

Block spin renormalization group approach and two-dimensional quantum gravity

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A block spin renormalization group approach is proposed for the dynamical triangulation formulation of two-dimensional quantum gravity. The idea is to update link flips on the block lattice in response to link flips on the original lattice. Just as the connectivity of the original lattice is meant to be a lattice representation of the metric, the block links are determined in such a way that the connectivity of the block lattice represents a block metric. As an illustration, this approach is applied to the Ising model coupled to two-dimensional quantum gravity. The correct critical coupling is reproduced, but the critical exponent is obscured by unusually large finite-size effects.

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I. DYNAMICAL TRIANGULATIONS AND QUANTUM GRAVITY

Dynamical triangulations are by now well established as a discretization of two-dimensional quantum gravity, at least when the central charge of the matter it is coupled to is less than one [1]. A two-dimensional triangulation is characterized by its points i , its links $\langle ij \rangle$, and its triangles $\langle ijk \rangle$. This information is uniquely determined by the adjacency matrix G_{ij} , which defines the nearest-neighbor pairs:

$$G_{ij} = \begin{cases} 1 & i, j \text{ are nearest neighbor,} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

All links are defined to be of length a , the lattice spacing, which is usually taken to be one. Since the triangles are, therefore, all equilateral, the coordination number at a site i , q_i , is related to the curvature R_i :

$$R_i = \pi(6 - q_i)/q_i. \quad (2)$$

Six is the average coordination number on a regular triangulation which has zero curvature. The adjacency matrix acts like a metric in the sense that it tells which points are near and which are far away. For points that are not nearest neighbors, the geodesic distance is defined to be the length of the shortest path between those points. In a quantum theory of gravity, a sum over all possible metrics is required to compute the partition function. On the lattice, this becomes a sum over all possible triangulations. If an additional statistical mechanical variable is placed on the nodes to represent matter, then a sum over all possible matter configurations is required for each triangulation. In a number of cases, these quantum gravity plus matter systems can be solved analytically.

The Ising model illustrates this and is the only example that will be discussed in this paper. In the absence of quantum gravity, i.e., on a fixed regular triangulation, the partition function is defined as

$$Z = \sum_S \exp \left(k \sum_{\langle ij \rangle} S_i S_j \right), \quad (3)$$

where the sum is over all configurations of the spins, $S_i = \pm 1$. This, the usual Ising model [2] has a second-order phase transition at $k = k^c$ where

$$k^c = \frac{1}{2} \operatorname{arcsinh} \frac{1}{\sqrt{3}} = 0.2746 \dots \quad (4)$$

and it has critical exponent

$$\nu = 1, \quad (5)$$

where ν is defined by the behavior of the correlation length, ξ , near the critical coupling

$$\xi(k - k^c) \approx (k - k^c)^{-\nu}. \quad (6)$$

In the presence of quantum gravity, the definition of the partition function includes the sum over triangulations (labeled by T):

$$Z = \sum_T \sum_S \exp \left(k \sum_{i,j} G_{ij}(T) S_i S_j \right). \quad (7)$$

This theory also has a second-order phase transition [3,4], this time at

$$k^c = -\frac{1}{2} \ln \tanh \frac{1}{2} \ln \left(\frac{108}{23} \right) = 0.2162 \dots, \quad (8)$$

with the critical exponent

$$\nu d = 3. \quad (9)$$

The exponent now appears in combination with the Hausdorff dimension d , which is dynamically determined. This combination appears because the correlation length (the object whose behavior ν describes) is a distance whereas in quantum gravity direct control over length

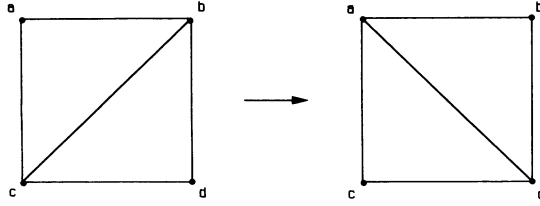


FIG. 1. A link flip.

scales is lost. The relation between volume (still under control) and distance is given by the Hausdorff dimension. It has been shown that for matter with central charge less than one, the scaling dimensions at a critical point of the pure matter theory are “dressed” by quantum gravity so that the critical behavior of the two theories is related [5].

