Quantization of 2+1 gravity for genus 2

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In previous papers we established and discussed the algebra of observables for 2+1 gravity at both the classical and quantum levels, and gave a systematic discussion of the reduction of the expected number of independent observables to $6g-6$ $(g>1)$. In this paper the algebra of observables for the case $g=2$ is reduced to a very simple form. A Hilbert space of state vectors is defined and its representations are discussed using a deformation of the Euler Γ function. The deformation parameter θ depends on the cosmological and Planck's constants.

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I. INTRODUCTION

In a previous paper [1] we presented the abstract quantum commutator algebra for 2+1 gravity with cosmological constant Λ:

$$[a_{mk},a_{jl}]=[a_{mj},a_{kl}]=0\ , \tag{1.1}$$

$$[a_{jk},a_{km}]=\left(\frac{1}{K}-1\right)(a_{mk}-a_{jk}a_{mk})\ , \tag{1.2}$$

$$[a_{jk},a_{kl}]=\left(1-\frac{1}{K}\right)(a_{jl}-a_{jk}a_{jk})\ , \tag{1.3}$$

$$[a_{jk},a_{lm}]=\left(K-\frac{1}{K}\right)(a_{jl},a_{km}-a_{kl}a_{jm})\ , \tag{1.4}$$

where $K=(4\alpha-ih)/(4\alpha+ih)=e^{i\theta},\Lambda=1/3\alpha^2$ is the cosmological constant and h is Planck's constant. In (1.1)–(1.4) m,j,l,k are four anticlockwise points of Fig. 1. $m,j,l,k=1,\ldots,n$, and the time-independent quantum operators a_{lk} correspond to the classical $n(n-1)/2$ gauge-invariant trace elements

$$\alpha_{ij}=\alpha_{ji}=\frac{1}{2}\mathrm{Tr}[S(t_it_{i+1}\cdots t_{j-1})]\ ,\quad S\in\mathrm{SL}(2,R)\ . \tag{1.5}$$

For $n=2g+2$ the map S: $\pi_1(\Sigma)\xrightarrow{S}\mathrm{SL}(2,R)$ is defined by the integrated anti–de Sitter connection in the initial data Riemann surface Σ of genus g, and refers to one of the two spinor components, say the upper component, of the spinor group $\mathrm{SL}(2,R)\otimes\mathrm{SL}(2,R)$ of the gauge group SO(2,2) of 2+1 gravity with a negative cosmological constant [2]. The algebra (1.1)–(1.4) is invariant under the quantum action of the mapping class group on traces [1]; the lower component yields an independent algebra of traces b_{ij} identical to (1.1)–(1.4) but with $K\to 1/K$. Moreover $[a_{ij},b_{kl}]=0\ \forall i,j,k,l$. Here we discuss only the upper component. The homotopy group $\pi_1(\Sigma)$ of the surface is defined by generators t_i, $i=1,\ldots,2g+2$ and presentation:

$$t_1t_2\cdots t_{2g+2}=1,\quad t_1t_3\cdots t_{2g+1}=1,\quad t_2t_4\cdots t_{2g+2}=1\ . \tag{1.6}$$

The first relator in (1.6) implies that Σ is closed. The operators in (1.1)–(1.4) are ordered with the convention that $t(a_{ij})$ is increasing from left to right where $t(a_{ij})=[(i-1)(2n-2-i)/2+j-1$.

The case of $g=1$, the torus, has been studied extensively, both in this approach [2], and others [3,4]. In this approach the algebra (1.1)–(1.4) is isomorphic to the quantum algebra of $\mathrm{SU}(2)_q$ when $\Lambda\neq 0$ [2]. For $\Lambda=0$ is has been shown [5] that the metric approach to determining the complex modulus of the torus [3] is classically equivalent to the classical limit of (1.1)–(1.4) for $n=4$. There are similar, recent results for $\Lambda\neq 0$ [6].

For $g>1$ there are very few results apart from those

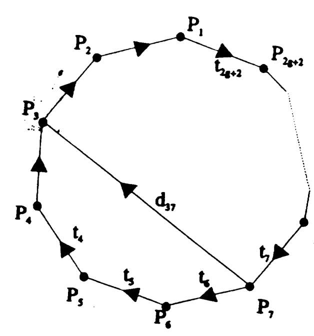

FIG. 1. $2g+2$ polygon representing a Riemann surface of genus g.

