PHYSICAL REVIEW D VOLUME 50, NUMBER 8 15 OCTOBER 1994

Tunneling geometries: Analyticity, unitarity, and instantons in quantum cosmology

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(Received 20 August 1993; revised manuscript received 13 June 1994)

We present the theory of tunneling geometries, which describes in the language of analytic continuation the nucleation of the Lorentzian universe from the Euclidean spacetime. We reformulate the underlying no-boundary wave function in the manifestly unitary representation of true physical variables and calculate it in the one-loop approximation. For this purpose a special technique of complex extremals is developed, which reduces the formalism of complex tunneling geometries to real ones, and also the method of collective variables is applied, separating the macroscopic degrees of freedom from the perturbative microscopic modes. The quantum distribution of Lorentzian universes on the space of collective variables incorporates the probability conservation and boils down to the partition function of quasi-de Sitter gravitational instantons weighted by their Euclidean effective action. The over-Planckian behavior of their distribution is determined by the anomalous scaling of the theory, which serves as a criterion for the high-energy normalizability of the cosmological wave function and the validity of the semiclassical expansion. It also provides a calculational scheme for obtaining the quantum scale of inflation which was recently shown to establish a sound link between quantum cosmology, inflation theory, and particle physics in the model with a nonminimally coupled inflaton field.

PACS number(s): 04.60.Ds, 04.62.+v, 98.80.Hw

I. INTRODUCTION

Quantum cosmology of the past decade developed essentially in the framework of the new paradigm – the origin of the inflationary universe from the two conceptually similar quantum states: no-boundary [1,2] and tunneling [3,4]. Their invention was followed by a great uprise of interest in the Euclidean quantum gravity in connection with the ideas of the third quantization and the Coleman theory of the cosmological constant [5]. However, an extensive accumulation of applications in this field (see the bibliography in Ref. [6]), did not raise the scope of the gravitational tunneling beyond the semiclassical level. In particular, such a problem as a generation of chaotic inflationary cosmologies [7] at a preferable grand unified theory (GUT) scale of the effective Hubble constant (necessary for observationally approved applications of inflation theory) remained open, having a negative solution in the tree-level approximation. Another important issue, the validity of the semiclassical expansion, also remained negative, because neither no-boundary nor tunneling wave functions are normalizable in the tree-level approximation at over-Planckian energy scales [8,9], and special assumptions are necessary to establish a Planckian "boundary" [10] to protect semiclassical inflation physics from the nonperturbative realm of quantum gravity.

The key to the solution of these problems, not resorting to the conjectures on a hypothetical over-Planckian phase of the theory, may consist in the semiclassical $\hbar$ expansion and the search for mechanisms that could justify this expansion. Despite the perturbative nonrenormalizability of quantum gravity, this approach makes sense in problems with quantum states peaked at sub-Planckian energies. In particular, it would work in quantum cosmology with the no-boundary or tunneling wave functions, provided they suppress the contribution of Planckian energies and generate the probability peaks at the lower (preferably GUT) scale compatible with microwave background observations [11]. As shown in authors' papers [12–14], the quantum gravitational corrections for tunneling geometries really allow one to reach this goal and, as a by-product, work out a selection criterion for viable particle-physics models. One can get a sharp probability peak with characteristic parameters of GUT and, thus, provide a numerically sound link between quantum cosmology, inflation theory, and the particle physics of the early universe [14]. Thus, the goal of this work is to present a formalism of tunneling geometries used in these papers.

A simple picture of tunneling geometry, demonstrating the purposes of this work, is shown in Fig. 1. The de Sitter solution with the cosmological constant $\Lambda = 3H^2$,

$$ds_L^2 = -dt^2 + a_L^2(t)\, c_{ab}\, dx^a dx^b, \tag{1.1}$$

$$a_L(t) = \frac{1}{H}\cosh(Ht), \tag{1.2}$$

describes the expansion of a spherical hypersurface with a round three-metric $a_L^2(t)\, c_{ab}$ and the scale factor $a_L(t)$. Its Euclidean counterpart with the positive-signature de Sitter metric

$$ds^2 = g^{\rm dS}_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + a^2(\tau)\, c_{ab}\, dx^a dx^b, \tag{1.3}$$

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$$a(\tau) = \frac{1}{H}\sin(H\tau), \tag{1.4}$$

describes the geometry of the four-dimensional sphere of radius $R = 1/H$ with spherical three-dimensional sections labeled by the latitude angle $\theta = H\tau$. Both metrics are related by the analytic continuation into the complex plane of the Euclidean "time" τ [15,16]

$$\tau = \pi/2H + it, \quad a_L(t) = a(\pi/2H + it). \tag{1.5}$$

This analytic continuation can be interpreted as a quantum nucleation of the Lorentzian spacetime from the Euclidean one and shown on Fig. 1 as a matching of the two manifolds (1.1)–(1.4) across the equatorial section $\tau = \pi/2H$ $(t = 0)$ – the bounce surface Σ_B.

The first difficulty with this simple picture is that this idea was never pushed beyond the tree-level approximation. At best, the quantum fields were considered on a completely classical tunneling background [17–19]. The origin of this difficulty can be generally formulated as a controversy between covariance and unitarity. The covariant Euclidean quantum gravity has powerful algorithms for the calculation of the loop corrections, but it is not closed as a self-contained physical theory, for it lacks good principles of associating the Euclidean amplitudes and, especially, their loop part with physical states, the physical inner products for the latter being not defined, etc. As a counterpart to it, there exists the quantization in physical [Arnowitt-Deser-Misner (ADM)] variables [20], which is equivalent in Lorentzian spacetime (at least perturbatively) to the Dirac-Wheeler-DeWitt scheme [21–24] and can be used for constructing the theory of tunneling geometries extending beyond the tree level.

The second difficulty is that this picture is applicable only to a limited class of *real* tunneling geometries [25,26]. In these problems the metric and matter fields are real on both the Lorentzian and Euclidean parts of spacetime and smoothly match across the nucleation surface with a vanishing extrinsic curvature and vanishing normal derivative of matter fields. This requirement is very strong and in many applications cannot be satisfied. The most important example is the chaotic inflation [2] driven by the effective Hubble constant $H(\varphi)$ which is generated by the inflaton scalar field φ. In this model the spacetime geometry is approximately described by Eqs. (1.1)–(1.4) with $H = H(\varphi)$. For large $H(\varphi)$ the scalar field is nearly conserved in time, but its derivative never exactly vanishes for solutions satisfying appropriate boundary conditions. Therefore, no nucleation surface exists at which a *real* Euclidean solution can be smoothly matched to the *real* Lorentzian one [27]. This is a general case in contrast to a rather narrow class of real tunneling geometries [28].

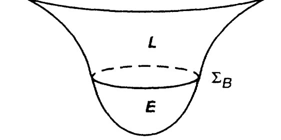

FIG. 1. Graphical representation of the Lorentzian spacetime L nucleating at the bounce surface Σ_B from the Euclidean manifold E of the no-boundary type, having the topology of the four-dimensional ball.

Here we undertake several steps toward the resolution of the above difficulties. Our starting point, partly formulated in [23,29], will be a manifestly unitary gravity theory in Lorentzian spacetime. This theory produces by certain analytic continuation the Euclidean quantum gravity – the auxiliary tool for describing the classically forbidden states of the gravitational field [30,31]. The necessity of such a continuation originates from the fact that families of classical solutions may have caustics. In order to extend these solutions beyond caustics, one has to continue them analytically into the complex plane of time. The structure of the caustic surfaces for the Wheeler-DeWitt equations will be considered in Ref. [32], while here we develop elements of the general theory and the one-loop approximation for quantum systems penetrating beyond these caustics, which we shall call the tunneling geometries.

In Sec. II we present the Euclidean quantum gravity as an analytic continuation from its Lorentzian counterpart. In Sec. III we discuss this analytic continuation for the no-boundary wave function $\Psi(q,t)$ in the representation of true physical variables q. It is intrinsically noncovariant, but this disadvantage is justified by a simplicity of its inner product

$$\langle\Psi_1 \,|\, \Psi_2\rangle = \int dq\, \Psi_1^*(q,t)\, \Psi_2(q,t). \tag{1.6}$$

The one-loop approximation for $\Psi(q,t)$ is also simple in this representation for it involves the functional determinant of the wave operator only on the space of physical modes. We use a reduction method of Ref. [33] to convert this determinant to a special form which, together with (1.6), is used later for the proof of unitarity. Section IV deals with the separation of physical fields into the collective variables and microscopic modes treated perturbatively. Section V contains the method generalizing the semiclassical expansion to the case of complex extremals. Its application in Sec. VI shows that the complex metric does not prevent from interpreting the no-boundary wave function as a special Euclidean vacuum of physical modes. Section VII presents the quantum distribution function for tunneling geometries on the space of collective variables, which boils down to the distribution of the quasispherical gravitational instantons weighted by their Euclidean *effective* action. It establishes the link between the noncovariant manifestly unitary Lorentzian theory with its covariant Euclidean counterpart – one of the main purposes of this paper. In Sec. VIII we present the first application of this theory, briefly reported in Refs. [12,23] – the over-Planckian behavior of the distribution function for the chaotic [7,2] inflationary cosmologies. In contrast to the tree-level approximation [8,9], it can suppress the over-Planckian energies, depending on the anomalous scaling of the Euclidean theory, and generate a sharp probability peak compatible with the needs of inflation theory and its observational status [14].

II. EUCLIDEAN QUANTUM GRAVITY AS AN ANALYTIC CONTINUATION FROM ITS LORENTZIAN COUNTERPART

Let us denote the collection of the three-dimensional metric $g_{ab}(\mathbf{x})$ and matter fields $\phi(\mathbf{x})$ in the canonical quantization of gravity by

$$\boldsymbol{q} = (g_{ab}(\mathbf{x}),\, \phi(\mathbf{x})). \tag{2.1}$$

Then, the transition amplitude from the configuration $\boldsymbol{q}_-$ at spatial hypersurface Σ_- to the configuration $\boldsymbol{q}_+$ at Σ_+ is given by the path integral over Lorentzian spacetime geometries and histories of matter fields $\boldsymbol{g} = (g_{\mu\nu}(\mathbf{x},t),\, \phi(\mathbf{x},t))$ interpolating between Σ_- and Σ_+

$$\boldsymbol{K}(\boldsymbol{q}_+,\boldsymbol{q}_-) = \int D\boldsymbol{\mu}[\boldsymbol{g}]\, e^{\frac{i}{\hbar}\boldsymbol{S}[\boldsymbol{g}]}. \tag{2.2}$$

Here $\boldsymbol{S}[\boldsymbol{g}]$ is the gravitational action in spacetime domain sandwiched between the hypersurfaces Σ_- and Σ_+, which takes the form

$$\boldsymbol{S}[\boldsymbol{g}] = \int_{t_-}^{t_+} dt\, \boldsymbol{\mathcal{L}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, N) \tag{2.3}$$

of the time integral with the Lagrangian $\boldsymbol{\mathcal{L}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, N)$ when spacetime is foliated by the t-parameter family of spatial surfaces, so that $\boldsymbol{g} = (\boldsymbol{q}(t), N(t))$ is decomposed into spatial three-metric and matter fields (2.1) and lapse and shift functions $N(t) = (N^\perp(\mathbf{x},t),\, N^a(\mathbf{x},t))$.

The integration measure $D\boldsymbol{\mu}[\boldsymbol{g}]$ in (2.2) includes the Faddeev-Popov gauge fixing with the corresponding ghost contribution and implies integration over the histories $\boldsymbol{g} = (\boldsymbol{q}(t), N(t))$ matching the fixed fields $\boldsymbol{q}(t_\pm) = \boldsymbol{q}_\pm$ at $\Sigma_\pm$. Lapse and shift functions N are integrated over at the boundary surfaces and, therefore, do not enter the arguments of the transition kernel (2.2) and provide its $t_\pm$ independence [34].

The analytic continuation of the real Lorentzian quantum gravity implies that the time variable and integration fields become complex valued, and the expression (2.2) goes over into the integral over some complex contour $\boldsymbol{C}$ in the configuration space of the theory

$$\boldsymbol{K}(\boldsymbol{q}_+,\boldsymbol{q}_-) = \int_{\boldsymbol{C}} D\boldsymbol{\mu}[\boldsymbol{\Phi}]\, e^{-\frac{1}{\hbar}\boldsymbol{\mathcal{I}}[\boldsymbol{\Phi}(z)]}, \tag{2.4}$$

where z, $\boldsymbol{\Phi}(z)$ and $\boldsymbol{\mathcal{I}}[\boldsymbol{\Phi}(z)]$ are the results of such a twofold analytic continuation of the time, configuration-space fields and their gravitational action (2.3)

$$t \to z, \quad \boldsymbol{g}(t) \to \boldsymbol{\Phi}(z) = (\boldsymbol{q}(z),\, N_E(z)),$$
$$-i\boldsymbol{S}[\boldsymbol{g}] \to \boldsymbol{\mathcal{I}}[\boldsymbol{\Phi}(z)]$$
$$= \int_C dz\, \boldsymbol{\mathcal{L}}_E(\boldsymbol{q}(z), d\boldsymbol{q}(z)/dz, N_E(z)). \tag{2.5}$$

Here C is a continuous curve in the complex plane of time variable, and the complex Lagrangian $\boldsymbol{\mathcal{L}}_E(\boldsymbol{q}, d\boldsymbol{q}/dz, N_E)$ is related to the original Lagrangian by the equation

$$\boldsymbol{\mathcal{L}}_E(\boldsymbol{q}, d\boldsymbol{q}/d\tau, N_E) = -\boldsymbol{\mathcal{L}}(\boldsymbol{q}, i d\boldsymbol{q}/d\tau, N), \tag{2.6}$$
$$N \equiv (N^\perp, N^a) = (N_E^\perp,\, iN_E^a), \tag{2.7}$$

for arbitrary functions $\boldsymbol{q} = \boldsymbol{q}(\tau)$ and $N_E = (N_E^\perp(\tau),\, N_E^a(\tau))$.

The quantities labeled by E denote the objects in the Euclidean spacetime having τ as a time variable. In particular, Eq. (2.6) incorporates the Wick rotation which basically boils down to multiplying the velocities and lapse functions by i. This procedure makes a formal transition from the Lorentzian metric ds_L^2 to the metric of Euclidean spacetime ds_E^2:

$$ds_L^2 = -N^2dt^2 + g_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \tag{2.8}$$
$$ds_E^2 = N^2d\tau^2 + g_{ab}(dx^a + N_E^a\, d\tau)(dx^b + N_E^b\, d\tau), \tag{2.9}$$

the latter being foliated by surfaces of constant τ. One can regard z as a variable in a complex plane of the Euclidean time τ so that its imaginary part coincides with t and $dz = d\tau + i\,dt$ in the integral (2.5) over arbitrary contour C.

We shall consider two choices of such a contour C:

$$C_L : \{z = it,\, \mathrm{Im}\, t = 0,\, t_- \le t \le t_+\}, \tag{2.10}$$
$$C_E : \{z = \tau,\, \mathrm{Im}\, \tau = 0,\, \tau_- \le \tau \le \tau_+\}. \tag{2.11}$$

The Lorentzian contour C_L generates the Lorentzian gravitational action (2.3) provided the restriction of $\boldsymbol{\Phi}(z)$ to this contour gives real-valued variables of the Lorentzian gravity theory $\boldsymbol{g}(t)$: $-i\boldsymbol{S}[\boldsymbol{g}(t)] = \boldsymbol{\mathcal{I}}[\boldsymbol{\Phi}(z)]|_{C_L}, \boldsymbol{\Phi}(z)|_{C_L} = \boldsymbol{g}(t)$. The Euclidean contour C_E gives rise to the Euclidean action corresponding to the metric (2.9), $\boldsymbol{I}[\boldsymbol{g}(\tau)] = \boldsymbol{\mathcal{I}}[\boldsymbol{\Phi}(z)]|_{C_E}, \boldsymbol{\Phi}(z)|_{C_E} = \boldsymbol{g}(\tau)$, and generates the basic path integral of the Euclidean quantum gravity.

