minkowski space creation and annihilation operators through $a_k = \sqrt{1+n_w}d_k^\dagger + \sqrt{n_w}d_k$ with $d_k|0_M\rangle = 0$ and $n_w = (e^{2\pi w}-1)^{-1}$ [24].

The Green's functions are then given by

$$\langle 0_R|\phi(x)\phi(x')|0_R\rangle = \int_0^\infty dw \int \frac{d^{n-2}k}{(2\pi)^{n-2}} u_k(x)u_k^*(x')$$

and

$$\langle 0_M|\phi(x)\phi(x')|0_M\rangle = \int_0^\infty dw \int \frac{d^{n-2}k}{(2\pi)^{n-2}} [n_w u_k^*(x)u_k(x') + (1+n_w)u_k(x)u_k^*(x')]$$

so that

$$\bar{G}(x,x') = -\int_0^\infty dw \int \frac{d^{n-2}k}{(2\pi)^{n-2}} n_w u_k^*(x)u_k(x') + \text{c.c.} \tag{2.8}$$

The eigenfunctions in Rindler space are solutions to

$$\left(-\frac{1}{s^2}\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{\partial}{\partial x_\perp}\frac{\partial}{\partial x_\perp} - m^2\right)u_k = 0 \tag{2.9}$$

and one obtains

$$u_k = \frac{1}{\pi}(\sinh\pi w)^{1/2} K_{iw}(s\sqrt{k^2+m^2})\exp i(kx_\perp - w\tau) \tag{2.10}$$

with $K_{iw}(z)$ the modified Bessel function. Now one uses this soluton to compute the Green's function $\bar{G}(x,x)$ obtaining the stress tensor from (2.6). For large mass, such as the massive states in a string theory, the last term in (2.6) is dominant and we find, for the stress energy,

$$\bar{T}_0^0 \approx m^2 \int_0^\infty dw(e^{2\pi w}-1)^{-1}\frac{1}{\pi^2}\sinh\pi w \times \int \frac{d^{n-2}k}{(2\pi)^{n-2}} |K_{iw}(s\sqrt{k^2+m^2})|^2. \tag{2.11}$$

The fermionic contribution to the vacuum stress has a similar form except for a Fermi-Dirac factor $(e^{2\pi w}+1)^{-1}$ [22]. Using the asymptotic expansion for the Bessel function $K_{iw}(x) \approx \sqrt{\frac{\pi}{2z}}e^{-z}$ and integrating over w and k one obtains, for the vacuum stress energy at large mass,

$$\bar{T}_0^0(s,m) \sim (m/s)^{n/2}e^{-2ms}, \tag{2.12}$$

which is in agreement with Sakai [13] and Takagi [23].

Now one forms the vacuum stress for a string theory $\bar{T}_0^{0S}$ by multiplying (2.12) by the number of string states at a given mass $\rho(m) = m_0^{n_s-1}m^{-n_s}e^{bm}$ and integrating over the mass to obtain

$$\bar{T}_0^{0S} = \int_{m_0}^\infty dm\,\rho(m)\bar{T}_0^0(s,m) \sim \int_{m_0}^\infty dm\, m_0^{n_s-1}m^{-n_s}e^{bm}(m/s)^{n/2}e^{-2ms}. \tag{2.13}$$

This clearly diverges $s < b/2 = s_{\min}$. Now since $s = 1/a$, the string stress energy diverges for accelerations $a > 2/b = a_{\max}$ and we have the limiting acceleration of Sakai [13] and Parentani and Potting [14]. The physical interpretation of this result can be inferred from the work of Candelas and Deutsch [22], who showed that $\bar{T}_{00}$ represents the absence of Hawking radiation from the vacuum and the presence of $T_{00}^{\text{thermal}} = \bar{T}_0^0$ thermal energy density in Rindler space. Therefore the divergence in $\bar{T}_{00}^S$ for $s < s_{\min}$ represents the absence from the vacuum of an infinite amount of Hawking radiation and there is a infinite wall of thermal stress energy $T_{00}^{\text{thermal}} = \bar{T}_0^{0S}$ a finite distance from the horizon. Since the energy of the thermal bath of radiation is taken from the external source accelerating the observer [19], it follows that a string cannot accelerate into that region of Rindler space. The energy required to accelerate further cannot be produced and the string can only continue at a uniform velocity into that region (see Fig. 1). Also by the equivalence principle, a string cannot remain stationary at a distance $s < s_{\min}$ from the horizon of a Schwarzschild geometry. The energy required to support the infinite thermal stress present there cannot be produced and the string simply slips into the black hole.