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of Moncrief [3] who studies the second-order, metric formalism and achieves very general results. In this paper the case $g = 2$ of the algebra (1.1)–(1.4) is studied in detail. In [7] we determined for $n \leq 6$, i.e., $g \leq 2$ a set of p linearly independent central elements $A_{n,m}$, $m = 1, \ldots, p$ where $n = 2p$ or $n = 2p + 1$, and analyzed the trace identities which follow from the presentation (1.5) of the homotopy group $\pi_1(\Sigma)$ and a set of rank identities. These identities together generate a two-sided ideal. For generic g there are precisely $6g - 6$ independent elements which satisfy the algebra (1.1)–(1.4). The reduction from $n(n-1)/2 = (g+1)(2g+1)$ to $6g - 6$ results from the use of the above-mentioned identities [7] but is highly nonunique. For $g = 2$ the reduction from the original 15 elements a_{ij} to six independent elements has been the subject of a long study. Here this reduction is implemented explicitly in terms of a set of six independent operators which satisfy a particularly simple algebra. There are many such possibilities but a convincing set is described as follows:

We group the vertices of the hexagon into three sectors, see Fig. 2, the vertices labeled $2b$ and $2b-1$ belonging to the sector b, $b = 1, \ldots, 3$. Accordingly, we define the sector function $s[2b] = s[2b-1] = b$. A convenient choice for the six independent elements is given by three commuting angles $\varphi_{-b} = -\varphi_b, b = \pm 1, \ldots, \pm 3$ defined by

$$a_{2b-1,2b} = \frac{\cos\varphi_b}{\cos\frac{\theta}{2}}, \quad b = 1, \ldots, 3 , \tag{1.7}$$

and commuting operators M_{ab} with the properties

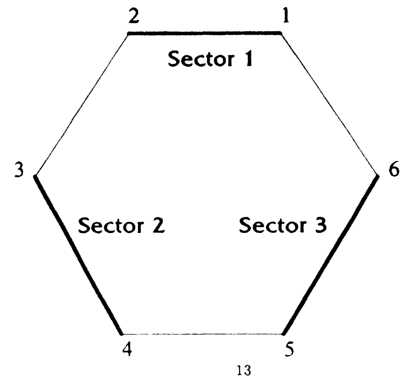

FIG. 2. Hexagon representing a $g = 2$ Riemann surface, with three commuting sectors.

$$M_{ab} = M_{ba}, \quad a, b = \pm 1, \ldots, \pm 3 ,$$
$$M_{a,-a} = 1, \ M_{a,-b}M_{b,c} = M_{ac} . \tag{1.8}$$

The M_{ab} act as raising and lowering operators on the φ_a:

$$M_{\pm a,b\varphi a} = (\varphi_a \mp \theta)M_{\pm a,b} . \tag{1.9}$$

It can be checked that the 12 remaining a_{ik} are represented by

$$a_{kj} = \frac{1}{K+1} K^{\frac{\tilde{k}+\tilde{j}}{2}+1} \sum_{n,m=\pm 1} \exp\{-i[n(\tilde{k}+1)\varphi_a + m\tilde{j}\varphi_b]\} \frac{\sin(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2})\sin(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2})}{\sin(n\varphi_a)\sin(m\varphi_b)} M_{na,mb} , \tag{1.10}$$

where we set $\tilde{k} = k \bmod 2 + \frac{1}{2}$ and $a = s(k), b = s(j), a, b, c$ in cyclical order. Under these conditions the a_{ik} satisfy the trace and rank identities. These identities can all be derived from

$$a_{12}a_{34} + K^{-2}a_{23}a_{14} - K^{-1}a_{13}a_{24} - (1 + K^{-2} - K^{-1})a_{56} = 0 \tag{1.11}$$

by repeated commutation with the elements of the algebra (1.1)–(1.4). For example two useful identities are

$$K^3 a_{12}a_{46} + K a_{24}a_{16} - (1 + K^3)a_{34}a_{45} - K^2 a_{14}a_{26} + (1 - K + K^2)a_{35} = 0 , \tag{1.12}$$

$$(1 + K^3)[(1 + K)a_{34}a_{56}a_{45} - K a_{34}a_{46} - a_{56}a_{35}] + K^2 a_{14}a_{25} - K^3 a_{12}a_{45} - K a_{24}a_{15} + K(1 + K^2 - K)a_{36} = 0 ,$$

and their images under cyclical permutations of the indices $1, \ldots, 6$. These identities are certainly not all independent. By heavy use of computer algebra we were able to show that (1.11) and (1.12) and their images follow from (1.7)–(1.10).