A. The no-boundary proposal

In contrast to the transition amplitude (2.2), the wave function has one argument associated with the spacelike hypersurface to which the quantum state is ascribed. It can be obtained from (2.2) or (2.4) by shrinking Σ_- to a point and integrating over all physical fields regular at this point. In the no-boundary proposal the cosmological wave function is constructed by integrating over Euclidean geometries and matter fields $\boldsymbol{g}$ on spacetime $\boldsymbol{M}$ which has a topology of a compact four-dimensional ball $\boldsymbol{B}^4$ bounded by a three-dimensional hypersurface Σ_+ with the boundary fields $\boldsymbol{q}_+$:

$$\boldsymbol{\Psi}(\boldsymbol{q}_+) = \int_{\boldsymbol{C}} D\boldsymbol{\mu}[\boldsymbol{g}]\, e^{-\frac{1}{\hbar}\boldsymbol{I}[\boldsymbol{g}]}. \tag{2.12}$$

The Euclidean gravitational action $\boldsymbol{I}[\boldsymbol{g}]$ in this equation is a particular case of (2.5) corresponding to the Euclidean contour C_E (2.11) with $\tau_- = 0$

$$\boldsymbol{I}[\boldsymbol{g}] = \int_0^{\tau_+} d\tau\, \boldsymbol{\mathcal{L}}_E(\boldsymbol{q}, d\boldsymbol{q}/d\tau, N_E), \tag{2.13}$$

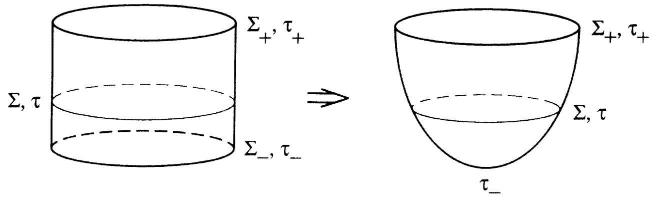

FIG. 2. Euclidean spacetime of the no-boundary type originating from the tubelike manifold $\Sigma \times [\tau_-, \tau_+]$ by shrinking one of its boundaries Σ_- to a point. It inherits the foliation with slices of constant Euclidean time τ in the form of quasispherical surfaces of "radius" τ with the center at $\tau_- = 0$.

because the manifold $\boldsymbol{M}$ can be viewed as originating from the tubelike spacetime $\Sigma \times [\tau_-, \tau_+]$ by the procedure of the above type: shrinking Σ_- to a point and inhabiting it by a positive-signature metric and matter fields (see Fig. 2). In such a manifold the role of a radial coordinate is played by the Euclidean time τ with the origin at $\tau_- = 0$.

The rest part of the no-boundary proposal is the choice of $\boldsymbol{C}$. Since the work [35], revealing the indefiniteness of the Euclidean gravitational action, it is known that integration cannot run over real four-geometries: to make the path integral formally convergent one should rotate the integration contour for the conformal mode into the complex plane. Unfortunately, at present, there is no theory which could have uniquely fixed this contour in the no-boundary proposal. Its choice can be constrained by a number of compelling but disjoint arguments, including convergence of the path integral, the recovery of quantum field theory in a semiclassical curved spacetime, the enforcement of quantum constraints, etc. [36], but still has essential freedom demonstrated in several minisuperspace models [37]. The general conclusion [36] was that, in spatially closed cosmology, the unique integration contour cannot be fixed on the basis of reduced phase space quantization [38–40], in contrast to asymptotically flat gravitational systems subject to positive-energy and positive-action theorems [41,42]. Here we shall not select this contour and assume that its choice has already been done by this or that rule, so that we have at our disposal a fixed class of topologically equivalent contours within which we can freely deform $\boldsymbol{C}$. In particular, we suppose that we can pass it through the saddle point $\boldsymbol{g}$ of the action, which gives the dominant semiclassical contribution $\boldsymbol{\Psi}(\boldsymbol{q}_+) \sim \exp\{-\boldsymbol{I}[\boldsymbol{g}]/\hbar\}$.

B. Nucleation of the Lorentzian universe from the Euclidean spacetime

The saddle point of the Euclidean action in (2.12) is a regular solution of classical equations subject to boundary data $\boldsymbol{q}_+$. In spherical coordinates with the geodetic radius τ near $\tau_- = 0$ the no-boundary regularity conditions imply the behavior of the four-metric

$$ds^2 = d\tau^2 + \tau^2 c_{ab}\, dx^a dx^b + O(\tau^3), \quad \tau \to 0. \qquad (2.14)$$

When $\boldsymbol{q}_+$ is close to the above three-geometry of "small size," one is granted to have a real solution of the Euclidean equations. For a theory with the cosmological constant $\Lambda = 3H^2$ and the round three-metric $\boldsymbol{q}_+ = a^2 c_{ab}$ with $a \leq 1/H$ this solution represents the four-geometry (1.3)–(1.4). However, when the three-geometry $\boldsymbol{q}_+$ is big enough, such a real solution may not exist, as it happens in the above example for $a \geq 1/H$. This is a manifestation of the fact that the solutions of Einstein equations have caustics in superspace of $\boldsymbol{q}$ and cannot regularly be continued into its "shadow" domains. But such a solution, which we shall denote by $\boldsymbol{\Phi}(z)$, can exist when the Euclidean contour (2.11) is replaced, via the procedure of analytic continuation, by some contour in the complex plane of time $z = \tau + it$, $C_+ : \{z = z(\sigma),\, 0 \leq \sigma \leq 1,\, z(0) = 0,\, z(1) = z_+\}$, starting at the vanishing Euclidean "radius" and ending at $z_+ = \tau_B + it_+$. For reasons of good physical interpretation it is worth breaking this contour into the union of the Euclidean (2.11) and Lorentzian (2.10) segments (see Fig. 3) with $\tau_- = 0$, $t_- = 0$, and $\tau_+ = \tau_B$:

$$C_+ = C_E \cup C_L\,. \qquad (2.15)$$

When the solution on C_E and C_L is *real*, then these two segments describe the real Euclidean and Lorentzian sections of one complex spacetime. They are analytically matched across the bounce surface $\tau = \tau_B$ giving rise to the "beginning of time" [26]. This is the case of the de Sitter Lorentzian spacetime (1.1) and (1.2) with the Hubble parameter $H = \sqrt{\Lambda/3}$, nucleating from the Euclidean four-dimensional hemisphere (1.3) and (1.4) of radius $R = 1/H$ at its equator $\tau_B = \pi/2H$ (see Fig. 1). For real tunneling geometries, the saddle-point action is the complex functional $\boldsymbol{\mathcal{I}}[\,\boldsymbol{\Phi}(z)\,] = \boldsymbol{I} - i\boldsymbol{S}$ with real and imaginary parts contributed, respectively, by the Euclidean and Lorentzian domains. The resulting wave function $\boldsymbol{\Psi}(\boldsymbol{q}_+) \sim \exp\{-\boldsymbol{I}/\hbar + i\boldsymbol{S}/\hbar\}$ describes the family of semiclassical Lorentzian universes characterized by the Hamilton-Jacobi function $\boldsymbol{S}$ and weighted by the exponentiated Euclidean action $\boldsymbol{I}$. Under certain positive-energy assumptions for matter fields a real tunneling so-

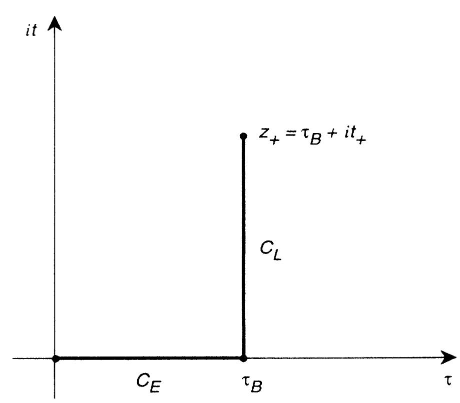

FIG. 3. The contour $C_+ = C_E \cup C_L$ of integration over complex time in the action, corresponding to the splitting of the whole spacetime into the combination of Euclidean (C_E) and Lorentzian (C_L) domains matched at the nucleation (bounce) point τ_B ($t = 0$).

lution is most likely to nucleate only a single connected Lorentzian spacetime with the topology $\mathbf{R}\times S^3$ and the de Sitter metric [25]. The rest of interpretation is based on the derivation of quantum field theory of matter fields in curved spacetime from the semiclassically approximated Wheeler-DeWitt equations [17–19,43]. Its application shows that matter fields on such a background are in the state of the Euclidean vacuum [43,16,26] which generates the large-scale cosmological structure compatible with observations [7,46].

III. NO-BOUNDARY WAVE FUNCTION IN THE REPRESENTATION OF PHYSICAL VARIABLES

A. ADM reduction and path-integral quantization

The basis of quantization of the physical variables is the ADM reduction to dynamically independent degrees of freedom [20,21,23]. It consists in imposing the gauge conditions on the constrained variables $\boldsymbol{q}$ and $\boldsymbol{p}$ ($\boldsymbol{p}$ is a set of canonical momenta conjugated to $\boldsymbol{q}$). The full system of the gravitational constraints and gauges can then be solved for $(\boldsymbol{q},\boldsymbol{p})$ in terms of independent canonical coordinates $q = q^i$ and their conjugated momenta $p = p_i$ which we shall label by a condensed index i. The conservation of gauge conditions also yields the lapse and shift functions and thus specifies a concrete spacetime foliation. Substituting $\boldsymbol{q} = \boldsymbol{q}\,(q,p)$, $\boldsymbol{p} = \boldsymbol{p}\,(q,p)$ into the canonical action produces the reduced action in terms of the unconstrained variables (q,p). It contains the nonvanishing, but generally time-dependent, Hamiltonian and by a standard procedure of the Legendre transform from p to $\dot{q} = dq/dt$ generates the Lagrangian $\mathcal{L}(q, dq/dt, t)$ and the Lagrangian action

$$S[\,q(t)\,] = \int_{t_-}^{t_+} dt\,\mathcal{L}(q, dq/dt, t). \tag{3.1}$$

According to Refs. [44,45] the path integral (2.2) over $\boldsymbol{g} = (\boldsymbol{q}, N)$ with the Faddeev-Popov integration measure coincides with a path integral over ADM variables of their exponentiated canonical action [47]. The transition from this phase-space path integral to its Lagrangian version amounts to the expression (2.2) with the covariant action replaced by its reduced version $S\,[\,q\,]$ and the new integration measure $D\mu[\,q\,]$. The latter accumulates the result of the non-Gaussian [48] integration over p and has an $\hbar$ expansion

$$D\mu[\,q\,] = \prod_t dq(t)\,[\,\det a(t)\,]^{1/2} + O\,(\,\hbar\,),$$

$$dq = \prod_i dq^i, \tag{3.2}$$

$$\det a = \det a_{ik}, \quad a_{ik} = \frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^k}. \tag{3.3}$$

Here the determinant of the Hessian matrix a_{ik} is understood with respect to condensed indices i and k. They include, depending on the representation of field variables, either continuous labels of spatial coordinates or discrete quantum numbers labeling some complete infinite set of functions on a three-dimensional space. Therefore the above determinant is functional, but its functional nature is restricted to a spatial slice of constant time t. The product over time points of $\det a(t)$ can be regarded as a determinant of higher functional dimensionality associated with the whole spacetime, if we redefine a_{ik} as a time-ultralocal operator $\boldsymbol{a} = a_{ik}\delta(t-t')$. We shall denote such functional determinants for both ultralocal and differential operators in time by Det. In view of the ultralocality of $\boldsymbol{a}$, the one-loop measure

$$\prod_t [\,\det a\,]^{1/2}(t) = [\,\mathrm{Det}\,\boldsymbol{a}\,]^{1/2} = \exp\left\{\frac{1}{2}\int_{t_-}^{t_+} dt\,\delta\,(0)\,\ln\det a(t)\right\}, \tag{3.4}$$

represents a pure power divergence. This contribution identically cancels the strongest (quartic) divergences of one-loop Feynman diagrams [49] – the property which will be demonstrated in Ref. [33] within the canonical framework.

In the representation of q the transition kernel (2.2) and the wave function (2.12) explicitly depend on time. Within the ADM reduction the role of time is played by some functional combinations of the phase-space coordinates $\boldsymbol{q}$ (and/or momenta $\boldsymbol{p}$), so that the arguments $\boldsymbol{q}_\pm$ of $\boldsymbol{K}(\boldsymbol{q}_+,\boldsymbol{q}_-)$ after the reduction give rise to time variables $t_\pm$.

B. Analytic continuation technique

The analytic continuation technique in physical variables is rather straightforward. Mainly it repeats Eqs. (2.2)–(2.6) with the original variables $\boldsymbol{g} = (\boldsymbol{q}, N)$, their Lagrangian $\boldsymbol{\mathcal{L}}\,(\boldsymbol{q},\dot{\boldsymbol{q}},N)$, and action $\boldsymbol{S}\,[\,\boldsymbol{g}\,]$ replaced by their ADM counterparts $q = q^i$, $\mathcal{L}\,(q, dq/dt, t)$ and $S[\,q(t)\,]$. In contrast to bold letters for the objects in the original formulation we shall use the usual letters for their ADM analogues. In particular, the physical kernel $K(q_+,t_+|q_-,t_-)$ and the wave function $\Psi(q_+,t_+)$ will replace $\boldsymbol{K}(\boldsymbol{q}_+,\boldsymbol{q}_-)$ and $\boldsymbol{\Psi}\,(\boldsymbol{q}_+)$. For a unitary map between transition kernels and wave functions in the ADM and the Dirac-Wheeler-DeWitt schemes see Refs. [22–24].