Not all models of interest can be solved exactly. Numerical methods provide another option. The key fact that makes simulations of dynamical triangulations practical is that any triangulation (for a given topology) can be reached from any other triangulation by a sequence of link flips [6]. A link flip is defined in Fig. 1 and is the replacement of a link with another connecting the two other points in the two triangles it shares. Matter can be simulated in the usual way.

In numerical simulations of lattice theories for high energy physics, the primary difficulty is often dealing with critical phenomena. When the lattice spacing is taken to zero, the correlation length (in physical units) is zero unless the correlation length in lattice units is infinite. Thus the location of continuous phase transitions and the study of their universal behavior are generically problems of interest for such theories. Finite-size scaling provides one method of computing critical couplings and exponents. This method has been successfully applied to systems coupled to quantum gravity such as the crumpling transition [7] and the Ising model [8].

Another method, the block spin renormalization group approach, is the subject of this paper. In this method, an effective theory is constructed by partitioning the lattice into blocks and averaging the degrees of freedom within each block. Starting with a critical theory and iterating this procedure produces a sequence of theories which flow toward a fixed point. Critical exponents can be calculated using expectation values of operators determined at this fixed point. This approach was first developed on spin theories (hence the name) and applied also to lattice gauge theories. It has not yet been applied to (the more recently invented) dynamically triangulated random surfaces [9]. The objective of this paper is to apply the block spin approach to this newer type of theory.

II. THE BLOCK SPIN RENORMALIZATION GROUP APPROACH

First, consider in more detail the block spin approach (see [10–16]). Denote a lattice spin variable by S so that the partition function is

$$Z = \sum_S e^{H[S]}. \quad (10)$$

Now average over blocks of spins as illustrated in Fig. 2. A square lattice is used, but the same scheme works on a triangulation which can be obtained by drawing in all of the up and to the right diagonals. Label each block by m and assign it a new spin t_m . More generally, a probability can be assigned for every possible value of t_m :

$$P(t_m) = K(t_m, |S|_m), \quad (11)$$

where $|S|_m$ denotes the set of spins within the block. Given a configuration of initial spins, the probability for a configuration of block spins is

$$P[t] = \prod_m K(t_m, |S|_m). \quad (12)$$

Multiplying by the probability of the initial configuration and summing over all possible initial configurations gives the total probability of obtaining the block configuration, which can be viewed as resulting from an effective theory

$$e^{H'[t]} = \sum_S \prod_m K(t_m, |S|_m) e^{H[S]}. \quad (13)$$

The effective theory has (for the scenario of Fig. 2) twice the original lattice spacing. If the physics is unchanged by this operation, the correlation length is, in lattice units, half its previous value. The physics is unchanged if the partition functions of the blocked and original theories are identical. This is assured if

$$\sum_{t_m} K(t_m, |S|_m) = 1. \quad (14)$$

An example of a solution to this equation is

$$K(t_m, |S|_m) = \delta(t_m - f(|S|_m)). \quad (15)$$

For instance, the “majority rule” for the Ising model is of this form: for an odd number of spins if the majority are up the block spin is up. Otherwise it is down. For

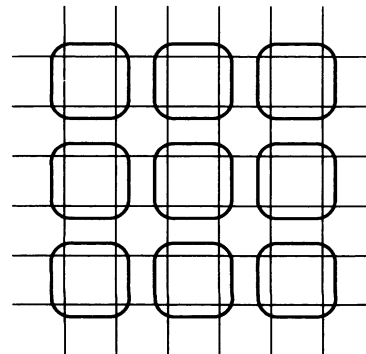


FIG. 2. A simple blocking scheme for spins. The nodes of the original lattice are represented by the intersections of the straight lines. Block spins are formed by averaging [as with Eq. (15)] the spins grouped with the rounded squares.

an even number of Ising spins, a tie breaker is required (trivially modifying the previous equation).

In general, if the block spins are the same type of variable as the original spins and if the blocked lattice is isomorphic to the original, then K can be applied iteratively to produce a sequence of effective Hamiltonians. Distances must be rescaled at each step. If the scale factor is b , the ratio of the number of points in the original lattice, N , to the number of points in the block lattice, N' , is

$$N/N' = b^D, \quad (16)$$

where D is the dimensionality of the lattice. The correlation length is reduced by b :

$$\xi' = \xi/b. \quad (17)$$

More abstractly, the renormalization group transformation acts on the space of Hamiltonians

$$H' = R[H]. \quad (18)$$

If this transformation has a fixed point

$$R[H^*] = H^*, \quad (19)$$

then

$$\xi^* = \xi^*/b, \quad (20)$$

which has solutions $\xi^* = 0$ and $\xi^* = \infty$. The former corresponds to a (trivial) system at infinite temperature. The latter corresponds to a (nontrivial) critical point. There are many possible renormalization group transformations for a given system and not all of them iterate to a nontrivial fixed point for the critical Hamiltonian at hand. Aside from the constraint of Eq. (14) and the requirement that the transformation reduces the number of degrees of freedom, it is also necessary to ask that it be apt, i.e., that it focuses on the critical phenomena of interest.