The relations (1.8) and (1.9) follow from the single sector factorization for all $a, b = \pm 1, \ldots, \pm 3$:

$$M_{ab} = M_a M_b = M_{ba} = M_b M_a . \tag{1.13}$$

$$M_{-a} = M_a^{-1} . \tag{1.14}$$

$$M_{\pm a}\varphi_a = (\varphi_a \mp \theta)M_{\pm a} . \tag{1.15}$$

It is clear that (1.13)–(1.15) can be formally satisfied by setting $M_a = \exp(-\theta\frac{\partial}{\partial\varphi_a})$ and therefore $M_{ab} = \exp[-\theta(\frac{\partial}{\partial\varphi_a} +$

$\frac{\partial}{\partial\varphi_b})]$, in turn (1.10) becomes

$$a_{kj} = \frac{1}{2\cos(\frac{\theta}{2})} \sum_{n,m=\pm1} \frac{\sin(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}) \sin(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2})}{\sin n\varphi_a \sin m\varphi_b} \exp\left(- i[n(\tilde{k}+1)\varphi_a + m\tilde{j}\varphi_b + \theta(np_a + mp_b)]\right) , \tag{1.16}$$

where we have used the Baker-Hausdorff formula [8]

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{AB - BA}{2}\right) = \exp(AB - BA)\exp(B)\exp(A) \tag{1.17}$$

valid when $AB - BA$ is a c number and $M_a = \exp(-i\theta p_a)$. Note that, from (1.7) and (1.16), all of the 15 original a_{ij} are expressed in terms of the three angles φ_a and their conjugate momenta p_a.

The treatment of (1.16) can be further simplified by noting that $a_{kj} = U_{\tilde{k}\tilde{j}}^{-1} A_{ab} U_{\tilde{k}\tilde{j}}$ where

$$U_{\tilde{k}\tilde{j}} = \exp\left(i\frac{(\tilde{k}+1)\varphi_a^2 + \tilde{j}\varphi_b^2}{2\theta}\right) , \tag{1.18}$$

$$A_{ab} = \frac{1}{2\cos(\frac{\theta}{2})} \sum_{n,m=\pm1} \frac{\sin(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}) \sin(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2})}{\sin n\varphi_a \sin m\varphi_b} \exp[-i\theta(np_a + mp_b)] .$$

A_{ab} is an operator which is a function of the sectors a, b only and is independent of the position of k, j within a, b. It can alternatively be expressed as

$$A_{ab} = \cos(\theta p_a)\cos(\theta p_b) + \left(\cot\varphi_a \cot\varphi_b - \frac{\cos\varphi_c}{\cos(\frac{\theta}{2})\sin\varphi_a \sin\varphi_b}\right)\sin(\theta p_a)\sin(\theta p_b)$$
$$-i\tan\left(\frac{\theta}{2}\right)[\cot\varphi_b \cos(\theta p_a)\sin(\theta p_b) + \cot\varphi_a \cos(\theta p_b)\sin(\theta p_a)].$$

The discussion of the representations of (1.13)–(1.15) is considerably simplified by the introduction of the deformed Euler gamma function $\Gamma(z,\theta)$ (see Appendix for the definition and a list of properties) which extends to the complex domain the symbol:

$$[n]! = \prod_{p=1}^{n} \frac{\sin\frac{p\theta}{2}}{\sin\frac{\theta}{2}} . \tag{1.19}$$

In particular $\Gamma(n+1,\theta) = [n]!$ and $\Gamma(z,\theta)$ is a meromorphic analytic function of z with poles at $z = -s-(2\pi r/\theta)$, $s, r \geq 0$, and integer and zeros at $z = s+(2\pi r/\theta)$, $s, r \geq 1$ and integer.

II. REPRESENTATIONS

The a_{ij} expressed by (1.7) are by definition all Hermitian operators. We denote by ϕ_a the generic eigenvalue of the operator φ_a and set $\phi = \{\phi_1, \phi_2, \phi_3\}$, $z_a = \cos\phi_a$, $z = \{z_1, z_2, z_3\}$ where the z_a are real and restricted to a domain $D^3 \subset R^3$. Let T with $T^2 = 1$ be the antilinear conjugacy operator $\Psi(z) \xrightarrow{T} \Psi^*(z)$. A measure $\sigma(z)d^3z$ with $\sigma(z) \geq 0$ and real turns H into a Hilbert space H with the norm

$$|\Psi|^2 = \int_{D^3} |\Psi(z)|^2 \sigma(z) d^3z . \tag{2.1}$$

The weight function $\sigma(z)$ can be determined from the Hermiticity of the a_{ij} (1.7) as follows.