In the no-boundary construction the contour C_+ (2.15) runs in the action functional from $z = 0$ to some complex point z_+. In the ADM quantization the wave function explicitly depends on time, and according to the above method of analytic continuation real and imaginary ranges of its argument can be associated, respectively, with the classically allowed and forbidden transitions of the system. This makes us to identify the final point of the integration contour $z_+ = \tau_B + it_+$ with the complex time argument of $\Psi(q_+,z_+)$. Breaking the contour C_+ into the union (2.15) of the Euclidean C_E and Lorentzian C_L segments implies that the no-boundary state is a result of the underbarrier penetration along C_E

followed by the evolution along the segment of real time C_L. An obvious choice of the center of Wick rotation is the intersection point $z = \tau_B$ of these two segments. Under this choice the Euclidean Lagrangian of complex ADM variables $\Phi(z)$ looks like

$$\mathcal{L}_E(\Phi(z), d\Phi(z)/dz, z) = -\mathcal{L}(\Phi(z), id\Phi(z)/dz, (z-\tau_+)/i), \quad (3.5)$$

because it generates from the universal complex action functional

$$\mathcal{I}[\Phi(z)] = \int_C dz\, \mathcal{L}_E(\Phi(z), d\Phi(z)/dz, z) \quad (3.6)$$

the Lorentzian action (3.1) on the contour C_L and the Euclidean physical action on the contour C_E

$$iS[q(t)] = -\mathcal{I}[\Phi(z)]\,|_{C_L}, \quad q(t) = \Phi(z)\,|_{C_L}, \quad (3.7)$$

$$I[\phi(\tau)] = \mathcal{I}[\Phi(z)]\,|_{C_E}, \quad \phi(\tau) = \Phi(z)\,|_{C_E}, \quad (3.8)$$

$$I[\phi(\tau)] = \int_0^{\tau_+} d\tau\, \mathcal{L}_E(\phi, d\phi/d\tau, \tau). \quad (3.9)$$

Now we can write the ADM no-boundary wave function as a path integral

$$\Psi(q_+, z_+) = \int D\mu[\Phi(z)]\; e^{-\frac{1}{\hbar}\mathcal{I}[\Phi(z)]}\Big|_{\Phi(z_+)=q_+} \quad (3.10)$$

with the complex action (3.6) and a local measure (3.2) defined on the contour C_+ joining the points $z = 0$ and z_+. The integration here goes over physical fields matching q_+ at the boundary $z = z_+$ of complex spacetime $\boldsymbol{M} = \Sigma \times C_+$ and satisfying the no-boundary regularity conditions in its center $z = 0$. The wave function (3.10) can be regarded as a result of the analytic continuation into the complex plane of time of the Euclidean path integral which corresponds to $z_+ = \tau_+$ and the choice of the contour $C_+ = C_E$ in (3.10). When $z_+ = \tau_B + it_+$ in $\Psi(q_+, z_+)$ it makes sense to identify this function with the *Lorentzian* quantum state of the system $\Psi_L(q_+, t_+)$ evolving in the real physical time t_+ and originating from $\Psi(q_+, \tau_+)$ by this analytic continuation

$$\Psi_L(q_+, t_+) = \Psi(q_+, \tau_B + it_+). \quad (3.11)$$

The semiclassical expansion of the Euclidean version of (3.10) is based on the classical extremal ϕ:

$$\Psi(q_+, \tau_+) = \left(\frac{\mathrm{Det}\,\boldsymbol{F}[\phi]}{\mathrm{Det}\,\boldsymbol{a}[\phi]}\right)^{-1/2} e^{-\frac{1}{\hbar}I[\phi]}\,[\,1 + O(\hbar)\,], \quad (3.12)$$

$$\frac{\delta I[\phi]}{\delta\phi(\tau)}\Big|_{\boldsymbol{M}} = 0, \quad \phi(\tau)\,|_{\boldsymbol{M}} = \mathrm{reg}, \quad (3.13)$$

$$\phi(\tau)\,|_{\partial\boldsymbol{M}} \equiv \phi(\tau_+) = q_+,$$

The one-loop factor here is a combination of the local measure (3.4) and contribution of the Gaussian functional integration over quantum fields around this extremal. This contribution is given by the functional determinant of $\boldsymbol{F}$ – the kernel of the quadratic part of the action

$$\boldsymbol{F} \equiv \boldsymbol{F}(d/d\tau)\,\delta(\tau - \tau') = \frac{\delta^2 I[\phi]}{\delta\phi(\tau)\,\delta\phi(\tau')}. \quad (3.14)$$

Because the Lagrangian $\mathcal{L}_E(\phi, d\phi/d\tau, \tau)$ contains at most first-order time derivatives of fields, this is a second-order matrix-valued differential operator $\boldsymbol{F}(d/d\tau) = \boldsymbol{F}_{ik}(d/d\tau)$

$$\boldsymbol{F}(d/d\tau) = -\frac{d}{d\tau}\,a\,\frac{d}{d\tau} - \frac{d}{d\tau}\,b + b^T\frac{d}{d\tau} + c, \quad (3.15)$$

where the coefficients $a = a_{ik}$, $b = b_{ik}$, and $c = c_{ik}$ are the (functional) matrices acting in the space of field variables $\phi(\tau) = \phi^k(\tau)$, and the superscript T denotes their (functional) transposition $(b^T)_{ik} \equiv b_{ki}$. These coefficients originate from the second-order derivatives of the Euclidean Lagrangian with respect to ϕ^i and $\dot\phi^i = d\phi^i/d\tau$. In particular, the matrix a_{ik} is given by the Euclidean version of the Hessian matrix (3.3): $a_{ik} = \partial^2\mathcal{L}_E/\partial\dot\phi^i\partial\dot\phi^k$.

C. The choice of gauge and nature of physical variables

The general scheme of the above type has a freedom in the choice of physical variables q^i. This choice specifies the way these variables are disentangled from the initial phase space of $\boldsymbol{q}$ and $\boldsymbol{p}$ and fixes the spacetime foliation by surfaces of constant t. This freedom shows up in the physical wave function and its analytic continuation $\Psi(q_+, z)$ into the complex plane of t. There exist several requirements which can restrict this excessive freedom.

First of all, it makes sense to define such an ADM reduction that yields $\mathcal{L}_E(\phi, d\phi/d\tau, \tau)$ real at real values of the Euclidean time τ and $\phi(\tau)$. Another important property is the boundedness from below for the Euclidean action of physical variables (3.9). It provides the formal (mode by mode) convergence of the Euclidean path integral over real fields $\phi(\tau)$, establishes the normalizability of the wave function $\Psi(q,t)$ on the real section of the q-configuration space (and, therefore, the possibility to regard physical variables as Hermitian operators) and provides a special technique of complex extremals which we develop in Sec. V. This property of the *physical* Euclidean action can be a consequence of a successful reduction and the properties of the original *covariant* action which, as is known, suffers from the indefiniteness in the conformal sector.

The basic approximation to the chaotic inflationary cosmology consists in the minisuperspace model with the metric (1.1) and (1.2), scale factor $a_L(t)$, and the effective Hubble constant H generated by the spatially homogeneous (inflaton) scalar field φ, $H = H(\varphi)$. Other inhomogeneous fields of all possible spins are treated as perturbations on this background and, therefore, the full set of the initial phase-space coordinates of the theory can be represented as $\boldsymbol{q} = (a,\ \varphi,\ \phi(\mathbf{x}),\ \psi(\mathbf{x}),\ A_a(\mathbf{x}),\ \psi_a(\mathbf{x}),\ h_{ab}(\mathbf{x}), ...)$. A physically meaningful reduction from $\boldsymbol{q}$ to q goes separately in the minisuperspace sector of the full superspace $(a,\ \varphi)$ and the sector of spatially inhomogeneous modes. There is only one Hamiltonian constraint which is effectively imposed on homogeneous modes $(a,\ \varphi)$ in the minisuperspace model of Bianchi IX type [50]. Therefore, there can be only one physical degree of freedom among these two minisuperspace variables $(a,\ \varphi)$. The second of these variables has to be fixed by the gauge condition which simultaneously disentangles time. It is useful to choose the inflaton field φ as this physical degree of freedom and interpret the approximate solution of classical equations of motion (1.2) with $H = H(\varphi)$ as a gauge which, thus, plays the role of parametrization of the initial phase-space coordinates in terms of the physical ones in the minisuperspace sector of the theory.

This gauge is very convenient because it corresponds to the choice of cosmic time with the lapse $N = 1$ in classical solutions [32] and, what is most important for our purposes, provides the reality of the Euclidean Lagrangian (3.5). Indeed, choosing $\tau_B = \pi/2H(\varphi)$ and analytically continuing the gauge (1.2) along the contour $C_+ = C_E \cup C_L$ onto the real axis of τ, one finds that the scale factor remains real and coincides with the Euclidean solution (1.4). Therefore, the physical Lagrangian on the Euclidean hemisphere $0 \leq \tau \leq \pi/2H(\varphi)$ is also real and even positive-definite in the sector of transverse-traceless graviton modes [29].

The ADM reduction for other fields can be performed in very many different ways. In the linearization approximation it mainly boils down to selecting the transverse (T), transverse-traceless (TT), etc., modes of spatial components of the corresponding tensor fields, so that the full set of physical variables can be written as

$$q^i = (\varphi,\ \phi(\mathbf{x}),\ \psi(\mathbf{x}),\ A_a^T(\mathbf{x}),\ \psi_a^T(\mathbf{x}),\ h_{ab}^{TT}(\mathbf{x}), ...). \quad (3.16)$$

Here the index i is an element of condensed DeWitt notations which we shall intensively use throughout the paper. It includes discrete spin labels of field components and also continuous labels of spatial coordinates $\mathbf{x}$. The functions of spatial coordinates $\mathbf{x}$ in (3.16) can be decomposed as infinite series in the complete set of some spatial harmonics, which in view of the compactness of a spatial section is discrete and countable. Then the continuous label $\mathbf{x}$ in i will be replaced by the discrete quantum numbers enumerating these spatial harmonics. In both cases, however, the operations of integration over $\mathbf{x}$ or infinite summation over these numbers will be a part of contracting the condensed indices.

The no-boundary regularity conditions (2.14) must be reformulated in physical variables (3.16). Note that the gauge (1.2) with $H = H(\varphi)$ analytically continued to the Euclidean time, $\tau = \pi/2H(\varphi) + it$, gives the scale factor $a_E(\tau) = \tau + O(\tau^2)$ which automatically satisfies (2.14) [this gauge picks up the unit lapse $N_E = 1$ and therefore guarantees that τ measures the proper radial distance in the center of the Euclidean ball $\mathcal{B}^4$ free from the conical singularity: $a_E(\tau) \sim \tau$ at $\tau \to 0$]. Thus, it remains to check that all physical fields (3.16) are regular at $\tau = 0$. For spatially homogeneous modes $\varphi = \varphi(\tau)$ this condition implies that their radial derivative should vanish at this point $(d\varphi/d\tau)(0) = 0$, while inhomogeneous modes should disappear themselves. These properties are a direct corollary of the direction independent limit of these modes or their derivatives at $\tau = 0$.

IV. THE METHOD OF COLLECTIVE COORDINATES

In field-theoretical models the variables q represent the continuous infinitude of modes (3.16). Their constructive treatment is possible only in certain approximations. The idea of such approximations consists in disentangling from the set of q a certain finite subset which plays the most important role in the dynamics of the system and exactly or approximately decouples from the rest of degrees of freedom. Then these distinguished variables are treated exactly, while the rest of the modes are either frozen out or considered perturbatively. Let us make the splitting of the argument q_+ in (3.12) into the collective variables φ and the rest of fields f

$$q_+ = \begin{bmatrix} \varphi \\ f \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \end{bmatrix} + \eta,\ \eta = \begin{bmatrix} 0 \\ f \end{bmatrix}. \quad (4.1)$$

Here, in general, φ can be regarded as subcolumn of q of finite dimensionality (as, for example, a finite set of the scale factor and anisotropy parameters in homogeneous Bianchi models). On the contrary, f is an infinite-dimensional vector (representing, in the same example, all spatially inhomogeneous field harmonics on a symmetric background).

Let us suppose that $\phi(\tau)$ is a solution of the classical Euclidean equations (3.13) with the unperturbed boundary condition at τ_+, $q_+ = (\varphi, 0)$, which is determined entirely by the collective variables φ. Similarly, we denote the solution of (3.13) with the perturbed boundary conditions (4.1) as $\phi(\tau) + \eta(\tau)$, so that $\eta(\tau)$ satisfies up to quadratic terms the linearized equations of motion and the condition of regularity at $\tau = 0$

$$\boldsymbol{F}(d/d\tau)\,\eta(\tau) = O(\eta^2),\ \eta(\tau_+) = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad \eta(0) = \text{reg.} \quad (4.2)$$

Thus the perturbation $\eta(\tau_+)$ of the boundary conditions $q = q(\tau_+)$ generates the perturbation $\eta(\tau)$ of the classical extremal and the perturbation expansion of the action

$$I[\phi + \eta] = I[\phi] + \delta I + \tfrac{1}{2}\delta^2 I + O(\eta^3), \quad (4.3)$$

$$\delta I = \int_0^{\tau_+} d\tau\, \frac{\delta I}{\delta\phi(\tau)}\,\eta(\tau) + \frac{\partial \mathcal{L}_E}{\partial\dot\phi}\,\eta\,\bigg|_{\tau_+}, \quad (4.4)$$

$$\delta^2 I = \int_0^{\tau_+} d\tau\, \eta^T(\boldsymbol{F}\eta) + \eta^T(\boldsymbol{W}\eta)\,\bigg|_{\tau_+}. \quad (4.5)$$

The integration by parts in the first-order variation yields the integral term containing the left-hand side of equations of motion and the surface term – the contribution of the boundary at $\tau = \tau_+$. In view of the no-boundary prescription the contribution at $\tau_- = 0$ is vanishing because of the regularity conditions. The second-order variation (4.5), with T denoting the transposition of the column $\eta = \eta^i$, follows from varying Eq. (4.4) on account of the variational relations

$$\delta \frac{\delta I}{\delta \phi\,(\tau)} = \boldsymbol{F}\,(d/d\tau)\,\eta(\tau), \tag{4.6}$$

$$\delta \frac{\partial \mathcal{L}_E}{\partial \dot{\phi}} = \boldsymbol{W}(d/d\tau)\,\eta\,(\tau). \tag{4.7}$$

Here $\boldsymbol{F} = \boldsymbol{F}\,(d/d\tau)$ is a differential operator of linearized equations (3.14) and (3.15), while $\boldsymbol{W} = \boldsymbol{W}(d/d\tau)$ we shall call the *Wronskian* operator which enters the relation

$$\varphi_1^T\,(\boldsymbol{F}\varphi_2) - (\boldsymbol{F}\varphi_1)^T\varphi_2$$

$$= -\frac{d}{d\tau}\left[\varphi_1^T\,(\boldsymbol{W}\varphi_2) - (\boldsymbol{W}\varphi_1)^T\varphi_2\right] \tag{4.8}$$

valid for arbitrary test functions φ_1 and φ_2 and usually used for the construction of the simplectic inner product. For $\boldsymbol{F}\,(d/d\tau)$ of the form (3.15) this operator equals

$$\boldsymbol{W}(d/d\tau) = a\,\frac{d}{d\tau} + b. \tag{4.9}$$

A. Basis functions of linearized field modes

The perturbation $\eta\,(\tau)$ satisfies the equations of motion (4.2) which can be solved by iterations in $\eta\,(\tau_+) = (0, f)$. In the linear approximation this solution can be represented in terms of regular basis functions of the Euclidean "wave" operator $\boldsymbol{F}\,(d/d\tau)$. They form the full set $\boldsymbol{u}(\tau)$ of solutions of the homogeneous differential equation

$$\boldsymbol{F}\,(d/d\tau)\,\boldsymbol{u}(\tau) = 0, \quad \boldsymbol{u}\,(0) = \text{reg}, \tag{4.10}$$

which are regular in the Euclidean spacetime ball $0 \leq \tau \leq \tau_+$.