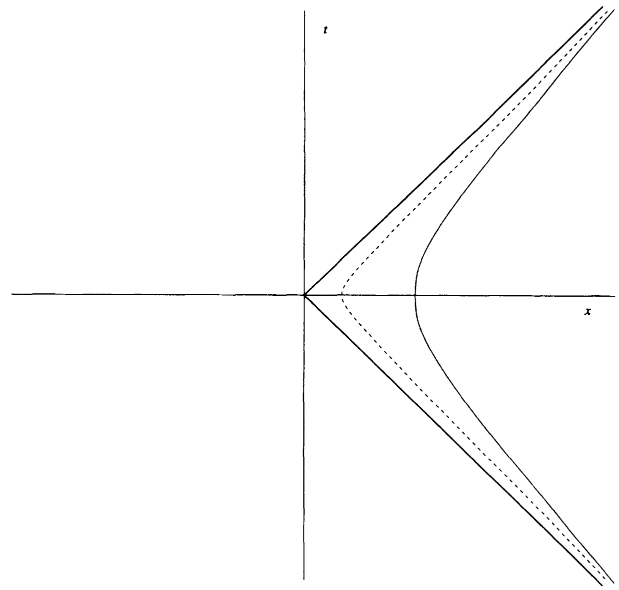

FIG. 1. Representation of the structure of Rindler space in string theory. The dashed line indicates the $s = b/2 = s_{\min}$ boundary inside of which the vacuum energy diverges and accelerations exceed $a = 2/b = a_{\max}$. The outside curved line is a trajectory of an accelerated observer with $a < a_{\max}$ and the dark 45 degree lines show the horizon $x = |t|$.

III. MAXIMUM ACCELERATION AND FINITE ENTROPY

The presence of an infinite wall of stress energy in Rindler space introduces a forbidden region and cutoff $s > s_{\min}$ which has important implications for finite entropy. Consider the single particle density of states of a scalar particle of mass m. Susskind and Uglum [6] took the eigenvalue equation (2.9) and solving for the radial momentum $p_s = (\frac{w^2}{s^2} - k^2 - m^2)^{1/2}$ with turnaround points $s_{\max} = \frac{w}{\sqrt{k^2+m^2}}$ and $s_{\min}$ obtained

$$n\pi = \int_{s_{\min}}^{s_{\max}} p_s ds = \int_{s_{\min}}^{w/\sqrt{k^2+m^2}} \left(\frac{w^2}{s^2} - k^2 - m^2\right)^{1/2} ds$$

for quantum number n. The single particle density of states represents the Jacobian $g(w,k,m) = dn/dw$ between the discrete index n and the quantity w and Susskind and Uglum find

$$g(w,k,m) = \frac{1}{2\pi}\ln\left(\frac{\frac{w}{s_{\min}} + p_s(s_{\min})}{\frac{w}{s_{\min}} - p_s(s_{\min})}\right), \tag{3.1}$$

where $p_s(s_{\min}) = (\frac{w_2}{s_{\min}^2} - k^2 - m^2)^{1/2}$. Integrating over the transverse momenta for $n = 4$ they obtain

$$\begin{aligned} g(w,m) &= \int A\frac{d^2k}{(2\pi)^2} g(w,k) \\ &= \frac{A}{(2\pi)^2}\left[\frac{w}{s_{\min}}\left(\frac{w^2}{s_{\min}^2} - m^2\right)^{1/2} \right. \\ &\left. + \frac{m^2}{2}\ln\left(\frac{\frac{w}{s_{\min}} - \left(\frac{w^2}{s_{\min}^2} - m^2\right)^{1/2}}{\frac{w}{s_{\min}} - \left(\frac{w^2}{s_{\min}^2} - m^2\right)^{1/2}}\right)\right]. \end{aligned} \tag{3.2}$$

The second contribution vanishes for a massless particle but is quite significant for a very massive particle as we shall see. Clearly the single particle density of states diverges on the horizon if $s_{\min} = 0$. To obtain the single string density of states we multiply $g(w,m)$ by $\rho(m)$ and integrate over the mass just as we did for the stress energy.