Let $p_a = -i(\partial/\partial\phi_a)$, $a = 1,2,3$ satisfying the canonical commutation relations (CCR)

$$[\varphi_a, \varphi_b] = 0, \quad [\varphi_a, p_b] = i\delta_{ab},$$
$$[p_a, p_b] = 0, \quad a, b, = 1,2,3 ; \tag{2.2}$$

it follows by conjugation that

$$[\varphi_a^\dagger, \varphi_b^\dagger] = 0, \quad [\varphi_a^\dagger, p_b^\dagger] = i\delta_{ab},$$
$$[p_a^\dagger, p_b^\dagger] = 0, \quad a, b, = 1,2,3 , \tag{2.3}$$

but also that

$$\varphi_a^\dagger = T\varphi_a T, \quad a_{2a,2a-1} = Ta_{2a,2a-1}T, \quad a = 1,2,3 . \tag{2.4}$$

The Hermiticity relation between $\mathcal{O}, \mathcal{O}^\dagger$, namely, $\langle\Psi, \mathcal{O}\Phi\rangle = \langle\mathcal{O}^\dagger\Psi, \Phi\rangle$ implies $(-i\frac{\partial}{\partial z_a})^\dagger = -i\sigma^{-1}\frac{\partial}{\partial z_a}\sigma$. But $\frac{\partial}{\partial z_a} = \frac{-1}{\sin\phi_a}\frac{\partial}{\partial\phi_a}$ whereby

$$p_a^\dagger = \left(-i\frac{\partial}{\partial\phi_a}\right)^\dagger = \left(i\sin\phi_a\frac{\partial}{\partial z_a}\right)^\dagger = i\sigma^{-1}\frac{\partial}{\partial z_a}\sigma\sin\phi_a^\dagger$$
$$= i\sigma^{-1}\frac{\partial}{\partial z_a}\sigma T\sin\phi_a T = -iT\sigma^{-1}\frac{\partial}{\partial z_a}\sigma\sin\phi_a T = -T\rho^{-1}p_a\rho T \ , \tag{2.5}$$

where $\rho(\phi) = C\sin\phi_1\sin\phi_2\sin\phi_3\sigma(z)$, C being a normalization constant, the operator $\rho = \rho(\varphi_1,\varphi_2,\varphi_3) = \rho(\varphi)$ is now to be determined by extending (2.4) to all i,k as $a_{ik} = a_{ik}^\dagger = Ta_{ik}T$.

From (1.16)–(1.18) we obtain, by conjugation,

$$a_{kj} = \frac{1}{2\cos(\frac{\theta}{2})}U_{\tilde{k}\tilde{j}}^\dagger\sum_{n,m=\pm1}\exp\left(i\theta(np_a^\dagger + mp_b^\dagger)\right)\frac{\sin\left(\frac{\theta}{4}+\frac{n\varphi_a^\dagger+m\varphi_b^\dagger+\varphi_c^\dagger}{2}\right)\sin\left(\frac{\theta}{4}+\frac{n\varphi_a^\dagger+m\varphi_b^\dagger-\varphi_c^\dagger}{2}\right)}{\sin n\varphi_a^\dagger\sin m\varphi_b^\dagger}U_{\tilde{k}\tilde{j}}^{-1,\dagger} \ . \tag{2.6}$$

We apply now (1.17) and reorder the operators in (2.6) by bringing the exponential factor to the right thus finding

$$a_{kj} = \frac{1}{2\cos(\frac{\theta}{2})}U_{\tilde{k}\tilde{j}}^\dagger\sum_{n,m=\pm1}\frac{\sin\left(\frac{5\theta}{4}+\frac{n\varphi_a^\dagger+m\varphi_b^\dagger+\varphi_c^\dagger}{2}\right)\sin\left(\frac{5\theta}{4}+\frac{n\varphi_a^\dagger+m\varphi_b^\dagger-\varphi_c^\dagger}{2}\right)}{\sin(n\varphi_a^\dagger+\theta)\sin(m\varphi_b^\dagger+\theta)}\exp[i\theta(np_a^\dagger + mp_b^\dagger)]U_{\tilde{k}\tilde{j}}^{-1,\dagger}$$