In view of functional matrix nature of the operator $\boldsymbol{F}\,(d/d\tau) = \boldsymbol{F}_{ik}\,(d/d\tau)$ its basis functions also form a matrix $\boldsymbol{u}(\tau) = \boldsymbol{u}^k_A(\tau)$. The condensed upper index k (acted upon by indices of the matrix operator) labels the components of a given basis function and includes its dependence on spatial coordinates, while the lower index A enumerates the basis functions themselves. The infinite ranges and the (discrete or continuous) nature of these indices k and A can be different depending on the parametrization of basic physical variables (3.16) and their possible decomposition in spatial harmonics. What is, however, in common to all field parametrizations is that there is a one to one map between the sets $\{k\}$ and $\{A\}$, so that the matrix $\boldsymbol{u}^k_A$ can be regarded as nonsymmetric but *quadratic* and *invertible* matrix having with respect to its infinite-dimensional indices the inverse $\boldsymbol{u}^{-1}(\tau) = (\boldsymbol{u}^{-1})^A_i$:

$$\boldsymbol{u}^i_A\,(\boldsymbol{u}^{-1})^A_k = \delta^i_k\,. \tag{4.11}$$

To illustrate the use of condensed DeWitt indices in the functional matrix $\boldsymbol{u}(\tau)$, consider a simple example of a scalar field $q^i = \phi\,(\mathbf{x})$ in flat spacetime, when the condensed index reduces to the set of continuous spatial coordinates $i = \mathbf{x}$. The linear equation of motion (4.10) in this case is the Euclidean Klein-Gordon equation which has a set of basis functions regular at past infinity $\tau \to -\infty$

$$\boldsymbol{u}^i_A\,(\tau) \equiv \boldsymbol{u}_{\mathbf{k}}(\mathbf{x},\tau), \quad i = \mathbf{x}, A = \mathbf{k}, \tag{4.12}$$

$$\boldsymbol{u}_{\mathbf{k}}(\mathbf{x},\tau) = e^{\,\omega(\mathbf{k})\,\tau + i\mathbf{k}\mathbf{x}},$$

$$\omega\,(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}, \tag{4.13}$$

enumerated by the continuous set of spatial momentum vectors $\mathbf{k}$. Every square-integrable function of spatial coordinates $h^i = h(\mathbf{x})$ can be decomposed in plane waves of the above type in the form

$$h\,(\mathbf{x}) = \int d^3\mathbf{k}\,\mathrm{e}^{\,\omega\,(\mathbf{k})\,\tau + i\mathbf{k}\mathbf{x}}\,h_{\mathbf{k}}, \tag{4.14}$$

which can be rewritten in condensed notations as $h^i = \boldsymbol{u}^i_A h^A$, $h^A \equiv h_{\mathbf{k}}$. Thus, this equation provides a linear one to one map between h^i and h^A. The inverse transformation

$$h_{\mathbf{k}} = \frac{1}{(2\pi)^3}\int d^3\mathbf{x}\,\mathrm{e}^{-\omega\,(\mathbf{k})\tau - i\mathbf{k}\mathbf{x}}\,h\,(\mathbf{x}), \tag{4.15}$$

in condensed indices has a simple form $h^A = (\boldsymbol{u}^{-1})^A_i\,h^i$, where $(\boldsymbol{u}^{-1})^A_i$ denotes the inverse of $\boldsymbol{u}^i_A$ (4.11) with a kernel given by Eq. (4.15).

It is also possible to decompose a scalar field in spherical-wave basis functions, enumerated instead of a momentum vector by its continuous norm $k = |\mathbf{k}|$ and discrete orbital $l = 0, 1, 2, \ldots$, and azimuthal m, $-l \leq m \leq l$, quantum numbers. In this case the condensed label $A = (k, l, m)$ will be of mixed continuous-discrete nature. In spatially closed cosmology, the set of harmonics is discrete and countable. For a general set of linearized fields (3.16), the condensed index A includes three discrete quantum numbers and the corresponding spin label $A = (n,\, l,\, m,\, \text{spin})$ which are again in one-to-one correspondence with $i = (\mathbf{x}, aT, abTT, \ldots)$, and so on. But the above peculiarities of the "fine" structure of various field models can always be encoded in the above universal relations, as (4.11), written in DeWitt's notations which we shall imply throughout the paper.

In view of the decomposition (4.1) of q^i, the full set of $\boldsymbol{u}\,(\tau)$ contains the modes of the collective variables φ and the rest of degrees of freedom f. Physically the decomposition (4.1) makes sense when they decouple at least in the linearized approximation, which means that in the basis of φ and f the differential operator $\boldsymbol{F}\,(d/d\tau)$ has a block-diagonal structure

$$\boldsymbol{F}\,(d/d\tau) = \begin{bmatrix} F_\varphi\,(d/d\tau) & 0 \\ 0 & F\,(d/d\tau) \end{bmatrix}, \tag{4.16}$$

with $F_\varphi\,(d/d\tau)$ and $F\,(d/d\tau)$ acting, respectively, in subspaces of φ and f. This has a simple illustration when

φ represents a spatially homogeneous background for inhomogeneous modes f. The latter are given by spatial harmonics which are orthogonal to the homogeneous linear modes of φ.

The block-diagonal structure (4.16) implies a similar form of all the matrix coefficients of the operator $\boldsymbol{F}(d/d\tau)$, its Wronskian operator (4.9)

$$\boldsymbol{W}(d/d\tau) = \begin{bmatrix} W_\varphi(d/d\tau) & 0 \\ 0 & W(d/d\tau) \end{bmatrix} \quad (4.17)$$

and also allows one to choose the matrix $\boldsymbol{u}(\tau)$ in the block-diagonal form with the basis functions $u_\varphi(\tau)$ and $u(\tau)$ of the linearized modes of φ and f, respectively,

$$\boldsymbol{u}(\tau) = \begin{bmatrix} u_\varphi(\tau) & 0 \\ 0 & u(\tau) \end{bmatrix}, \quad F_\varphi(d/d\tau)\, u_\varphi(\tau) = 0,$$
$$F(d/d\tau)\, u(\tau) = 0. \quad (4.18)$$

Finally, let us consider the regularity conditions for basis functions of the operator $\boldsymbol{F}$. Due to the no-boundary nature of $\boldsymbol{M}$, its point of vanishing coordinate radius $\tau = \tau_- \equiv 0$ is a singular point of the radial part of $\boldsymbol{F}(d/d\tau)$. Indeed, for physical fields (3.16) of all possible spins, $s = 0, 1/2, 1, 3/2, 2, \ldots$, the coefficient $a = a_{ik}$ in $\boldsymbol{F}(d/d\tau)$ can be written as

$$a_{ik} = ({}^4g)^{1/2} g^{\tau\tau} g^{a_1 a_2} \cdots g^{a_s a_{2s}}\, \delta(\mathbf{x}_i - \mathbf{x}_k),$$
$$i = (a_1, \ldots a_s, \mathbf{x}_i),\ k = (a_2, \ldots a_{2s}, \mathbf{x}_k), \quad (4.19)$$

and in the regular metric (2.14) has the behavior

$$a = a_0\, \tau^k + O(\tau^{k+1}),\ k = 3 - 2s,\ \tau \to \tau_- = 0, \quad (4.20)$$

where a_0 is defined by Eq. (4.19) with respect to the round metric c_{ab} on a three-sphere of the unit radius and the unit lapse $g^{\tau\tau} = N^{-2} = 1$. Therefore the coefficients of Eqs. (4.10) for basis functions have the following asymptotic behavior

$$\left(\frac{d^2}{d\tau^2} + f\frac{d}{d\tau} + g\right)\boldsymbol{u}(\tau) = 0, \quad f = \frac{k}{\tau}\boldsymbol{I} + O(\tau^0),$$
$$g = \frac{g_0}{\tau^2} + O(\tau^{-1}). \quad (4.21)$$

Here the leading singularity in the potential term g originates from the spatial Laplacian $g^{ab}\nabla_a\nabla_b$ entering the operator $\boldsymbol{F}$, which scales in the metric (2.14) as $1/\tau^2$, and the leading term of f is always a multiple of the unity matrix $\boldsymbol{I}$ with the same parameter $k = 3 - 2s$ as in (4.20). In the representation of the eigenfunctions of a spatial Laplacian the (functional) matrix g_0 can be also diagonalized, $g_0 = \mathrm{diag}\{-\omega_i^2\}$, so that, without the loss of generality, both singularities in (4.21) can be characterized by simple numbers k and $\omega^2 = \omega_i^2$ for every component of $\boldsymbol{u} = \boldsymbol{u}^i$.

As it follows from the theory of differential equations with singular points [51], there are two types of solutions $\boldsymbol{u}_-(\tau)$ and $\boldsymbol{u}_+(\tau)$ differing by their behavior near $\tau_- = 0$:

$$\boldsymbol{F}\boldsymbol{u}_\pm = 0, \quad \boldsymbol{u}_-(\tau) = \boldsymbol{U}_-\,\tau^{\mu_-} + O(\tau^{1+\mu_-}),$$
$$\boldsymbol{u}_+(\tau) = \boldsymbol{V}_+\,\tau^{\mu_+} + O(\tau^{1+\mu_+}), \quad (4.22)$$

where $\mu_\pm$ are the roots of the quadratic equation involving only the coefficients of leading singularities $\mu^2 + (k-1)\,\mu - \omega^2 = 0$. In view of the non-negativity of ω^2 (the eigenvalue of $-c^{ab}\nabla_a\nabla_b$) these roots are of opposite signs, $\mu_-\mu_+ = -\omega^2 \leq 0$, and we can choose μ_- to be non-negative in order to have $\boldsymbol{u}(\tau) = \boldsymbol{u}_-(\tau)$ as a set of regular basis functions at $\tau_- = 0$, the remaining part of them $\boldsymbol{u}_+(\tau)$ being singular. By our assumption the operator $\boldsymbol{F}$ does not have zero eigenvalues on the Euclidean spacetime of the no-boundary type (otherwise, its functional determinant and the one-loop prefactor of the wave function are not defined). Therefore, there are no basis functions which are simultaneously regular at $\tau_- = 0$ and vanishing for positive $\tau \leq \tau_+$, and their matrix can be considered invertible everywhere in this range of τ except the origin [52] $\tau_- = 0$. Below we shall denote the regular basis functions either by $\boldsymbol{u}(\tau)$ or by $\boldsymbol{u}_-(\tau)$, when we prefer to emphasize their regularity at $\tau_- \equiv 0$.

B. Perturbation theory in microscopic variables and $\hbar$ expansion

The basis functions of the above type will serve us as a technical tool for two purposes: the perturbation theory in microscopic variables f and the reduction method for the one-loop functional determinants. We begin with this perturbation theory in powers of $\eta(\tau) = O(f)$ and show how it actually reduces to the expansion in $\hbar$. Note that in virtue of the invertibility of $\boldsymbol{u}_-(\tau)$ the linearized solution of the boundary-value problem (4.2) has the form

$$\eta(\tau) = \boldsymbol{u}(\tau)\,\boldsymbol{u}^{-1}(\tau_+)\,\eta(\tau_+) + O(\eta^3), \quad (4.23)$$

where we suppress the indices of matrices $\boldsymbol{u}(\tau) = \boldsymbol{u}^i_A(\tau)$, $\boldsymbol{u}^{-1}(\tau_+) = [\boldsymbol{u}^{-1}(\tau_+)]^A_i$ and columns $\eta(\tau) = \eta^i(\tau)$ implying again the DeWitt rule of summation-integration over supercondensed labels. After substituting this solution into the linear and quadratic terms of the perturbed Euclidean action (4.3), the volume contributions vanish in the quadratic approximation due to the background $\delta I/\delta\phi(\tau) = 0$ and linearized (4.2) equations of motion. The remaining terms give

$$I[\phi + \eta] = I[\phi] + \left[\frac{\partial \mathcal{L}_E}{\partial \phi}\,\eta + \frac{1}{2}\eta^T(\boldsymbol{W}\boldsymbol{u})\,\boldsymbol{u}^{-1}\eta\right]_{\tau_+}$$
$$+ O(\eta^3). \quad (4.24)$$

In view of the form of the boundary-value perturbation (4.2), only the quadratic form in η survives in (4.24) and takes the form

$$I[\phi + \eta] = I[\phi] + \tfrac{1}{2} f^T D(\tau_+)\, f + O(f^3), \quad (4.25)$$

$$D(\tau) = [\,W(d/d\tau)\,u(\tau)\,]\; u^{-1}(\tau). \quad (4.26)$$

This expression can be used in Eq. (3.12) for the wave function together with the preexponential factor $(\mathrm{Det}\,\boldsymbol{F}[\phi + \eta]/\mathrm{Det}\,\boldsymbol{a}[\phi + \eta])^{-1/2}$ expanded in powers of $\eta = O(f)$,

$$\Psi\,(q_+,\tau_+) = \left(\frac{\mathrm{Det}\,\boldsymbol{F}\,[\,\phi\,]}{\mathrm{Det}\,\boldsymbol{a}\,[\,\phi\,]}\right)^{-1/2} \exp\left\{-\frac{1}{\hbar}I\,[\,\phi\,] - \frac{1}{2\hbar}f^T D(\tau_+)\,f\right\}\,[\,1 + O\,(\,f\,) + O\,(\,f^3/\hbar\,)\,]\,. \qquad (4.27)$$

The Gaussian exponent suppresses the states with large f, because of the positive definiteness of the matrix $D\,(\tau_+)$, which is a direct corollary of the boundedness of the Euclidean action from below. If the extremal $\phi\,(\tau)$ realizes a minimum of the action, then its quadratic perturbation (4.5) is positive-definite for *arbitrary* $\eta\,(\tau)$, $\delta^2 I\,>\,0$. On linearized solutions with the boundary data (4.2) specified by f it reduces to

$$\delta^2 I = f^T D\,(\tau_+)\,f > 0 \qquad (4.28)$$

and, thus, provides the positive definiteness of the quadratic form in the exponential of (4.27). Therefore one can use the asymptotic bound

$$e^{-\frac{1}{2\hbar}f^T D f}\,f^n = O\,(\,\hbar^{n/2}),\quad \hbar \to 0, \qquad (4.29)$$

valid for a wide class of positive definite quadratic functionals [53]. Thus, the linear $O\,(\,f\,)$ and cubic $O\,(\,f^3/\hbar\,)$ corrections in (4.27) go beyond the one-loop approximation.

C. The basis-functions algorithm for the one-loop preexponential factor

As shown in Ref. [33] the regular basis functions $\boldsymbol{u}_-(\tau)$ can be used for the calculation of the one-loop preexponential factor in (4.27). This procedure consists in the reduction which allows to obtain the functional determinant $\mathrm{Det}\,\boldsymbol{F}$ in terms of the quantity of the lower functional dimensionality – the determinant of the nondegenerate matrix of regular basis functions $\boldsymbol{u}^i_A(\tau)$ taken with respect to its indices [54]. These basis functions have the behavior (4.22) and are defined up to linear τ-independent recombinations. The latter can be used to make the coefficient $\boldsymbol{U}_-$ completely independent of the background fields ϕ on $\boldsymbol{M}$ and, without loss of generality, equal the functional matrix unity $\boldsymbol{I}$. Then this algorithm for a one-loop prefactor takes the form [33]

$$\left(\frac{\mathrm{Det}\,\boldsymbol{F}}{\mathrm{Det}\,\boldsymbol{a}}\right)^{-1/2} = \mathrm{const}\;[\,\det \boldsymbol{u}_-(\tau_+)\,]^{-1/2}\,,$$
$$\boldsymbol{u}_-(\tau) = \boldsymbol{I}\,\tau^{\mu_-} + O\,(\,\tau^{1+\mu_-}). \qquad (4.30)$$

Combining Eqs. (4.27) and (4.29) with this reduction algorithm, we finally get the one-loop Euclidean wave function $\Psi\,(q_+,\tau_+) = \Psi\,(\varphi, f,\,\tau_+)$

$$\Psi\,(\varphi, f, \tau_+) = \Psi_{[\,\phi\,]}\,(f,\tau_+)\,\big|_{\,\phi=\phi\,(\tau\,|\,\varphi,\,\tau_+)}\,, \qquad (4.31)$$

$$\Psi_{[\,\phi\,]}\,(f,\tau_+) \equiv \mathrm{const}\,(\,\det \boldsymbol{u}_-(\tau_+)_{\,[\,\phi\,]})^{-1/2} \exp\left\{-\frac{1}{\hbar}\,I\,[\,\phi\,] - \frac{1}{2\hbar}f^T D\,(\tau_+)\,f\right\}\left[\,1 + O\,(\hbar^{1/2})\,\right]. \qquad (4.32)$$

In contrast to a Gaussian dependence on f, the variables φ enter $\Psi(\varphi, f,\,\tau_+)$ through the functional argument $\phi(\tau)$ of $\Psi_{[\,\phi\,]}\,(f,\tau_+)$, for they parametrize the extremal $\phi(\tau) = \phi(\tau|\,\varphi,\tau_+)$ of the Euclidean equations with the boundary data $q_+ = (\varphi, 0)$.