Now perturb H^* with an operator O so that the system is slightly off of criticality. It is convenient to reparametrize the scale factor

$$b = e^l \quad (21)$$

in order to discuss differential changes of scale (now mediated by a transformation R_l). Assume that

$$\begin{aligned} \frac{d}{dl}(H^* + O) &\equiv \lim_{l \rightarrow 0} \frac{R_l[H^* + O^*] - (H^* + O^*)}{l} \\ &= LO + \dots, \end{aligned} \quad (22)$$

where L is a linear operator and nonlinear terms (denoted by the ellipsis) are neglected. If the eigenoperators of L are defined by the equation

$$LO_i^* = \lambda_i O_i^* \quad (23)$$

and the O_i^* form a complete set of operators in the neighborhood of H^* , then, for H_0 in that neighborhood,

$$H_0 = H^* + \sum_i h_i O_i^*. \quad (24)$$

The evolution of the coefficients (called linear scaling fields) is determined by

$$\frac{dh_i(l)}{dl} = \lambda_i h_i(l), \quad (25)$$

implying

$$h_i(l) = h_i e^{\lambda_i l} \quad (26)$$

so that

$$H_l = H^* + \sum_i h_i e^{\lambda_i l} O_i^*, \quad (27)$$

where H_l represents the effective Hamiltonian after the scale is changed by $b = e^l$. The λ_i are critical exponents: if $\lambda_i > 0$, the i th perturbation grows (O_i^* is relevant) while if $\lambda_i < 0$, the i th perturbation diminishes (O_i^* is irrelevant). If $\lambda_i = 0$, O_i^* is marginal.

To make contact with traditional scaling ideas, consider a system with only one relevant operator whose corresponding linear scaling field is the coupling k . Take all of the irrelevant couplings to be zero. Then Eq. (17) can be written as

$$\xi(k - k^c) = e^l \xi(e^{\lambda l} (k - k^c)). \quad (28)$$

If l is chosen so that

$$e^l \propto (k - k^c)^{-1/\lambda}, \quad (29)$$

then

$$\xi(k - k^c) = \text{const} \times (k - k^c)^{-1/\lambda}, \quad (30)$$

which is the same as Eq. (6) after making the identification

$$\nu = 1/\lambda. \quad (31)$$

How can this approach be realized in a numerical setting? In such a setting, it is the expectation values of functions of products of the original spins,

$$\langle O \rangle = \sum_S O(S) e^{H[S]}, \quad (32)$$

that are readily accessible. Expectation values in the blocked theory can be computed with the original spins using the definition of blocked spins:

$$\langle O(t) \rangle = \sum_t O(t) e^{H'[t]} \quad (33)$$

$$= \sum_t O(t) \sum_S \prod_m K(t_m, |S|_m) e^{H[S]}. \quad (34)$$

If K is a δ function or majority rule then each configuration of spins specifies a configuration of block spins so that the sum on t is determined.

On an infinite lattice, the criticality of an initial Hamiltonian could be verified by observing that expectation

values of arbitrary operators approach fixed values as the renormalization group procedure is iterated. In practice, a computationally manageable lattice is typically small, especially after several iterations of the transformation. Finite size effects spoil the matching. This problem can be circumvented by comparing expectation values obtained on a lattice of N points blocked m times to those obtained on a lattice of N' points blocked $m - 1$ times with $N/N' = b^D$. The two block lattices are the same size so that differences in expectation values are due to the difference in the Hamiltonians alone.