and

$$a_{kj} = \frac{1}{2\cos(\frac{\theta}{2})}TU_{\tilde{k}\tilde{j}}^{-1}\sum_{n,m=\pm1}\frac{\sin\left(\frac{5\theta}{4}+\frac{n\varphi_a+m\varphi_b+\varphi_c}{2}\right)\sin\left(\frac{5\theta}{4}+\frac{n\varphi_a+m\varphi_b-\varphi_c}{2}\right)}{\sin(n\varphi_a+\theta)\sin(m\varphi_b+\theta)}\exp[i\theta(np_a + mp_b)]U_{\tilde{k}\tilde{j}}T \ . \tag{2.7}$$

We define the maps

$$\phi_a,\phi_b,\phi_c \overset{\Delta(na,mb)}{\longrightarrow} \phi_a+n\theta,\phi_b+m\theta,\phi_c \ , \tag{2.8}$$

where as before n,m takes all values ±1 and a,b,c are any permutation of 1,2,3. From $a_{ik} = a_{ik}^\dagger = Ta_{ik}T$ and by comparing (2.7) with (1.16) we find the recursion relation in the eigenvalues ϕ, z:

$$\Delta(na,mb)\sigma(z_1,z_2,z_3) = \sigma(z_1,z_2,z_3)\frac{\sin\left(\frac{\theta}{4}-\frac{n\varphi_a+m\varphi_b+\varphi_c}{2}\right)\sin\left(\frac{\theta}{4}-\frac{n\varphi_a+m\varphi_b-\varphi_c}{2}\right)}{\sin\left(\frac{5\theta}{4}+\frac{n\varphi_a+m\varphi_b+\varphi_c}{2}\right)\sin\left(\frac{5\theta}{4}+\frac{n\varphi_a+m\varphi_b-\varphi_c}{2}\right)} \ . \tag{2.9}$$

A solution of (2.9) is then provided by

$$\sigma(z_1,z_2,z_3) = P(\phi)\prod_{m_1m_2m_3=\pm1}\Gamma\left(-\frac{1}{4}+\frac{q\pi}{\theta}+\frac{m_1\phi_1+m_2\phi_2+m_3\phi_3}{2\theta},2\theta\right) \ , \tag{2.10}$$

where q is arbitrary and integer and $P(\phi)$ is invariant under (2.8), otherwise arbitrary, in (2.10) the product is carried on all independent sign choices of m_1,m_2,m_3.

By using (A7) we see that

$$E(\phi,\theta,q+1) = S(\phi)E(\phi,\theta,q) \ ,$$

where

$$S(\phi) = 2^8(2\sin\theta)^{-8\pi/\theta}\prod_{m_1,m_2,m_3=\pm1}\sin\pi\left(-\frac{1}{4}+\frac{q\pi}{\theta}+\frac{m_1\phi_1+m_2\phi_2+m_3\phi_3}{2\theta}\right) \ ,$$

$$E(\phi,\theta,q) = \prod_{m_1,m_2,m_3=\pm1}\Gamma\left(-\frac{1}{4}+\frac{q\pi}{\theta}+\frac{m_1\phi_1+m_2\phi_2+m_3\phi_3}{2\theta},2\theta\right) \ . \tag{2.11}$$

Since $S(\phi)$ is invariant under (2.8) it can be absorbed into $P(\varphi)$ hence the appearance of q does not signal any new arbitrariness. It is, however, convenient in our discussion to have a solution which depends explicitly on q.

The function $\rho(\phi)$ is periodic of period 2π and odd in ϕ_1,ϕ_2,ϕ_3 if we have [see (2.10) and (A7)]

$$\frac{\rho(\phi_1+2\pi,\phi_2,\phi_3)}{\rho(\phi_1,\phi_2,\phi_3)} = \frac{P(\phi_1+2\pi,\phi_2,\phi_3)}{P(\phi_1,\phi_2,\phi_3)}\prod_{m_2m_3=\pm1}\frac{\sin\pi\left(-\frac{1}{4}+\frac{q\pi}{\theta}+\frac{\phi_1+m_2\phi_2+m_3\phi_3}{2\theta}\right)}{\sin\pi\left(\frac{1}{4}-\frac{(q-1)\pi}{\theta}+\frac{\phi_1+m_2\phi_2+m_3\phi_3}{2\theta}\right)} = 1 \ .$$

This can be achieved by setting

$$\theta = \frac{2q-1}{2t+1}2\pi, \quad t \text{ is an integer} , \tag{2.12}$$

and $P(\varphi) = 1$.