The entropy is given in the canonical ensemble by forming $S = \beta U + \ln Z$, where $U = -\frac{\partial}{\partial\beta}\ln Z$ and

$$\ln Z = -\int_{m_0}^{\infty} dm \int_{m s_{\min}}^{\infty} dw\, \rho(m) g(w,m)\frac{1}{2}\ln\left(\frac{1-e^{-\beta w}}{1+e^{-\beta w}}\right) \tag{3.3}$$

with β set equal to 2π. For massless states the entropy was calculated by 't Hooft [1] and Susskind and Uglum [6] who showed that

$$S_{\text{massless}} = \frac{A n_0}{360\pi s_{\min}^2} \tag{3.4}$$

with n_0 related to the number of massless particles. To calculate the entropy of massive states it is convenient to define $E = w/s_{\min}$ and $p = \sqrt{E^2 - m^2}$. Then the single particle density of states in Rindler space becomes

$$g(E,m) = \frac{A}{(2\pi)^2}\left[Ep + \frac{m^2}{2}\ln\left(\frac{E-p}{E+p}\right)\right] \tag{3.5}$$

as compared with a Minkowski space density of states that goes like Ep. The partition function now becomes

$$\begin{aligned} \ln Z = &-\int_{m_0}^{\infty} dm \int_{m}^{\infty} dE\, \rho(m) s_{\min}\frac{A}{(2\pi)^2} \\ &\times\left[Ep + \frac{m^2}{2}\ln\left(\frac{E-p}{E+p}\right)\right]\frac{1}{2}\ln\left(\frac{1-e^{-\beta E s_{\min}}}{1+e^{-\beta E s_{\min}}}\right). \end{aligned} \tag{3.6}$$

For very massive string states $E \approx m + \frac{p^2}{2m}$ and a nonrelativistic approximation is appropriate. Defining $v = p/E$ the density of states takes the simplified form

$$g(E,m) \approx \frac{A}{(2\pi)^2} m^2\left(\frac{v}{1-v^2} - v - \frac{1}{3}v^3\right) \approx \frac{A}{(2\pi)^2}\frac{2}{3}m^2 v^3$$

for massive nonrelativistic states. Note that if the term involving the logarithm in (3.2) were not present the density of states would only go like a single power of the velocity.

With the simplified density of states the partition function becomes

$$\begin{aligned} \ln Z = &-\int_{m_0}^{\infty} dm \int_{0}^{\infty} dv\, \rho(m) s_{\min}\frac{A}{(2\pi)^2}\frac{2}{3}m^3 v^4 \frac{1}{2} \\ &\times\ln\left(\frac{1-\exp(-\beta s_{\min} m - \beta s_{\min}\frac{mv^2}{2})}{1+\exp(-\beta s_{\min} m - \beta s_{\min}\frac{mv^2}{2})}\right). \end{aligned} \tag{3.7}$$

For very massive particles the argument of the logarithm is near 1, so expanding the logarithm and integrating over the velocity v we obtain

$$\begin{aligned} \ln Z = &\int_{m_0}^{\infty} dm\, m_0^{n_s - 1} m^{-n_s} e^{bm}\frac{1}{4} s_{\min}\frac{A}{(2\pi)^2} m^3\sqrt{\pi} \\ &\times(\beta s_{\min} m/2)^{-5/2} e^{-\beta s_{\min} m}. \end{aligned} \tag{3.8}$$

The free energy $U = -\frac{\partial}{\partial\beta}\ln Z$ is then given by

$$\begin{aligned} U = &\int_{m_0}^{\infty} dm\, m_0^{n_s - 1} m^{-n_s} e^{bm}\frac{1}{4} s_{\min}\frac{A}{(2\pi)^2} m^3\sqrt{\pi} \\ &\times(\beta s_{\min} m/2)^{-5/2}\left(s_{\min} m + \frac{5}{2\beta}\right) e^{-\beta s_{\min} m}. \end{aligned} \tag{3.9}$$

The entropy is then $S = \beta U + \ln Z$, and by performing the integral over mass using the incomplete Γ function $\Gamma(a,z) = \int_z^{\infty} dt\, t^{a-1} e^{-t}$ it can be expressed as

$$\begin{aligned} S_{\text{massive}} = &\frac{A}{s_{\min}^2}\frac{\sqrt{\pi}}{16\pi^2(\beta/2)^{5/2}}\left[\beta(m_0 s_{\min})^{3/2}[m_0(\beta s_{\min} - b)]^{-5/2+n_s}\Gamma\left(\frac{5}{2} - n_s, m_0(\beta s_{\min} - b)\right)\right. \\ &\left. + \frac{7}{2}(m_0 s_{\min})^{1/2}[m_0(\beta s_{\min} - b)]^{-3/2+n_s}\Gamma\left(\frac{3}{2} - n_s, m_0(\beta s_{\min} - b)\right)\right]. \end{aligned} \tag{3.10}$$