Once the couplings are near their critical values, they can be systematically improved. Near the fixed point Hamiltonian,

$$\langle O_i \rangle = \langle O_i \rangle_{H=H^*} + \sum_j \frac{\partial \langle O_i \rangle_{H=H^*}}{\partial k_j} \delta k_j + \dots, \quad (35)$$

where the k_j are the relevant bare parameters in the theory and $\delta k_j = k_j - k_j^c$. Formulating, as described above, the difference between expectation values taken from original lattices of different sizes blocked down to lattices the same size gives

$$\begin{aligned} \langle O_i^{(n)} \rangle_L - \langle O_i^{(n-1)} \rangle_S \\ = \sum_j \left[\frac{\partial \langle O_i^{(n)} \rangle_L}{\partial k_j} - \frac{\partial \langle O_i^{(n-1)} \rangle_S}{\partial k_j} \right] \delta k_j, \end{aligned} \quad (36)$$

where L (large) denotes an initial lattice with some volume N and S (small) denotes an initial lattice with volume N/b^d . The superscripts indicate the blocking level. This equation predicts the adjustment to the bare parameters necessary to achieve criticality. The term in brackets can be computed from expectation values using expressions of the form

$$\frac{\partial \langle O_i^{(n)} \rangle}{\partial k_j^{(m)}} = \langle O_i^{(n)} O_j^{(m)} \rangle - \langle O_i^{(n)} \rangle \langle O_j^{(m)} \rangle. \quad (37)$$

The critical exponents can also be obtained numerically. If, after n iterations of the renormalization group transformation,

$$H^{(n)} = H^* + O_i(k_i^{(n)} - k_i^*), \quad (38)$$

then

$$R[H^* + (k_i^{(n)} - k_i^*)O_i] = H^* + T_{ij}^*(k_j^{(n)} - k_j^*)O_j + \dots, \quad (39)$$

where now the scale factor b is fixed and the stability matrix T_{ij}^* gives

$$(k_i^{(n)} - k_i^*) = T_{ij}^*(k_j^{(n-1)} - k_j^*) \quad (40)$$

or

$$T_{ij}^* = \left. \frac{\partial k_i^{(n)}}{\partial k_j^{(n-1)}} \right|_{H^*}. \quad (41)$$

Diagonalizing this matrix gives linear scaling fields h (lin-

ear combinations of the k_i) and eigenvalues Λ which obey

$$h_i^{(n)} = \Lambda_i h_i^{(n-1)}. \quad (42)$$

Comparing with Eq. (26) reveals that these eigenvalues are related to the critical exponents:

$$\Lambda_i = b^{\lambda_i}. \quad (43)$$

The stability matrix can be determined from correlations of the type in Eq. (37) by using the chain rule

$$\frac{\partial \langle O_i^{(n)} \rangle}{\partial k_j^{(n-1)}} = \sum_l \frac{\partial k_l^{(n)}}{\partial k_j^{(n-1)}} \frac{\partial \langle O_i^{(n)} \rangle}{\partial k_l^{(n)}}. \quad (44)$$

Arbitrary couplings are implicated, so T_{ij}^* is an infinite matrix. In practice, only a finite number of expectation values can be handled, so it must be truncated. This is a source of error.

III. A BLOCK SPIN TRANSFORMATION FOR DYNAMICAL TRIANGULATIONS

The block spin renormalization group approach has been successfully applied to spin models and gauge theories. How can it be applied to dynamical triangulations? Since the Hausdorff dimension is dynamically determined, there is no direct access to the length scale. Further, the degrees of freedom do not consist of elements of an algebra or manifold residing on a regular lattice so that they can simply be averaged in some way and projected back onto the algebra or manifold. A block spin approach for triangulations will have to look a little different.

Imagine a triangulation and another triangulation that is somehow a result of blocking the original one. The blocked lattice should have fewer points, but it should still be a triangulation. If the original triangulation is regular (implying toroidal topology), it is clear how to arrange this. For example, see Fig. 3. The volume (number of points) is decreased by a factor of 4. Now, any other triangulation (of the torus) can be reached from the original triangulation by a sequence of link flips. Likewise, any other block triangulation can be reached from the blocked one in Fig. 3 by a sequence of block link flips. The central idea of this paper is this: any rule that dic-

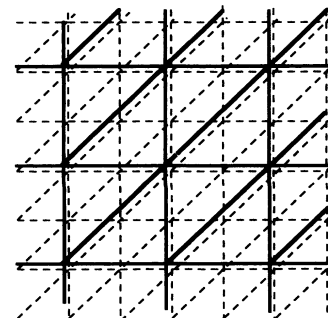


FIG. 3. A blocking scheme for a regular triangulation.

tates when a block link should be flipped in terms of when the original links are flipped is equivalent to a definition of a renormalization group kernel K .