We list here the basic properties of $E(\phi, \theta, q)$.

(1) $E(\phi, \theta, q)$ is even in each of the ϕ_1, ϕ_2, ϕ_3.

(2) $E(\phi, \theta, q)$ is periodic of period 2π in each of the ϕ_1, ϕ_2, ϕ_3.

(3) $E(\phi, \theta, q)$ is real but not necessarily positive for ϕ_1, ϕ_2, ϕ_3 all real. It follows by analytic continuation that $E(\phi^\dagger, \theta, q) = E(\phi, \theta, q)^\dagger$.

(4) $E(\phi, \theta, q)$ is real and positive if at least one of the ϕ_1, ϕ_2, ϕ_3 is imaginary and the others real. This follows from the possibility of arranging (2.11) in pairs of conjugate factors.

In this case we may choose $\sigma(z) = E(\phi, \theta, q)$. The discussion of the positivity of the function $\sigma(z)$ for arbitrary z is rather involved. A particular solution is provided by restricting all z_a to the hyperbolic domain $z_a > 1$, i.e., all ϕ_a pure imaginary. In this case all a_{kj} from (1.7) and (1.16) are represented by unbounded Hermitian operators. This, and the inclusion of the other SL$(2, R)$ component, will be discussed elsewhere [9].

APPENDIX

Here we give the definition and a comprehensive list of properties of the deformed Euler Γ function:

$$\Gamma(z,\theta) = \left(\frac{\theta}{2\sin\frac{\theta}{2}}\right)^{z-1} \Gamma(z) \prod_{n=1}^{\infty} \left(\frac{\theta}{2\pi n}\right)^{2z-1} \frac{\Gamma(z+\frac{2\pi n}{\theta})}{\Gamma(1-z+\frac{2\pi n}{\theta})}, \quad |\arg\theta| < \pi , \tag{A1}$$

$$\lim_{\theta\to 0} \Gamma(z,\theta) = \Gamma(z) , \tag{A2}$$

$$\Gamma(1,\theta) = 1 , \tag{A3}$$

$$\Gamma(z+1,\theta) = \Gamma(z,\theta)\frac{\sin\frac{\theta z}{2}}{\sin\frac{\theta}{2}}, \quad \Gamma(n+1,\theta) = [n]!, \quad n \text{ is an integer} > 0 , \tag{A4}$$

$$\Gamma\left(z+\frac{2\pi}{\theta},\theta\right) = 2\sin(\pi z)\left(2\sin\frac{\theta}{2}\right)^{-2\pi/\theta} \Gamma(z,\theta) , \tag{A5}$$

$$\Gamma(z,\theta)\Gamma(1-z,\theta) = \frac{2\pi\sin\frac{\theta}{2}}{\theta\sin(\pi z)} , \tag{A6}$$

$$\Gamma(z,\theta)\Gamma\left(\frac{2\pi}{\theta}-z,\theta\right) = \frac{\pi}{\theta\sin\frac{\theta z}{2}}\left(2\sin\frac{\theta}{2}\right)^{2-2\pi/\theta} , \tag{A7}$$

$$\Gamma(z,\theta)\Gamma\left(1+\frac{2\pi}{\theta}-z,\theta\right) = \frac{2\pi}{\theta}\left(2\sin\frac{\theta}{2}\right)^{1-2\pi/\theta} . \tag{A8}$$

Setting $\theta' = 4\pi^2/\theta$ we have the duality property

$$\Gamma(z\cdot\theta) = \Gamma\left(\frac{\theta z}{2\pi},\theta'\right)\left(2\sin\frac{\theta'}{2}\right)^{z\theta/2\pi-1}\left(2\sin\frac{\theta}{2}\right)^{1-z}\frac{\theta'}{2\pi} . \tag{A9}$$

Equation (A9) is meaningless in the limit $\theta \to 0$ and therefore the standard Euler gamma function $\Gamma(z)$ has no dual symmetry. From (A9) it follows that the function $\Gamma(z,a,b) = a\Gamma[bz,(2\pi a/b)][2\sin(\pi a/b)]^{bz-1}$ is symmetrical, i.e., $\Gamma(z,a,b) = \Gamma(z,b,a)$. Duality exchanges (A4) with (A5) and (A6) with (A7).

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