V. THE METHOD OF COMPLEX EXTREMALS

A. Matching conditions between the Euclidean and Lorentzian spacetimes

According to Sec. III the "Lorentzian" wave function is the analytic continuation (3.11) of (4.31) into the complex plane of the Euclidean time:

$$\Psi_L\,(\varphi, f, t_+) = \Psi_{[\,\Phi\,]}\,(f, z_+)\,\big|_{\,\Phi=\Phi\,(z\,|\,\varphi,\,z_+)}\,, \qquad (5.1)$$

which results in the complex extremal. Here we present the semiclassical technique that allows one to handle this case within the scope of real solutions of Euclidean and Lorentzian equations.

To begin with, we reserve the notations $q\,(t)$ and $\phi\,(\tau)$ for real parts of $\Phi\,(z)$, respectively, on Lorentzian and Euclidean segments of C_+, denote their imaginary parts by $h(t)$ and $\eta(\tau)$, and also introduce the notation $Q\,(t)$ for the full complex field on C_L:

$$\Phi(\tau) = \phi\,(\tau) + i\eta\,(\tau),\;\; Q\,(t) = q\,(t) + ih\,(t),$$
$$Q(t) \equiv \Phi\,(z)\,|_{C_L} = \Phi\,(\tau_B + it). \qquad (5.2)$$

Then the complex action (3.6) on this contour takes the form

$$\mathcal{I}\,[\,\Phi(z)\,]\,|_{C_+} = I\,[\,\Phi(\tau)\,] - iS\,[\,Q(t)\,], \qquad (5.3)$$

where $I\,[\,\Phi(\tau)\,]$ and $S\,[\,Q(t)\,]$ are the Euclidean and Lorentzian actions (3.9) and (3.1) as functions of their complex functional arguments.

Let us consider the variational principle for this Lorentzian-Euclidean action which gives the saddle point of the path integral (3.10). Its first-order variation is similar to (4.4)

$$\delta\mathcal{I} = \int_0^{\tau_+} d\tau\,\frac{\delta I}{\delta\Phi}\,\delta\Phi + \frac{\partial\mathcal{L}_E}{\partial\dot\Phi}\,\delta\Phi\,\bigg|_{\tau_B}$$
$$-i\int_0^{t_+} dt\,\frac{\delta S}{\delta Q}\,\delta Q + i\,\frac{\partial\mathcal{L}}{\partial\dot Q}\,\delta Q\,\bigg|_{t=0}. \qquad (5.4)$$

The fields and their variations satisfy the regularity conditions at $\tau = 0$ and fixed boundary conditions at $t = t_+$, whence the total-derivative term survives only at τ_B. In view of the analytic continuation (5.2), the Euclidean and Lorentzian fields satisfy the matching conditions $\Phi(\tau_B) = Q(0)$, which means that $\delta\Phi(\tau_B) = \delta Q(0)$. Therefore, equating to zero separately the volume and surface terms in the variational equation $\delta\mathcal{I} = 0$, one gets the system of Euclidean and Lorentzian equations of motion

$$\frac{\delta I}{\delta\Phi} = 0, \quad \frac{\delta S}{\delta Q} = 0, \tag{5.5}$$

for fields subject to special matching conditions at the nucleation point $\tau = \tau_B$ $(t = 0)$

$$\frac{\partial\mathcal{L}_E}{\partial\dot\Phi}\bigg|_{\tau_+} + i\,\frac{\partial\mathcal{L}}{\partial\dot Q}\bigg|_{t=0} = 0. \tag{5.6}$$

These show that the tunneling geometries with real fields exist only when both the Euclidean $\partial\mathcal{L}_E/\partial\dot\Phi$ and Lorentzian $\partial\mathcal{L}/\partial\dot Q$ momenta separately vanish at the nucleation point. The covariant version of this statement in the gravitational sector of all fields sounds as a vanishing of the extrinsic curvature K_{ab} of the nucleation surface [36,25,26]. In the example of the de Sitter universe generated by the inert cosmological constant this surface coincides with the equator of the Euclidean four-dimensional sphere with the vanishing time derivative of the scale factor (1.4).

In the general case of nonzero momenta at $\tau = \tau_B$, the fields $\Phi(\tau)$ and $Q(t)$ become complex, and the very notion of the Euclidean-Lorentzian transition becomes questionable, because complex physical fields generate complex-valued metric tensors which can hardly be ascribed to spacetimes of either Euclidean or Lorentzian signature. We shall show, however, that the Euclidean-Lorentzian decomposition still makes sense within the $\hbar$ expansion. For this purpose we shall develop the expansion in imaginary parts $\varepsilon = (\eta, h)$ of the complex fields (5.2) and demonstrate that it corresponds to the asymptotic expansion in $\hbar^{1/2}$.

B. Perturbation theory in the imaginary corrections and the $\hbar$ expansion

This perturbation expansion begins with substituting (5.2) into the classical equations and matching conditions (5.6) and expanding the result in powers of $\varepsilon = (\eta, h)$. The separation of real and imaginary parts in (5.5) leads to the equations

$$\frac{\delta I[\phi]}{\delta\phi(\tau)} = O(\eta^2), \quad \boldsymbol{F}(d/d\tau)\,\eta(\tau) = O(\eta^3), \tag{5.7}$$

$$\frac{\delta S[q]}{\delta q(t)} = O(h^2), \quad \boldsymbol{F}_L(d/dt)\,h(t) = O(h^3), \tag{5.8}$$

where $\boldsymbol{F}_L(d/dt)$ is the Lorentzian wave operator of linearized equations [at the background of $q(t)$], analogous to its Euclidean version (3.14)

$$\boldsymbol{F}_L \equiv \boldsymbol{F}_L(d/dt)\,\delta(t-t') = \frac{\delta^2 S[q]}{\delta q(t)\,\delta q(t')}. \tag{5.9}$$

The expansion of the matching condition (5.6) up to linear terms in ε can be performed with the aid of the variational equation (4.7) and its Lorentzian version

$$\delta\,\frac{\partial\mathcal{L}_L}{\partial\dot q} = \boldsymbol{W}_L(d/dt)\,\delta q(t), \tag{5.10}$$

which serves as a definition of the *Wronskian* operator $\boldsymbol{W}_L = \boldsymbol{W}_L(d/dt)$ for (5.9). The separation of the real and imaginary parts of Eq. (5.6) then yields the system of matching conditions coupling the dynamics of Euclidean and Lorentzian variables

$$\begin{aligned}\frac{\partial\mathcal{L}_E}{\partial\dot\phi}\bigg|_{\tau_+} &= \boldsymbol{W}_L\,h\,\bigg|_{t=0} + O(\varepsilon^2),\\ \frac{\partial\mathcal{L}_L}{\partial\dot q}\bigg|_{t=0} &= -\,\boldsymbol{W}\,\eta\,\bigg|_{\tau_+} + O(\varepsilon^2).\end{aligned} \tag{5.11}$$

Now we can calculate the complex action (5.3) up to quadratic terms in ε. Linear terms follow from (5.4), while the quadratic ones can be obtained by using Eq. (4.5) for the Euclidean action and its Lorentzian analogue. In virtue of Eqs. (5.7) and (5.8) all the volume terms turn to be $O(\varepsilon^3)$, so that the quadratic contribution of imaginary corrections reduces to the sum of surface terms at the nucleation point $\tau = \tau_+$ $(t = 0)$

$$\begin{aligned}\mathcal{I}[\Phi] = I[\phi] - i\,S[q] + i\,\frac{\partial\mathcal{L}_E}{\partial\dot\phi}\,\eta\,\bigg|_{\tau_+} - \frac{\partial\mathcal{L}_L}{\partial\dot q}\,h\,\bigg|_{t=0}\\ -\frac{1}{2}\,\eta^T(\boldsymbol{W}\eta)\,\bigg|_{\tau_+} - \frac{i}{2}\,h^T(\boldsymbol{W}_L h)\,\bigg|_{t=0} + O(\varepsilon^3).\end{aligned} \tag{5.12}$$

Here we took into account the reality of q_+ and the no-boundary regularity conditions leading to vanishing surface terms at $t = 0$ and $\tau = 0$. Then the use of Eqs. (5.11) and the relation $h(0) = \eta(\tau_+)$ allows us to rewrite the above expression with $\varepsilon^T(\boldsymbol{\mathcal{W}}\varepsilon)$ denoting the full quadratic form in the variables $\varepsilon \equiv \operatorname{Im}\Phi(z) = (h(t), \eta(\tau))$

$$\mathcal{I}[\Phi] = I[\phi] - i\,S[q] + \tfrac{1}{2}\,\varepsilon^T(\boldsymbol{\mathcal{W}}\varepsilon) + O(\varepsilon^3), \tag{5.13}$$

$$\varepsilon^T(\boldsymbol{\mathcal{W}}\varepsilon) = \eta^T(\boldsymbol{W}\eta)\,\big|_{\tau_+} + i\,h^T(\boldsymbol{W}_L\,h)\,\big|_{t=0}. \tag{5.14}$$

The crucial point of our derivations is that the net effect of the Lorentzian-Euclidean matching conditions (5.11) and linear terms in the expression (5.12) consists in changing the overall sign of the quadratic form in η and h. This has a drastic consequence for the asymptotic $\hbar$ expansion of the wave function (5.1) with the complex extremal $\Phi(z\,|\,\varphi, z_+)$. Indeed, substituting (5.13) into the functional $\Psi_{[\,\Phi\,]}(f, z_+)$ given by (4.32) and reexpanding everything, except this exponentiated quadratic form, in powers of ε, one has

$$\Psi_L(\varphi, f, t_+) = e^{-\frac{1}{2\hbar}\varepsilon^T(\mathcal{W}\varepsilon)}\left\{\Psi_{[\,\mathrm{Re}\,\Phi\,]}(f, z_+) + O(\varepsilon) + O(\varepsilon^3/\hbar)\right\}. \tag{5.15}$$

Since $\eta(\tau)$ satisfies up to higher order terms the linearized equations (5.7) and no-boundary regularity conditions, the real part of this quadratic form coincides with the part of the Euclidean action quadratic in η and is positive definite by the assumption of good convexity properties of the Euclidean action

$$\mathrm{Re}\,\left[\varepsilon^T(\mathcal{W}\varepsilon)\right] = \delta_\eta^2 I + O(\eta^4). \tag{5.16}$$

Therefore, one can use the analogue of the asymptotic bound (4.29) to show that $\exp\left[-\frac{1}{2\hbar}\varepsilon^T(\mathcal{W}\varepsilon)\right]\varepsilon^n = O(\hbar^{n/2})$, $\hbar \to 0$, whence it follows that all the perturbation corrections of (5.15) in powers of ε actually belong to higher orders of a semiclassical expansion. Thus, despite the complex nature of $\Phi(z)$, the semiclassical expansion can still be performed on the real-valued background $\mathrm{Re}\,\Phi(z)\,|_{C_+} = (\phi(\tau), q(t))$, and with the corresponding elements of the Feynman diagrammatic technique – the inverse propagator, its basis functions and the matrix (4.26) of quantum dispersions for microscopic variables:

$$\boldsymbol{F} = \boldsymbol{F}_{[\,\mathrm{Re}\,\Phi\,]}, \quad D_L(t) \equiv D(\tau_+ + it)_{[\,\mathrm{Re}\,\Phi\,]}. \tag{5.17}$$

Imaginary corrections everywhere except the quadratic form of the action (5.13) can be treated by perturbations generating in higher orders additional set of Feynman diagrams.

One should emphasize a crucial role of the convexity of the Euclidean action, which provides the positivity of the form (5.16). In Einstein gravity theory this property is violated in the sector of the conformal mode which is widely believed to enter the set of physical variables in spatially closed cosmology [55]. The only known procedure of handling this mode consists in the rotation of its integration contour to the complex plane [35]. This means that the same conformal rotation must be done in the argument of the wave function, which implies the complexification of the configuration-space point q_+, $h(t_+) \neq 0$, and the corresponding modification of the formalism of complex extremals [32]. We shall not consider this modification here, and in what follows assume good properties of the Euclidean action. In this paper this will be justified by isolating the conformal mode into the sector of collective variables and considering (see Sec. VIII) only the high-energy behavior of their quantum distribution. This behavior is unaffected by the tree-level properties of the classical action and is determined by the quantum anomalous scaling of the theory (see discussion in Sec. IX).

VI. EUCLIDEAN VACUUM VIA NUCLEATION OF THE LORENTZIAN UNIVERSE FROM THE EUCLIDEAN SPACETIME

Combining Eqs. (3.11), (4.32), and (5.15) one obtains the wave function in the Lorentzian regime. Under the analytic continuation (3.11) the real Euclidean modes $\boldsymbol{u}_-(\tau)$ go over into complex functions $\boldsymbol{u}_-(z)$ on the contour C_+ (2.15). Thus, if we introduce the notation $(\boldsymbol{v}(t), \boldsymbol{v}^*(t))$ for the pair of complex conjugated functions originating from $\boldsymbol{u}_-(z)_{[\,\Phi\,]}$ on C_L, $z = \tau_B + it$,

$$\boldsymbol{v}(t) = (\boldsymbol{u}_-(\tau_B + it)_{[\,\Phi\,]})^*,$$
$$\boldsymbol{v}^*(t) = \boldsymbol{u}_-(\tau_B + it)_{[\,\Phi\,]}, \tag{6.1}$$

then the wave function of the Lorentzian universe (5.1) takes the form

$$\Psi_L(\varphi, f, t_+) = \mathrm{const}\,[\,\det \boldsymbol{v}^*(t_+)\,]^{-1/2}\exp\left\{-\frac{1}{2\hbar}f^T D_L(t_+)\,f\right\}\mathrm{e}^{-\frac{1}{\hbar}\mathcal{I}[\,\Phi\,]}\left[\,1 + O(\hbar^{1/2})\,\right], \tag{6.2}$$

where we have reabsorbed the quadratic form in ε into the full complex classical action.

The functions $(\boldsymbol{v}(t), \boldsymbol{v}^*(t))$ satisfy the complex conjugated equations – a direct corollary of the equation for $\boldsymbol{u}_-(z)_{[\,\Phi\,]}$. However, the above method allows us to treat $\varepsilon \equiv \mathrm{Im}\,\Phi(z)$ by perturbations and consider, instead of the complex operator $\boldsymbol{F}_{[\,\Phi\,]}(d/dz)$, the real Euclidean $\boldsymbol{F}(d/d\tau)$ and Lorentzian operators $\boldsymbol{F}_L(d/dt)$ related to $\boldsymbol{F}_{[\,\Phi\,]}(d/dz)$ by

$$\boldsymbol{F}_{[\,\Phi\,]}(d/dz)\,\big|_{C_E} = \boldsymbol{F}(d/d\tau) + O(\varepsilon), \;\; \boldsymbol{F}_{[\,\Phi\,]}(d/dz)\,\big|_{C_L} = -\boldsymbol{F}_L(d/dt) + O(\varepsilon). \tag{6.3}$$

Here the Euclidean operator is defined in (5.17), while the Lorentzian operator $\boldsymbol{F}_L(d/dt) = -\boldsymbol{F}_{[\,\mathrm{Re}\,\Phi\,]}(d/idt)\,\big|_{C_L}$ is calculated at the real background $q(t) = \mathrm{Re}\,Q(t)$ and has the form analogous to (3.15)

$$\boldsymbol{F}_L(d/dt) = -\frac{d}{dt}\,a\,\frac{d}{dt} - \frac{d}{d\tau}\,b_L + b_L^T\,\frac{d}{d\tau} - c. \tag{6.4}$$

Its coefficients a and c are identically related to their Euclidean versions, while the coefficient b_L represents a Wick rotation of its Euclidean counterpart: $b_L = ib$.