In order for a kernel to be apt it must preserve the important physics. Physically, the triangulation is viewed as a discretization of a Euclidean spacetime with the adjacency matrix acting as a metric: points connected by a link are “close,” while points not connected by a link are not close. The block lattice is produced by marking a subset of points on the original lattice and then connecting them (defining a blocked adjacency matrix) to form the blocked lattice. If the blocked adjacency matrix is to act as an effective metric on the block lattice, it should indicate which block sites are close and which are further away. A link on the block lattice has two neighboring triangles just as in Fig. 1. Define the geodesic distance between block points x and y , r_{xy} , as the minimum number of links that must be traversed on the underlying lattice to get from one point to the other. If block link flips are made whenever

$$r_{ad} < r_{bc}, \quad (45)$$

then the block lattice preserves the idea of a metric. This now defines a block spin renormalization group transformation for dynamical triangulations. It will turn out to be apt.

In order to handle matter coupled to this system, a block spin algorithm must be specified for the matter as well. Only Ising spins will be discussed in what follows. A rule must be specified defining which nodes are associated together in a block. Presumably one and only one of the nodes that are specified as block nodes will appear in each block. Then a rule (such as the majority rule) must be given relating the block spin to those spins in the block. For a regular triangulation (such as that of Fig. 3), the block node itself and the three neighboring nodes in the forward directions (right and up) could be defined as being in the block. This is the scheme of Fig. 2. On a dynamical lattice such an assignment is not always possible. There are often points on the original lattice that are greater than distance one from any block node. This means that either nodes further than distance one from the block node must be included in the block, or less than four nodes must sometimes be used.

The definition of block spin used here is as follows. Each block node is in its own block. Each block node is then allowed to pick (randomly) one nearest neighbor that has not yet been picked (two different block nodes sometimes share a neighbor). This is repeated twice so that three neighbors are chosen if there is no contention. If, at some point in the selection process, there are no neighbors that have not been spoken for, no node is selected and that block has fewer than four spins. The majority rule then determines the block spin. The weakness in this procedure is that the coordination number of a site influences the selection of the spins in a block and therefore also the determination of the block spin. This could effectively contribute a relevant perturbation to the Hamiltonian, spoiling the matching of expectation values. Expectation values in the gravitational sector

should not be influenced, because the selection of block links in no way depends on the selection of block spins. In the calculations discussed below, there is a small effect present in the spin sector (at the highest blocking level considered) but not in the gravitational sector that might be explained in this way.

These ideas are implemented using the Ising model coupled to quantum gravity with the critical value of the coupling constant as given in Eq. (8). The triangulations are updated using the link flip algorithm and the spins are updated using the Wolff algorithm [17]. The block links are updated using the algorithm based on Eq. (45). This is computationally intensive because it involves the calculation of the distance between block nodes in terms of the links of the underlying lattice. This distance can be large at the lower blocking levels. Instead of calculating both of the distances involved in Eq. (45), it is quicker to just determine which is smaller. This can be done in a loop that labels all of the neighbors of the four block sites involved, then all of their neighbors and so on until the neighbors from one of the diagonal pairs meet. The block link should join this pair. Five to ten passes through all of the links of the block lattice are made with this procedure in order to implement the triangulation blocking algorithm. The block spin procedure is then iterated by treating the block lattice as an original lattice.

All original lattice sizes are chosen so that a regular triangulation would block down to a 3×3 torus. This choice is made to ease comparisons between the cases with and without quantum gravity. The minimal triangulation of a torus requires seven points, so nine points is not wastefully large. The renormalization group transformation defined in this paper rescales the volume by a factor of 4, so possible volumes for the initial lattices are nine times powers of 4. The volumes used are 144, 576, and 2304. These lattices can be renormalized twice, three times, and four times, respectively. Runs on these lattices involved 10^5 passes through each lattice where a pass is defined as 16 sweeps through each link of the lattice with the link flip algorithm along with either 200 Wolff updates (for the smaller two lattices) or 800 (for the largest).