Now we take the inverse Hawking temperature $\beta = 2\pi$ and from Sec. II $s_{\rm min} = \frac{b}{2}$. Noting that $s_{\rm min}, m_0^{-1}$, and b are all of order $\sqrt{\alpha'}$, we set $m_0 = n_s'/s_{\rm min}$ with n_s' a number of order 1. The entropy due to the massive string states then becomes

$$S_{\rm massive} = \frac{A}{s^2_{\rm min}} r(n_s, n_s') \tag{3.11}$$

where

$$r(n_s,n_s') = \frac{1}{16\pi^4}\left[2\pi(n_s')^{3/2}(2n_s'(\pi-1))^{-5/2+n_s}\Gamma\left(\frac{5}{2}-n_s, 2n_s'(\pi-1)\right)\right.$$
$$\left.+\frac{7}{2}(n_s')^{1/2}(2n_s'(\pi-1))^{-3/2+n_s}\Gamma\left(\frac{3}{2}-n_s, 2n_s'(\pi-1)\right)\right]. \tag{3.12}$$

The total entropy is the sum of that due to the massless and massive states and is given by

$$S = S_{\rm massless} + S_{\rm massive} = \frac{A}{s^2_{\rm min}}\left(\frac{n_0}{360\pi} + r(n_s,n_s')\right), \tag{3.13}$$

where n_0 is related to the number of massless states of the string theory through $n_0 = n_{b0} + \frac{7}{8}n_{f0}$ with n_{b0} and n_{f0} the number of massless bosonic and fermionic modes.

Setting (3.13) equal to the Bekenstein-Hawking black hole entropy $S_{\rm BH} = \frac{A}{4G}$ yields the condition

$$\frac{G}{4} = s^2_{\rm min}\bigg/\left(\frac{n_0}{360\pi} + r(n_s,n_s')\right) \tag{3.14}$$

or, since $s_{\rm min} = b/2$,

$$G = b^2\bigg/\left(\frac{n_0}{360\pi} + r(n_s,n_s')\right). \tag{3.15}$$

For all string theories b is proportional to α' so that (3.15) gives a relation between Newton's constant and the string tension. Setting $b = n_s''\sqrt{\alpha'}$ we have

$$G = \alpha' n_s''^2\bigg/\left(\frac{n_0}{360\pi} + r(n_s,n_s')\right). \tag{3.16}$$

Therefore the relation between Newton's constant and the string tension is fixed by requiring the entropy of massless and massive string states to be equal to the entropy of a black hole. For heterotic superstring theory compactified to four dimensions $n_s = 4$ and $b = (2+\sqrt{2})\pi\sqrt{\alpha'}$ so $n_s'' = (2+\sqrt{2})\pi$. For open superstring theory compactified to four dimensions $n_s = \frac{5}{2}$ and $b = 2\sqrt{2}\pi\sqrt{\alpha'}$ so that $n_s'' = 2\sqrt{2}\pi$. In either case $m_0 = n_s'/s_{\rm min} = 2n_s'/b$ can be chosen so that n_s' is of order 1. The massless contribution to the entropy turns out to be much greater than that of the massive modes whose major effect is to set the cutoff length $s_{\rm min}$ as discussed in Sec. II. This being the case the relation between Newton's constant and the string tension is of order $G \sim \frac{10^5}{n_0}\alpha'$, where n_0 is related to the number of massless modes.

String theory itself relates Newton's constant to α' through gauge and string coupling constants. These relations depend on the type of string theory considered. For heterotic superstring theory compactified to four dimensions [25], setting $\kappa_4 = \sqrt{8\pi G}$, the gauge coupling to g_4, the string coupling to g, and the compactified six volume to V, the relation is $2\kappa_4 = \sqrt{2\alpha'}g_4 = \frac{(2\alpha')^2}{\sqrt{V}}g$. In terms of Newton's constant we have $G = \frac{g_4^2}{16\pi}\alpha' = \frac{g^2}{2\pi V}\alpha'^4$. For open superstrings [26] the relations are $\kappa_4 \sim \frac{\sqrt{V}}{\alpha'}g_4^2 = \frac{8\alpha'^2}{\sqrt{V}}g^2$ which can be put in the form for Newton's constant $G \sim \frac{Vg_4^4}{8\pi\alpha'^3}\alpha' = \frac{8g^4}{\pi V}\alpha'^4$. The open superstring gauge coupling can be weakly coupled with weakly coupled σ model $V \gg \alpha'^3$ and still be consistent with the relation (3.16) derived by setting the entropy of the string states to the Bekenstein-Hawking formula. All other couplings must either be strong or have a strongly coupled σ model $V \ll \alpha'^3$ to be consistent with (3.16).