The functions (6.1) satisfy the inhomogeneous equations with the operator $\boldsymbol{F}_L(d/dt)$, but the inhomogeneous terms $O(\varepsilon) = O(\hbar^{1/2})$ can be again discarded in the one-loop approximation. With this one-loop accuracy we, therefore, have the real-valued differential equation for the complex Lorentzian basis functions

$$\boldsymbol{F}_L\boldsymbol{v} = 0, \quad \boldsymbol{F}_L\boldsymbol{v}^* = 0. \tag{6.5}$$

The Wronskian operator $\boldsymbol{W}_L(d/dt)$ of (6.4) generates the variational equation (5.10) for the canonical momentum and also enters the Lorentzian analogue of the relation (4.8) valid for arbitrary test functions $h_1(t)$ and $h_2(t)$

$$h_1^T (\boldsymbol{F}_L h_2) - (\boldsymbol{F}_L h_1)^T h_2 = -\frac{d}{dt}\left[h_1^T (\boldsymbol{W}_L h_2) - (\boldsymbol{W}_L h_1)^T h_2 \right], \quad (6.6)$$

$$\boldsymbol{W}_L (d/dt) = a\frac{d}{dt} + b_L,$$
$$\boldsymbol{W}_L (d/dt) = i\boldsymbol{W}_{[\mathrm{Re}\,\Phi]} (d/idt) \,\big|_{C_L}. \quad (6.7)$$

The reality of Lorentzian wave and Wronskian operators is crucial for establishing the complex structure on the space of classical solutions, which gives rise to the "beginning of time" [26]. For a second-order differential equation $\boldsymbol{F}_L h = 0$ with real coefficients, the complex linear space of solutions $h(t)$ can be equipped with the conserved inner product

$$\langle h_1, h_2\rangle = i\left[h_1^\dagger (\boldsymbol{W}_L\, h_2) - (\boldsymbol{W}_L\, h_1)^\dagger h_2 \right], \quad (6.8)$$

where $h^\dagger \equiv (h^*)^T$ is a Hermitian conjugation involving both the transposition and the complex conjugation. When the functions $h(t)$ are related by the analytic continuation to their Euclidean counterparts $\varphi(\tau)$, this inner product reduces to the Wronskian construction of the Euclidean operator $\boldsymbol{F}$

$$\varphi_1^T (\boldsymbol{W}\varphi_2) - (\boldsymbol{W}\varphi_1)^T \varphi_2 = -\langle h_1^*, h_2\rangle,$$
$$h_{1,2}(t) = \varphi_{1,2}(\tau_B + it). \quad (6.9)$$

Now, if we take as $\varphi_{1,2}(\tau)$ two Euclidean basis functions $\boldsymbol{u}_-(\tau)$, having a vanishing Wronskian $\boldsymbol{u}_-^T (\boldsymbol{W}\boldsymbol{u}_-) - (\boldsymbol{W}\boldsymbol{u}_-)^T \boldsymbol{u}_- = 0$ in view of their regular behavior at $\tau = 0$ (4.30), then the complex conjugated Lorentzian basis functions (6.1) satisfy the orthogonality relation

$$\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle = 0. \quad (6.10)$$

On the other hand, Lorentzian basis functions of one "positive frequency" have the conserved matrix of inner products

$$\boldsymbol{\Delta} = \langle \boldsymbol{v}, \boldsymbol{v}\rangle, \quad \boldsymbol{\Delta} \equiv \boldsymbol{\Delta}_{AB}, \quad (6.11)$$

which can be calculated at the point of nucleation $t = 0$ ($\tau = \tau_B$) where the following matching conditions hold between the Lorentzian and Euclidean modes: $\boldsymbol{u}_-(\tau_B) = \boldsymbol{v}(0) = \boldsymbol{v}^*(0)$, $(\boldsymbol{W}\boldsymbol{u}_-)(\tau_B) = i\boldsymbol{W}_L\, \boldsymbol{v}^*(0)$. In virtue of these matching conditions this matrix

$$\boldsymbol{\Delta} = 2\, \boldsymbol{u}_-^T(\boldsymbol{W}\boldsymbol{u}_-)\,\big|_{\tau_B}, \quad \boldsymbol{\Delta} = \boldsymbol{\Delta}^T, \quad \boldsymbol{\Delta} = \boldsymbol{\Delta}^*, \quad (6.12)$$

coincides with the kernel of the positive definite part of the Euclidean action quadratic in the fields $\delta\phi(\tau) = \boldsymbol{u}(\tau)\,\eta$, $\delta^2 I = \eta^T \left[\boldsymbol{u}_-^T(\boldsymbol{W}\boldsymbol{u}_-)\,|_{\tau_B}\right]\eta = \frac{1}{2}\eta^T \boldsymbol{\Delta}\,\eta > 0$. Therefore, $\boldsymbol{\Delta}$ is a real positive-definite symmetric matrix, which together with the orthogonality relations (6.10) implies that $\boldsymbol{v}(t)$ and $\boldsymbol{v}^*(t)$ can be regarded as a set of Lorentzian positive and negative frequency modes.

According to the decomposition (4.1) and the corresponding block-diagonal structure of the relevant Euclidean operators and modes (4.16)–(4.18), a similar decomposition properties hold for their Lorentzian counterparts

$$\boldsymbol{F}_L = \mathrm{diag}\,(F_{\varphi\, L}, F_L), \quad \boldsymbol{W}_L = \mathrm{diag}\,(W_{\varphi\, L}, W_L), \quad (6.13)$$
$$\boldsymbol{v}(t) = \mathrm{diag}\,(v_\varphi(t), v(t)), \quad \boldsymbol{\Delta} = \mathrm{diag}\,(\Delta_\varphi, \Delta).$$

Therefore all the Wronskian orthogonality relations of the above type hold separately in the sector of collective variables and the sector of microscopic ones. The operators $F_{\varphi\, L}$, $W_{\varphi\, L}$, and their modes $v_\varphi(t)$ determine the quantum properties of φ and of the background $\Phi(z)$.

The properties of the microscopic modes f in (6.2) are determined by the matrix of quantum dispersions D_L defined by Eqs. (4.26) and (5.17) in terms of $W_L(d/dt)$ and $v^*(t)$:

$$D_L(t) = -i\,[\,W_L(d/dt)\,v^*(t)\,]\,[v^*(t)\,]^{-1}. \quad (6.14)$$

In virtue of (6.10) this complex matrix is symmetric and has a positive-definite real part derivable from the corollary $W_L v^* = (v^\dagger)^{-1}(W_L v)^\dagger v^*$ of Eq. (6.10):

$$D_L^T - D_L = (v^\dagger)^{-1}\langle v, v^*\rangle\,(v^*)^{-1} = 0, \quad (6.15)$$
$$D_L^* + D_L = (v^\dagger)^{-1}\Delta\, v^{-1}. \quad (6.16)$$

As a consequence, the Gaussian state (6.2) is a vacuum of linearized modes f relative to the Lorentzian basis functions (6.1). Consider the Heisenberg operator of the linear quantum field $\hat f(t)$ decomposed into a set of $(v(t),\, v^*(t))$

$$\hat f(t) = v\,(t)\,\hat a + v^*(t)\,\hat a^* \equiv v_A(t)\,\hat a^A + v_A^*(t)\,\hat a^{*A} \quad (6.17)$$

with the operator Hermitian-conjugated coefficients $\hat a = \hat a^A$, and $\hat a^* = \hat a^{*A}$. In virtue of (5.10), the canonical momentum $\hat p\,(t)$ for this linear field equals

$$\hat p\,(t) = W_L(d/dt)\,\hat f(t) = (W_L v)\,(t)\,\hat a + (W_L v)^*(t)\,\hat a^*. \quad (6.18)$$

Then, the orthogonality (6.10) allows one to solve the system of Eqs. (6.17)–(6.18) for the operators $(\hat a,\, \hat a^*)$:

$$\hat a = i\,\Delta^{-1} v^\dagger \hat p - i\,\Delta^{-1}(W_L v)^\dagger \hat f,$$
$$\hat a^* = -i\,\Delta^{-1} v^T \hat p + i\,\Delta^{-1}(W_L v)^T \hat f. \quad (6.19)$$

In view of the commutation relations $[\,\hat f,\, \hat p\,] = i\,\hbar\, I$ [I is a unit matrix in the f sector of $\boldsymbol{I} = \mathrm{diag}\,(I_\varphi, I)$, and all the other commutators vanish], $\hat a$ and $\hat a^*$ have the only nonvanishing commutator

$$[\,\hat a^A, \hat a^{*B}\,] = \hbar\,(\Delta^{-1})^{AB}. \quad (6.20)$$

Since Δ is a real positive definite matrix, it can be diagonalized by linear transformations of basis functions $v(t)$ making their set orthonormal $\langle v_A, v_B\rangle = \Delta_{AB} = \delta_{AB}$, so that $\hat{a}$ and $\hat{a}^*$ become the usual annihilation and creation operators. In particular, the operator $\hat{a}$ in the coordinate representation, $\hat{f} = f$, $\hat{p} = \hbar\partial/i\partial f$, annihilates the Gaussian quantum state (6.2) of linearized perturbations

$$\hat{a}\,(f,\partial/\partial f) = \hbar v^\dagger \frac{\partial}{\partial f} - i\,(W_L v)^\dagger f,$$
$$\hat{a}\,(f,\partial/\partial f)\,\Psi_L\,(\varphi, f, t_+) = 0. \tag{6.21}$$

Thus, the Lorentzian universe nucleates from the Euclidean spacetime with the vacuum state, corresponding to the field decomposition in special positive and negative frequency basis functions [43,16]. They originate by the analytic continuation (6.1) from regular modes in the Euclidean ball. This decomposition and, therefore, the definition of the vacuum is unique, because the only admissible freedom consists in unitary rotations of the positive-frequency subset $v(t)$, preserving its orthonormality and not mixing $v(t)$ and $v^*(t)$. In de Sitter models this special vacuum state coincides with the so-called Euclidean vacuum [56] which has important applications in the theory of the inflationary universe [46], because it provides the spectrum of density fluctuations responsible for the formation of the large scale structure of the observable universe.

VII. QUANTUM DISTRIBUTION OF TUNNELING UNIVERSES

In applications, one needs the density matrix which can be obtained from the wave function by tracing out the unmeasured degrees of freedom. This procedure represents a nontrivial step toward quantities having a direct physical interpretation. Usually the degrees of freedom which carry the most important information about the system are some macroscopic collective variables of the type considered in Sec. IV, that is why the matter of primary interest within the above method of collective coordinates is their density matrix

$$\hat{\rho}\,(t) = \mathop{\rm Tr}_f |\,\Psi_L\,(t)\rangle\!\rangle\langle\!\langle\Psi_L\,(t)\,|. \tag{7.1}$$

This object encodes all the correlation functions of the collective variables φ. In particular, it includes the density of their probability distribution which is just the diagonal element $\rho\,(\varphi,\,t) = \rho\,(\varphi, \varphi\,|\,t)$ of $\hat{\rho}\,(t) = \rho\,(\varphi, \varphi'|\,t)$ in the coordinate representation of φ. In the physical Hilbert space with the trivial inner product (1.6), this diagonal element equals

$$\rho\,(\varphi,\,t) = \int df\,|\Psi_L\,(\varphi, f, t)\,|^2. \tag{7.2}$$

The knowledge of (6.2) allows us to calculate the full density matrix (7.1), but here we shall mainly concentrate on this quantity. It plays an important role in quantum cosmology of tunneling universes, for it determines their probability distribution in the space of such macroscopic variables as a Hubble constant, parameters of anisotropy, etc.

Substituting (6.2) into (7.2) and taking into account Eq. (6.16), one immediately finds the following answer for the Gaussian integral over f:

$$\rho\,(\varphi,\,t) = \mathrm{const}\;\frac{(\det\Delta_\varphi)^{1/2}}{|\det v_\varphi\,(t)\,|}\;(\det\boldsymbol{\Delta})^{-1/2}\,e^{-\frac{2}{\hbar}\mathrm{Re}\,\mathcal{I}\,[\,\Phi\,]}\left[\,1 + O\,(\hbar^{1/2})\,\right], \tag{7.3}$$

where the real part of the complex action (5.13)

$$\mathrm{Re}\,\mathcal{I}[\,\Phi\,] = I\,[\,\phi\,] + \tfrac{1}{2}\,\eta^T(\boldsymbol{W}\eta)\,(\tau_B) + O\,(\,\hbar^{3/2}), \tag{7.4}$$

must include the positive definite real part of the quadratic form (5.14) damping the contribution of the imaginary corrections $\varepsilon = O\,(\hbar^{1/2})$. Here we took into account the block-diagonal form of the matrices $\boldsymbol{v}$ and $\boldsymbol{\Delta}$, $\det\boldsymbol{v}\,(t) = \det v_\varphi\,(t)\det v\,(t), \det\boldsymbol{\Delta} = \det\Delta_\varphi\det\Delta$, due to which the one-loop preexponential factor of (7.3) includes the determinant of the *full* Wronskian matrix $\boldsymbol{\Delta}$ and the determinant of the Lorentzian modes of the collective variables $v_\varphi\,(t)$ normalized by $(\det\Delta_\varphi)^{1/2}$ to unity in the inner product (6.8). The emergence of the full Wronskian matrix in this algorithm in conjunction with the reduction method for functional determinants on closed spacetimes [33] make us to consider in the next section a special geometric interpretation of (7.3).

A. Doubling the Euclidean spacetime

The doubled Euclidean spacetime serves for the interpretation and covariant calculation of the partition function (7.3). It was proposed for these purposes in Ref. [12] and also used for the general analysis of real tunneling geometries in Ref. [25]. To begin with, note that the matrix $\boldsymbol{\Delta}$ in (7.3) can be represented by Eq. (6.12) in terms of the regular basis functions on the Euclidean spacetime ball $\boldsymbol{M} \equiv \boldsymbol{M}_- = \boldsymbol{B}^4$ with the "center" at τ_-. This spacetime carries a real Euclidean metric and matter fields characterized by $\phi\,(\tau) = \mathrm{Re}\,\Phi\,(\tau)$ and has as a boundary the nucleation surface Σ_B, $\tau = \tau_B$. Now consider its orientation reversed copy $\boldsymbol{M}_+$, which can be regarded as mirror image of $\boldsymbol{M}_-$ with respect to this boundary. One can now construct the doubled manifold $\boldsymbol{2M}$ by joining $\boldsymbol{M}_-$ and $\boldsymbol{M}_+$ across their common boundary Σ_B (see Fig. 4)

$$\boldsymbol{2M} = \boldsymbol{M}_- \cup \boldsymbol{M}_+. \tag{7.5}$$

This doubled manifold is closed, has a topology of a four-dimensional sphere, and admits an isometry θ mapping its two halves $\boldsymbol{M}_\pm$ onto one another

$$\boldsymbol{M}_\pm = \theta\boldsymbol{M}_\mp: \quad \{\,x \in \boldsymbol{M}_\pm,\; \theta\,x \in \boldsymbol{M}_\mp\,\}. \tag{7.6}$$

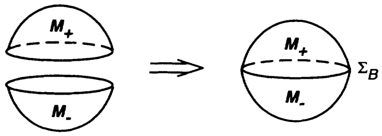

FIG. 4. The doubling of the Euclidean manifold, which arises in the calculation of the quantum distribution function for Lorentzian universes. The Euclidean spacetime of the no-boundary type M_- matched across the nucleation surface Σ_B with its orientation reversed copy M_+ gives rise to a closed manifold $2M = M_- \cup M_+$ – the gravitational instanton of spherical topology. For generic complex tunneling geometries the matching of M_- with M_+ is not smooth, which is shown on the picture by the edge at Σ_B.