Performing the renormalization group procedure at the critical coupling should produce an expectation value of any given operator that approaches a constant up to finite size effects which are controlled by only comparing numbers obtained on effective lattices of the same size. Three operators are used from the spin sector and three from the gravity sector. O_1 is defined as the product of spins at the opposite ends of a link. O_2 is defined as the product of spins at the opposite ends of the conjugate link, the link that would result from a flip. O_3 is defined as the product of all four spins involved in O_1 and O_2 . O_4 , O_5 , and O_6 are defined just like the previous three, except instead of using the spin, the coordination number minus six is used. This gives information about the gravitational sector. Another gravity sector operator computed (O_7) is the maximum coordination number on the lattice. Table I shows the resulting expectation value of the Ising term (O_1) as a function of the volume and blocking level n . This table demonstrates both the finite-

TABLE I. The expectation value of the Ising term as a function of the number of iterations of the renormalization group transformation, n , and the volume of the initial lattice.

n	$V = 2304$	$V = 576$	$V = 144$
0	0.4946(1)	0.4989(2)	0.5093(4)
1	0.2304(4)	0.3066(5)	0.3971(7)
2	0.2450(7)	0.3517(9)	0.481(1)
3	0.350(1)	0.482(2)	
4	0.489(2)		

size effects and the way they are countered by always comparing numbers obtained on lattices of the same size. Reading across the rows, the numbers vary due to finite-size effects. Reading along diagonals, down and to the left, the numbers are all obtained from effective lattices that are the same size and, in the bottom two diagonals, they (nearly) approach a constant. The bottom number in each column is from a lattice with volume nine. If Eq. (36) is used to predict how far the input coupling is from the true critical one, the prediction for $\delta k = k - k^c$ is

$$\delta k = 0.0002(3) \quad (46)$$

comparing the $V = 144$ and $V = 576$ data at the highest blocking levels (two and three, respectively). This number should be zero, since the input coupling was the known critical coupling. Results from similar runs, but with the system slightly off of criticality, are shown along with this data in Fig. 4. For large enough iterations of the renormalization group transformation and for input couplings close enough to the critical value, the results from Eq. (36) would fall on a straight line (as indicated in the figure). The plotted points, obtained with a relatively small number of iterations, are close to this line.

Thus, at this level of iteration, the critical coupling can be estimated quite accurately. What happens at higher

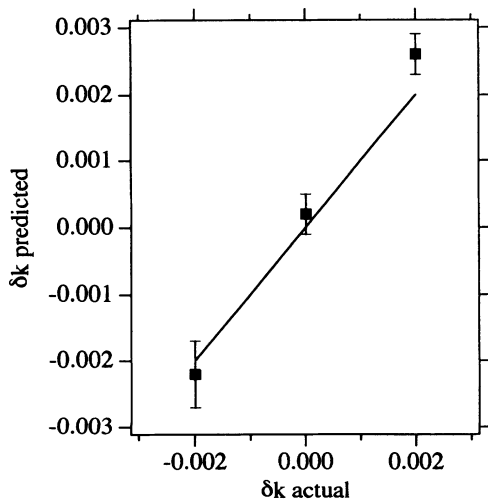


FIG. 4. Predicted versus actual values for $\delta k = k - k^c$ obtained using Eq. (36). A line with slope one is drawn for comparison.

levels of iteration? Here, a hint of trouble begins to appear. At $k = k^c$, the prediction for δk is

$$\delta k = 0.0008(3) \quad (47)$$

comparing the $V = 576$ and $V = 2304$ data at the highest blocking levels. This number is statistically different from zero and signals some kind of trouble. This same difficulty can be seen in the data in Table I, where the $n = 4$ number does not match as well with the $n = 3$ number on the same diagonal as the $n = 3$ and $n = 2$ numbers did. This result is not improved by longer runs or more effort updating the block links. It could be a result of correlations induced in the block spins by the blocking algorithm as discussed earlier. Evidence that the problem is in the spin sector rather than the gravity sector is given by Table II, which gives the behavior of all of the operators blocked to a lattice with volume nine. All of the gravity sector operators match within statistics. So, while the blocking procedure for triangulations appears to be satisfactory, Eq. (47) signals a possible flaw in the blocking procedure for matter that needs to be investigated more carefully.