In [1] 't Hooft's studied the entropy of a particle theory of fixed mass and infinite numbers of degrees of freedom and found an unreasonably large value of the cutoff length. Indeed, requiring that the entropy agrees with the Bekenstein-Hawking formula fixes the cutoff length

$$s_{\rm min} = \frac{\sqrt{G}}{2}\left(\frac{n_0}{360\pi} + r(n_s,n_s')\right)^{1/2}. \tag{3.17}$$

Taking the limit $n_0 \to \infty$ tells us that $s_{\rm min} \sim \sqrt{n_0}$ if the entropy is to reproduce the Bekenstein-Hawking entropy. Thus for an infinite number of degrees of freedom the cutoff $s_{\rm min}$ would have to move infinitely far away from the horizon yielding an unphysical picture of a macroscopic black hole. String theory, on the other hand, has an infinite number of degrees of freedom of increasing mass and a fixed value for $s_{\rm min} = b/2$ with a forbidden region only very close to the horizon. This region corresponds to Planckian acceleration, and it is physically reasonable for string effects to play a role there.

Our calculation of the black hole entropy in string theory was done assuming the canonical ensemble. There are concerns about the validity of using the canonical ensemble because of the negative specific heat and loss of equilibrium of both black holes and massive string states. In this paper as in [6] the area of the horizon was taken to be L^2 with L the limit of transverse coordinates in Rindler space. Rindler space was identified with the spacetime of a very large black hole with horizon area $4\pi(2GM)^2$ and, as in Rindler space, was taken to be static. For a very large black hole, the variation in the horizon area with time is so slight that the identification with Rindler space is appropriate and the deviation from equilibrium relatively small. However, for small evaporating black holes the horizon area is rapidly varying, with a metric deviating strongly from Rindler space. In this regime a microcanonical counting of states is a more appropriate procedure far from equilibrium.

A microcanonical description leads to a number of states $\sigma(W) = \exp[S(W)]$ with W related to the mass of the black hole and

$$\sigma(W) = \sum_N \frac{1}{N!} \prod_{i=1}^{N} \int_{m_0}^{\infty} dm_i \int_{m_i s_{\min}}^{\infty} dw_i \, \rho(m_i) \times g(w_i, m_i) \delta(W - \Sigma_i w_i). \tag{3.18}$$

Here N represents the number of strings and $1/N!$ ensures the correct statistics. The single particle density of states $g(w, m)$ can be that relevant to a small evaporating black hole out of equilibrium. A similar microcanonical description has been used to describe string theories at high energy density [27,28], where a single massive string state can carry almost all the energy and represent a nonequilibrium configuration. In this way massive string states can also lead to a negative specific heat [29]. However, in the case of a very large black hole studied in this paper the canonical ensemble is valid because the limiting Hawking temperature is still less by a factor of $1/\pi$ from the Hagedorn temperature above which thermodynamic quantities can diverge.

IV. CONCLUSION

In this paper we have examined the arguments leading to a maximum acceleration in string theory and an infinite wall of stress a finite distance outside the horizon. We multiplied the single massive particle density of states of Refs. [1,6] by the number of string modes at a given mass and summed over all masses to obtain a single string density of states in Rindler space or about a very large black hole. We found that a cutoff on the density of states was introduced by the infinite wall of vacuum stress and maximum acceleration. The entropy of the string excitations was computed using the canonical ensemble at the Hawking temperature and was finite because of the natural cutoff. The massive string contribution preserved the relation that the entropy is proportional to the area of the horizon divided by the cutoff squared. We found that the string entropy agreed with the Bekenstein-Hawking formula if a particular relationship existed between Newton's constant and the string tension, and then compared this with similar relationships found in various string theories. Finally we discussed the validity of the canonical ensemble for evaporating black holes and massive string states.

The basic point is that there is a region outside the horizon of a black hole within which a string cannot remain stationary. Likewise, by the equivalence principle, there is a region outside the horizon of Rindler space within which a string cannot accelerate into. This suggests that an effective cutoff is introduced on the boundary of a region slightly away from the horizon. This can cut off the divergence in the single particle density of states and yield a finite answer for the entropy. 't Hooft's result that the brick wall cutoff should move infinitely away from the horizon in a theory with infinite numbers of degrees of freedom is avoided in string theory due to the infinite set of states of arbitrarily high mass. Further studies using a fundamental description of the string propagator [30], instead of the sum over modes approach we have taken here, are necessary to place the entropy calculation on a sound footing. In general, string calculations have features like modular invariance which are not obvious in sum over field theories and are important for a geometric understanding of the partition function and entropy.

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