The foliation of M_- by surfaces of constant τ can be continuously extended by this isometry to the whole of $2M$ with the parameter τ ranging from τ_- to τ_+ labeling the "center" of M_+ which coincides with the "north" pole of the doubled manifold. This foliation can be represented by the continuous one-parameter family of surfaces $\Sigma(\tau)$ which expand from zero volume at the south cap of $2M$ and then again shrink to a point at the north cap after passing the equatorial section $\Sigma_B = \Sigma(\tau_B)$ at $\tau_B = (\tau_- + \tau_+)/2$

$$\tau(x) = \tau,\;\; \mathbf{x}(x) = \mathbf{x},\;\; x \in M_-,\;\; \tau_- \le \tau \le \tau_B,$$
$$\tau(\theta x) = \tau_+ + \tau_- - \tau,\;\; \mathbf{x}(\theta x) = \mathbf{x},\;\; \theta x \in M_+. \tag{7.7}$$

This foliation explicitly demonstrates the reversal of Euclidean time on M_+ in contrast to spatial coordinates $\mathbf{x}$ identically related on surfaces Σ and $\theta\Sigma$. Obviously, the coordinate τ parametrizing the whole of $2M$ plays the role of the latitude angle θ on the four-dimensional sphere homeomorphic to $2M$, ranging from 0 to π, while $\mathbf{x}$ are the "angular" coordinates on quasispherical spatial sections Σ.

The four-geometry and matter fields on $2M$ are also a subject of the isometry map (7.6), which means that on M_+ they are defined as a reflection image of those on the original spacetime $M_- = M$. In the foliation (7.7) this fact can be easily represented as a following definition of the field $\phi(\tau)$ for $\tau_B \le \tau \le \tau_+$ in terms of those for $\tau_- \le \tau \le \tau_B$:

$$\phi(\tau) = \phi(\tau_+ + \tau_- - \tau). \tag{7.8}$$

Such fields are continuous but generally nonanalytic at the junction surface Σ_B unless their normal derivative $d\phi/d\tau(\tau_B)$ vanishes there. In case of real tunneling geometries [25,26] this condition is satisfied as a requirement of vanishing extrinsic curvature of Σ_B, $K_{ab}|_{\Sigma_B} = 0$, this fact providing the analytic matching of real Euclidean manifold M_- with its double M_+ and with the nucleating real Lorentzian spacetime M_L. For complex tunneling fields this condition is generally violated, because $\dot\phi(\tau_B) \sim \partial\mathcal{L}_E/\partial\dot\phi(\tau_B) = O(\varepsilon)$ in view of the matching condition (5.11).

The lack of smoothness of the background fields does not prevent us from extending the basis function $u_-(\tau)$ to the whole of $2M$ as a solution of the equation $Fu_-|_{2M} = 0$ with continuous zeroth and first order derivatives at Σ_B (the second-order derivatives will generally jump at Σ_B, because the coefficients of F are discontinuous at this surface). These basis functions are regular at the south pole τ_- of the doubled manifold, but singular at τ_+, because we assume that the Euclidean operator F does not have zero modes on $2M$. There exists another set of basis functions u_+ which are the reflection image of u_- defined in the foliation (7.7) by the relation [57]

$$u_+(\tau) = u_-(\tau_+ + \tau_- - \tau). \tag{7.9}$$

These basis functions are regular at τ_+, singular at τ_-, and satisfy the following matching conditions at the junction surface Σ_B:

$$u_-(\tau_B) = u_+(\tau_B),\quad Wu_-(\tau_B) = -Wu_+(\tau_B). \tag{7.10}$$

Therefore, the set of inner products Δ, given by Eq. (6.12), can be rewritten in the form of the Wronskian matrix $\Delta_{+-} = (\Delta_{+-})_{AB}$ of these two sets of Euclidean basis functions on the doubled manifold [independent of τ in virtue of the relation (4.8)]

$$\Delta = \Delta_{+-},\quad \Delta_{+-} = u_+^T\,(Wu_-) - (Wu_+)^T u_-. \tag{7.11}$$

This property serves for the following important observation. According to the reduction technique of Ref. [33] for functional determinants on a spacetime of spherical topology, the one-loop preexponential factor on such a spacetime can be generated by the determinant of the Wronskian matrix (7.11) of the two complete sets of basis functions $u_\pm(\tau)$ which, in the τ foliation of the above type, are regular, respectively, at τ_+ and τ_- and have the asymptotic behavior [cf. Eq. (4.30)]

$$u_-(\tau) = I\,(\tau - \tau_-)^{\mu_-} + O\left[(\tau - \tau_-)^{1+\mu_-}\right],\quad \tau \to \tau_-, \tag{7.12}$$

$$u_+(\tau) = I\,(\tau_+ - \tau)^{\mu_-} + O\left[(\tau_+ - \tau)^{1+\mu_-}\right],\quad \tau \to \tau_+ \tag{7.13}$$

with the field-independent coefficient – the matrix unity I. This reduction algorithm reads

$$(\det\Delta_{+-})^{-1/2} = \text{const}\left[\mathop{\mathrm{Det}}_{2M} F/\mathop{\mathrm{Det}}_{2M} a\right]^{-1/2} \tag{7.14}$$

and implies that the preexponential factor of our partition function (7.3) in the main boils down to the contribution of functional determinants on $2M$. Such determinants are calculated on the space of functions regular on closed compact manifold $2M$ and constitute the one-loop *effective* action of the Euclidean theory on this spacetime.

B. Covariant distribution function: Covariance versus unitarity

Using the above relation in (7.3) we arrive at the algorithm

$$\rho(\varphi, t) = \text{const}\, \frac{(\det \Delta_\varphi)^{1/2}}{|\det v_\varphi(t)|} \exp\left\{-\frac{1}{\hbar}\mathbf{\Gamma}_{1-\text{loop}}[\phi] - \frac{1}{\hbar}\eta^T(\boldsymbol{W}\eta)_B\right\}\left[1+O(\hbar^{1/2})\right]. \quad (7.15)$$

Here $v_\varphi(t)$ is a set of Lorentzian modes of collective variables φ which has a matrix of inner products (6.8) and $\mathbf{\Gamma}_{1-\text{loop}}[\phi]$ is the effective action on the doubled Euclidean spacetime:

$$F_{\varphi L}(d/dt)\, v_\varphi(t) = 0, \quad \langle v_\varphi, v_\varphi\rangle = \Delta_\varphi, \quad (7.16)$$

$$\mathbf{\Gamma}_{1-\text{loop}}[\phi] = I_{2M}[\phi] + \frac{\hbar}{2}\operatorname*{Tr}_{2M}\ln \boldsymbol{F} - \frac{\hbar}{2}\operatorname*{Tr}_{2M}\ln \boldsymbol{a}, \quad (7.17)$$

$$I_{2M}[\phi] = 2\, I_M[\phi]. \quad (7.18)$$

The background field $\phi(\tau)$ on $\boldsymbol{2M}$ is a real part of the exact complex extremal $\Phi(z) = \Phi(z|\varphi, t)$, parametrized by the boundary data (φ, t):

$$\phi(\tau) = \operatorname{Re}\Phi(\tau|\varphi, t), \quad 0 \le \tau \le \tau_B. \quad (7.19)$$

The effective action (7.17) includes the classical Euclidean action (7.18) on $\boldsymbol{2M}$ and the one-loop contribution given by the logarithm of (7.14).

It would seem that the new algorithm (7.15) does not have any advantages over the original expression (7.3), for the replacement of $\det\boldsymbol{\Delta}$ by the determinant of higher functional dimensionality complicates the calculations. However, the new form of $\rho(\varphi, t)$ is covariant and, therefore, subject to powerful manifestly covariant methods of calculating $\mathbf{\Gamma}_{1-\text{loop}}[\phi]$. They allow one to perform a covariant regularization of the divergent partition function and obtain its high-energy behavior, which will be considered below.

The usual price one pays for manifest covariance is the loss of manifest unitarity. The root of the difficulty is that, in covariant quantization, the physical sector is deeply hidden in the full space of the theory including ghosts, zero and negative norm states, etc., and very subtle methods are required to recover unitarity from the covariant formalism or, vice versa, render the unitary theory a manifestly covariant form [58,45,49,23]. A remarkable feature of the algorithm (7.15) is that, being formulated in terms of the physical degrees of freedom, it combines both of the desired properties: the covariance of radiative corrections in the Euclidean effective action (7.17) and its one-loop unitarity encoded in the preexponential factor.

C. Unitarity and partition function of gravitational instantons

To prove unitarity, which is basically the conservation of the total probability, note that a set of basis functions $v_\varphi(t)$ in (7.15) can be obtained from the family of classical extremals by differentiating them with respect to the constants of motion. The resulting functions satisfy the linearized equations of motion and the same regularity conditions as these classical extremals. In our case the extremals $\Phi(z|\varphi, t_+)$ are parametrized by their boundary conditions φ at the final moment of Lorentzian time t_+, but for our purposes it will be more convenient to parametrize them by the value ϕ_B of their real part at the moment of the Lorentzian nucleation:

$$\Phi(z) = \Phi(z, \phi_B) = \phi(z, \phi_B) + i\,\eta(z, \phi_B), \quad (7.20)$$

$$\phi_B = \phi(\tau_B, \phi_B),$$

$$v^*_\varphi(t) = \left.\frac{\partial\Phi(z, \phi_B)}{\partial\phi_B}\right|_{z=\tau_B+it}. \quad (7.21)$$

Since $\Phi(z_+, \phi_B) = \varphi$, the matrix of the above basis functions is real at t_+, and its determinant coincides with the Jacobian of transformation from ϕ_B to φ

$$\varphi \to \phi_B, \quad \det v_\varphi(t_+) = \det\,(\partial\varphi/\partial\phi_B). \quad (7.22)$$

Further proof is based on the above method according to which the imaginary part ε can be treated by perturbations in all the terms except the quadratic form in the total exponential of (7.15). When combined with the classical action (7.18), this form gives rise to the (doubled) imaginary part $I_{2M}[\phi] + \eta^T(\boldsymbol{W}\eta)_B + O(\eta^3) = 2I(t_+, \varphi) \equiv 2\operatorname{Im}\mathcal{S}[\Phi(z)]$ of the full complex action (5.13), $\mathcal{S}[\Phi(z)] \equiv i\mathcal{I}[\Phi(z)] = S(t_+, \varphi) + iI(t_+, \varphi)$. As a function of the data (t_+, φ) at the end point z_+ of the extremal, this action $\mathcal{S}[\Phi(z)] = \mathcal{S}(t_+, \varphi)$ can be regarded as a complex Hamilton-Jacobi function. It satisfies the Hamilton-Jacobi equation with the physical Hamiltonian $H(\varphi, p)$ which in its turn generates the following equation for the imaginary part $I = I(t_+, \varphi)$:

$$\frac{\partial I}{\partial t_+} + \frac{\partial I}{\partial\varphi}\left.\frac{\partial H}{\partial p}\right|_{p=\partial S/\partial\varphi} = O\left[\left(\frac{\partial I}{\partial\varphi}\right)^3\right]. \quad (7.23)$$

In view of the Euclidean-Lorentzian matching conditions of Sec. V, $\partial I/\partial\varphi = O(\varepsilon) = O(\hbar^{1/2})$, so that the solution of (7.23), $I(t, \bar\varphi(t)) = I(0, \bar\varphi(0)) + O(\hbar^{3/2})$, is practically a constant [36] along the real-valued trajectory $\bar\varphi(t)$ evolving according to

$$\dot{\bar\varphi} = \left.\frac{\partial H(\bar\varphi, p)}{\partial p}\right|_{p=\partial S(t,\bar\varphi)/\partial\bar\varphi}, \quad \bar\varphi(t_+) = \varphi. \quad (7.24)$$

The latter differs from the real part of the exact complex extremal at most by $O(\varepsilon) = O(\hbar^{1/2})$ terms, $\bar\varphi(t) = \phi(\tau_B + it, \phi_B) + O(\varepsilon)$.

Thus, the dependence on t_+ in $I(t_+, \varphi)$ can be completely absorbed into the redefinition of the field variable, $\varphi \to \varphi_B \equiv \bar\varphi(0)$. Correspondingly, the tree-level part of (7.15) can be regarded as the Euclidean action $I_{2M}[\bar\phi]$ on a new real classical background $\bar\phi(\tau)$ with the boundary condition φ_B at τ_B, the latter being determined as a function of (t_+, φ) from the solution of (7.24):

$$\delta I[\bar\phi]/\delta\bar\phi(\tau) = 0, \quad \bar\phi(\tau_B) = \varphi_B(t_+, \varphi). \quad (7.25)$$

With the same accuracy the full one-loop exponential in (7.15) turns out to be the effective action on this new background $\bar{\phi}(\tau)$ or can be regarded as a function of its boundary data $\varphi_B = \phi_B + O(\hbar^{1/2})$ at the nucleation surface

$$\mathbf{\Gamma}_{1-\text{loop}}[\phi] + \eta^T(\boldsymbol{W}\eta)_B = \mathbf{\Gamma}_{1-\text{loop}}[\bar{\phi}] = \mathbf{\Gamma}_{1-\text{loop}}(\varphi_B). \tag{7.26}$$

On the other hand, by differentiating the ε analogue of the asymptotic bound (4.29) one finds that $\partial\varepsilon/\partial\phi_B = O(\hbar^{1/2})$, whence (7.22) takes the form of the following change of variables:

$$\varphi \to \varphi_B, \quad [\det v_\varphi(t_+)]^{-1} = \det(\partial\varphi_B/\partial\varphi) + O(\hbar^{1/2}), \tag{7.27}$$

absorbing all the dependence of (7.15) on the Lorentzian time t_+.

Therefore, the quantum distribution of tunneling geometries reduces to the partition function $\rho_{2M}(\varphi_B)$ of the gravitational instantons $\boldsymbol{2M}$ with a special classical background field subject to the boundary conditions (7.25) at the junction surface Σ_B:

$$\rho(\varphi, t_+) = \rho_{2M}(\varphi_B) \det \frac{\partial\varphi_B}{\partial\varphi}\Big|_{\varphi_B=\varphi_B(\varphi,t_+)}, \tag{7.28}$$

$$\rho_{2M}(\varphi_B) = \text{const}\,[\det \Delta_\varphi(\varphi_B)]^{1/2}\, e^{-\frac{1}{\hbar}\mathbf{\Gamma}_{1-\text{loop}}(\varphi_B)}\left[1 + O(\hbar^{1/2})\right]. \tag{7.29}$$

As a result one has

$$\int d\varphi\, \rho(\varphi, t_+) = \int d\varphi_B\, \rho_{2M}(\varphi_B) = \text{const}, \tag{7.30}$$

which accomplishes the proof of unitarity for the distribution function [59].