With this caveat regarding the $n = 4$ spin operators, what about critical exponents? Table III lists estimates for $1/\nu d$ obtained using Eqs. (41) and (44). Since the volume is being rescaled here and not a length scale, Eq. (43) must be rewritten slightly. If the volume rescaling factor is $v = b^d$, then

$$\Lambda_i = v^{\lambda_i/d} \quad (48)$$

so that $1/\nu d$ is obtained from the maximum eigenvalue of the stability matrix using

$$\frac{1}{\nu d} = \frac{\ln \Lambda_{\max}}{\ln v} \quad (49)$$

with $v = 4$. The matrix T_{ij}^* can be truncated with different numbers of operators and those estimates are listed in each column of Table III. The operator order for truncations is O_1, O_3, O_2 , and O_4 . Satisfactory error estimates could not be obtained when O_5 and O_6 were included. Looking at the table, it is immediately clear that there are effects from both the finite size of the lattice (as seen by looking across rows) and the limited number of iterations of the renormalization group transformation. It

TABLE II. Expectation values of a variety of operators (defined in the text) as a function of the initial lattice's volume for n such that the blocked lattices have volume nine in each case.

Operator	$V = 2304, n = 4$	$V = 576, n = 3$	$V = 144, n = 2$
O_1	0.489(2)	0.482(2)	0.481(1)
O_2	0.488(2)	0.482(2)	0.479(1)
O_3	0.306(2)	0.293(2)	0.289(1)
O_4	-0.365(4)	-0.367(2)	-0.368(1)
O_5	0.347(8)	0.355(5)	0.360(2)
O_6	0.20(4)	0.15(1)	0.038(8)
O_7	7.879(5)	7.880(2)	7.867(1)

TABLE III. Estimates for $1/\nu d$ as a function of the volume of the initial lattice, the level n [from Eq. (41)], and the number of operators used in the truncation of the stability matrix. From one to four operators are used and the results are listed down for each blocking level. Errors listed in parentheses were obtained by binning the data into at least twenty bins. Those errors listed in square brackets were obtained using fewer bins.

n	$V = 2304$	$V = 576$	$V = 144$
1	-0.269(3)	-0.068(3)	0.083(2)
	-0.260(4)	-0.080(4)	0.076(2)
2	0.417(2)	0.378(3)	0.342(3)
	0.426(2)	0.403(3)	0.385(5)
	0.455[1]	0.432[3]	0.410[5]
	0.455[1]	0.432[3]	0.410[5]
3	0.493(1)	0.407(3)	
	0.534(2)	0.463(6)	
	0.545(1)	0.476(6)	
	0.545(1)	0.476[5]	
4	0.449(3)		
	0.505(5)		
	0.513(6)		
	0.513[7]		

is normal to have these effects, but they are unusually large here. Compare this data to that from the three-dimensional Ising model as studied in [18]. There, using the regularity and the smallness of the finite-size effects, they are able to give an infinite lattice estimate for each blocking level. Then, giving some thought to the magnitude of the subleading exponent, they are able to extrapolate to a large number of blockings and obtain an estimate of $1/\nu d$ in excellent agreement with others. In their analog of Table III, the difference between the largest and smallest numbers in the table is no more than 30%. Here the large variation precludes such an analysis.

It is possible that with sufficient data on larger lattices a similar procedure would produce the correct result [Eq. (9)] $1/\nu d = 1/3$.

There are systems where finite size effects and effects due to too few blockings are so small that the detailed procedure of [18] is virtually unnecessary. The two-dimensional Ising model in the absence of quantum gravity is an example. By taking the program that produced the previous results and deleting the line that calls the routine that updates the triangulations, it is easy to reproduce the correct pure Ising exponents. For instance, starting with a 24×24 lattice, setting the coupling to the value given by Eq. (4), and using two operators in the truncation of the stability matrix, the sequence of estimates of $1/\nu d$ for $n = 1, 2$, and 3 is, respectively, 0.467(7), 0.51(2), and 0.498(7). These compare favorably with the exact answer [from Eq. (5)], 0.5.

It should be possible to do block spin calculations similar to the one described in this paper, but on arbitrary topologies and with more general volume rescalings. Simply mark points on the original lattice and create a triangulation with them in order to provide an initial block lattice with the desired volume and topology. One way to do this, starting with an initial lattice of volume $V = N$ and genus g , is to knock out (do the inverse of barycentric subdivision on) n threefold coordinated nodes and their associated triangles to produce a volume $V' = N - n$. Once this initial block lattice exists, the simulation can be carried on as above by updating the block lattice according to the geodesic rule, Eq. (45). This defines the renormalized lattice. The matter field is another issue and is likely to be problematic if $V'/V \approx 1$. For $V'/V = 1/p$ with p an integer, a majority rule could be used to average the site and $p - 1$ randomly chosen neighbors.

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