The algorithms (7.15) or (7.28) and (7.29) have a good graphical illustration demonstrating their unitarity. The partition function, as an inner product of the wave function with itself, is shown in Fig. 5 as a composition of the two spacetime manifolds combined of Euclidean and Lorentzian domains and associated, respectively, with $\Psi_L(\varphi, f, t)$ and $\Psi_L^*(\varphi, f, t)$. Due to unitarity, which makes sense only in physical Lorentzian spacetime, the Lorentzian "brims" of these two "hats" cancel, because this portion of the spacetime is described by the unitary evolution operator. What remains is, in the main, the doubled Euclidean manifold $\boldsymbol{2M}$ – the gravitational instanton of spherical topology serving as a support for the Euclidean action (7.17). This nontrivial remnant can be explained by the fact that the dynamical "evolution" on Euclidean spacetime is described by the nonunitary heat equation rather than the Schrödinger one.

The analytic continuation from Euclidean spacetime (Wick rotation) for the matrix elements between different quantum states – in and out asymptotic vacua – is well known in the asymptotically flat case and usually serves as a calculational basis in scattering theory. A similar technique for expectation values, that is matrix elements of operators with respect to one and the same state, is much less known because of the obvious difficulties with analyticity. In contrast to the wave function, the corresponding expectation values can never be analytic for they involve both $\Psi_L(q,t)$ and its complex conjugate $\Psi_L^*(q,t)$. However, in the context of a special quantum state – the standard asymptotic in-vacuum – there exists a special technique relating the expectation values in Lorentzian spacetime to the Euclidean effective action [60]. Apparently, the algorithm (7.15) is the first analogue of this technique for spatially closed spacetime and with the no-boundary quantum state of the system. Obviously, this state plays the role of the standard in-vacuum of asymptotically flat worlds, its regular Euclidean modes being the counterparts of the positive energy plane waves which under the Wick rotation go over into the modes vanishing in the remote Euclidean "past." The same analogy also transpires in the reduction methods for functional determinants of Ref. [33] where the south and north poles of the compact sphere-like manifold were associated with the $\pm\infty$ of the asymptotically flat spacetime, while the corresponding regular modes $\boldsymbol{u}_\pm$ were associated with the basis functions of the in and out vacua.

In the general case the extremals are complex, and in the one-loop approximation the effect of their complexity boils down to the Gaussian factor in (7.15) damping the contribution of their imaginary part. Apparently, this property explains the lack of interest in literature to complex instantons, the contribution of which is always

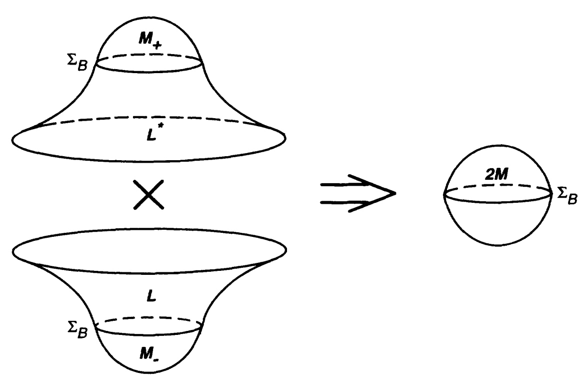

FIG. 5. The graphical representation of calculating the quantum distribution of tunneling Lorentzian universes: a composition of the combined Euclidean-Lorentzian spacetime $\boldsymbol{M}_- \cup \boldsymbol{L}$ with its orientation reversed and complex conjugated copy $\boldsymbol{M}_+ \cup \boldsymbol{L}^*$ results in the doubled Euclidean manifold $\boldsymbol{2M}$ – the gravitational instanton carrying the Euclidean effective action of the theory. The cancellation of the Lorentzian domains $\boldsymbol{L}$ and $\boldsymbol{L}^*$ reflects the unitarity of the theory in the physical spacetime of Lorentzian signature.

exponentially suppressed compared to real tunneling solutions. However, there are problems lacking the real solutions, in which case the technique of the above type becomes indispensable. The important example is the model of chaotic inflation driven by the inflaton scalar field. We shall consider this model in much detail in a forthcoming paper [32] and also use it in the next section to illustrate the issue of the high energy behavior and normalizability of the no-boundary wave function.

VIII. HIGH-ENERGY BEHAVIOR

As discussed in the Introduction, the validity of a semiclassical expansion essentially depends on the energy scale of the problem. The latter is determined by the quantum state of the system and the location of maxima of the corresponding partition function for those variables which play the major role in its dynamics. Thus, the validity of the loop expansion in quantum cosmology can follow from the behavior of the partition function of collective variables constructed above. This partition function includes the contribution of these variables themselves and also of the infinite set of microscopic field modes. Therefore, it suffers from the ultraviolet divergences and requires regularization and renormalization. In principle, these procedures must be done at all stages of calculating the wave function and partition function. Only in this case we would have the consistent and consecutive *operator* quantization. However, at the present state of art in high-energy physics, only in simple low-dimensional field models such an approach has been realized and has a well-established status. In realistic field theories we still have to skip the operatorial stage of quantization at the unregularized level and make regularization only in the final algorithms given by loop Feynman diagrams. Thus we shall consider the regularization of a partition function rather than the wave function itself.

Even apart from this liberty, there still remains a problem of whether the properly regularized infinities can be renormalized by physically sensible procedure. We do not discuss here this issue which is a subject of a vast literature on the over-Planckian structure of fundamental interactions. Instead, we assume that, whatever physical origin of this procedure is (either it is a fundamental finite string theory underlying its low-energy effective limit or the inclusion of the infinite set of counterterms), the correct renormalization consists in the subtraction of the covariantly regulated ultraviolet infinities. It is hard to perform a covariant regularization in the noncovariant ADM quantization. However, our partition function combines manifest unitarity with the Euclidean effective action, which can be rendered covariant form and, therefore, covariantly renormalized. Here we sketch this procedure which yields the high-energy behavior of the distribution function.

A. Covariant renormalization and anomalous scaling

Converting the effective action (7.17) into a covariant form consists in its transformation from physical variables ϕ to the original set of fields $\boldsymbol{g} = (\boldsymbol{q}, N_{\boldsymbol{E}})$ taken in some gauge which has a form of local conditions on $\boldsymbol{g}$ and its spacetime derivatives

$$\chi(\boldsymbol{g}, \partial\boldsymbol{g}) = 0. \tag{8.1}$$

Such a transformation is identical for the classical part of the effective action $I[\phi] = \boldsymbol{I}[\boldsymbol{g}]$, where the boldfaced notation is used for the classical action in the initial variables (cf. Sec. II). For its one-loop part it is given by the one-loop approximated Faddeev-Popov ansatz [61]

$$\frac{\hbar}{2}\,\mathrm{Tr}\ln\boldsymbol{F} - \mathrm{Tr}\ln\boldsymbol{a} = \frac{\hbar}{2}\,\mathrm{Tr}\ln\boldsymbol{\mathcal{F}} - \hbar\,\mathrm{Tr}\ln\boldsymbol{Q} + O\left(\delta\boldsymbol{I}/\delta\boldsymbol{g}\right). \tag{8.2}$$

It involves the wave operator of the full set of fields $\boldsymbol{g}$, determined by the total action $\boldsymbol{I}_{\rm tot}[\boldsymbol{g}]$ which includes the gauge breaking term with the gauge of the above type

$$\boldsymbol{\mathcal{F}} = \frac{\delta^2\boldsymbol{I}_{\rm tot}}{\delta\boldsymbol{g}\,\delta\boldsymbol{g}},\qquad \boldsymbol{I}_{\rm tot}[\boldsymbol{g}] = \boldsymbol{I}[\boldsymbol{g}] + \int d^4x\,\chi^2(\boldsymbol{g}, \partial\boldsymbol{g}) \tag{8.3}$$

and the corresponding ghost operator $\boldsymbol{Q}$ which is determined by the infinitesimal coordinate gauge transformation of gauge conditions (8.1).

On mass shell, that is on the solution of classical equations $\delta\boldsymbol{I}/\delta\boldsymbol{g} = 0$ for the background $\boldsymbol{g}$, the expression (8.2) is gauge independent [62] and exactly generates the one-loop effective action in physical variables [63]. This freedom in the choice of $\chi(\boldsymbol{g}, \partial\boldsymbol{g})$ allows us to choose them as belonging to the class of the background covariant gauge conditions [58,62], in which all the traces of the noncovariant (3+1)-splitting of the spacetime completely disappear and $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{Q}$ become local covariant differential operators of the second order. The functional determinants of such operators already admit the covariant regularization and have powerful calculational methods for their asymptotic scaling behavior [58,64,62].

For any local field theory, the classical action has the asymptotic scaling invariance under the global conformal transformations of the four-metric and matter fields $\boldsymbol{g} = (g_{\mu\nu}(x), \phi(x))$

$$\bar{g}_{\mu\nu}(x) = \Omega^2 g_{\mu\nu}(x),\quad \bar{\phi}(x) = \Omega^{c_\phi}\,\phi(x), \tag{8.4}$$

$$\boldsymbol{I}[\bar{\boldsymbol{g}}] \simeq \boldsymbol{I}[\boldsymbol{g}],\quad \Omega\to 0, \tag{8.5}$$

where c_ϕ represents the set of conformal weights of fields $\phi(x)$ [65]. On the contrary, covariantly regularized quantum corrections have the anomalous scaling behavior

$$\int d^4x\left(2\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}} + \sum_\phi c_\phi\,\bar{\phi}\frac{\delta}{\delta\bar{\phi}}\right)\frac{1}{2}\,\mathrm{Tr}\ln\boldsymbol{\mathcal{F}}[\bar{\boldsymbol{g}}] \simeq -\frac{1}{(4\pi)^2}A_2[\boldsymbol{g}],\quad \Omega\to 0, \tag{8.6}$$

defined by the coefficient A_2 of the Schwinger-DeWitt proper-time expansion for the functional trace of the heat kernel of the operator $\mathcal{F} = \mathcal{F}[g]$ [58,66]. It is given by volume and surface integrals over M and ∂M of local invariants constructed out of the coefficients of the operator $\mathcal{F}$, spacetime curvature and the extrinsic curvature of ∂M. Thus, in the limit of small distances the scaling behavior of the full effective action

$$\Gamma_{1-\text{loop}}[\bar{g}] \simeq -\frac{1}{(4\pi)^2} A_2^{\text{tot}}[g] \ln \Omega \Gamma_{1-\text{loop}}[g], \quad \Omega \to 0 \quad (8.7)$$

is determined by the total A_2 coefficient of both the gauge field $\mathcal{F}$ and ghost $\mathcal{Q}$ operators in (8.2). For theories invariant under local Weyl transformations (8.4) with the local parameter $\Omega = \Omega(x)$ the relations (8.5), (8.6), and (8.7) hold exactly, and the integrand on the left-hand side of (8.6) represents the conformal anomaly given by the volume density of A_2 – the coincidence limit of the DeWitt coefficient $a_2(x) = a_2(x,x)$ [58].

B. Anomalous scaling on the gravitational instanton and the normalizability of the Hartle-Hawking wave function

Application of the last equation with due regard for the algorithm (7.28) and (7.29) is of crucial importance in the model of the quantum birth of the chaotic inflationary universe. This model within a wide class of the field Lagrangians is considered in Refs. [14,32]. However, the issue of the high-energy behavior of the partition function, raised above, can be resolved in the universal and model-independent way.

As is well known [8,9] the tree-level Hartle-Hawking wave function is not normalizable in this model. As a function of the inflaton scalar field φ, it tends to a constant for $\varphi \to \infty$ and does not suppress the contribution of the over-Planckian energy scales. This follows from the algorithm (7.28) with the tree-level partition function

$$\rho_{\text{tree}}(\varphi_B) = \text{const}\, e^{-\frac{1}{\hbar} I_{2M}(\varphi_B)}, \quad (8.8)$$

$$I_{2M}(\varphi_B) = I_0 + I_1/\varphi_B^2 + O(1/\varphi_B^4), \quad \varphi_B \to \infty. \quad (8.9)$$

This behavior follows from the fact that the Euclidean segment of a complex classical history $\phi(\tau) = \Phi(\tau|\varphi, t)$, $0 \le \tau \le \tau_B$ has in the large-ϕ limit a simple form of a practically constant and real scalar field coinciding with its value at the nucleation point

$$\phi(\tau) \simeq \text{const} = \phi_B(\varphi, t), \quad \phi_B(\varphi, t) \simeq \varphi_B. \quad (8.10)$$

The value $\phi_B(\varphi, t)$ is parametrized in accordance with the form of Lorentzian extremal by its final point (φ, t) and always satisfies the inequality $\phi_B > \varphi$ because the scalar field slowly decreases during the inflationary stage. Therefore the limit $\varphi \to \infty$ guarantees large values of $\varphi_B \to \infty$. The field (8.10) generates an effective cosmological constant $\Lambda = 3H^2(\varphi_B)$. For Lagrangians viable from the viewpoint of the inflationary scenario, it grows monotonically, $H(\varphi_B) \to \infty$, for $\varphi_B \to \infty$. The corresponding solution is the metric of the Euclidean de Sitter space (1.3) and (1.4) with the radius $R = 1/H$, which generates the Lorentzian de Sitter universe by the nucleation at $\tau_B = \pi/2H$. The deviation of the full Euclidean-Lorentzian extremal from the exactly de Sitter form and its imaginary corrections are vanishing for large H and, therefore, in this high-energy limit the doubled manifold $2M$ is a four-dimensional sphere S^4 of vanishing radius R, carrying the four-geometry (1.3) and (1.4) and constant scalar field (8.10) $g_R = (g_{\mu\nu}^{DS}, \varphi_B)$

$$2M = S^4, \quad g_R = (g_{\mu\nu}^{DS}, \varphi_B), \quad R = 1/H(\varphi_B) \to 0. \quad (8.11)$$

The Euclidean action on this instanton has a form (VIII B) with the coefficients depending on the model of coupled gravitational and inflaton scalar field [14,32]. Therefore, the tree-level partition function (8.8), does not suppress the over-Planckian energy scales $\varphi_B \to \infty$.

The situation drastically changes in the one-loop approximation, when the classical action is replaced by the effective one (7.17). The asymptotic behavior of $\Gamma_{1-\text{loop}}$ follows from Eq. (8.7) with the parameter $\Omega = \mu^2/H^2(\varphi_B)$

$$\Gamma_{1-\text{loop}}[g_R] \simeq Z \ln \frac{H^2(\varphi_B)}{\mu^2}, \quad H(\varphi_B) \to \infty, \quad (8.12)$$

$$Z = \frac{1}{(4\pi)^2} A_2^{\text{tot}}[g_R] \Big|_{H(\varphi_B) \to \infty}, \quad (8.13)$$

where Z is a total anomalous scaling on the de Sitter instanton of vanishing size with the background fields (8.11) and μ is a mass parameter reflecting the renormalization ambiguity. Thus, the distribution (7.29) of the de Sitter instantons has the behavior

$$\rho(\varphi_B) \simeq \text{const}\,[H(\varphi_B)]^{-Z-2}, \quad H(\varphi_B) \to \infty, \quad (8.14)$$

where two extra negative powers of $H(\varphi_B)$ come from the Wronskian normalization $(\Delta_\varphi)^{1/2}$ for the inflaton mode $v_\varphi(t) = [\partial\phi_B(\varphi, t)/\partial\varphi]^{-1}$ (see Refs. [14,32]).

Therefore, depending on the value of Z, this partition function either suppresses the contribution of the over-Planckian energy scales or infinitely enhances it and serves as an applicability criterion for the semiclassical expansion. In particular, for theories with nonminimally coupled inflaton field with quartic self-interaction, for which $H(\varphi_B) \sim \varphi_B$, the high-energy normalizability of the distribution function, $\int^\infty d\varphi_B\, \rho(\varphi_B) < \infty$, implies [12]

$$Z > -1. \quad (8.15)$$

Since the value of Z is determined from (8.13) by the full field content of the universe [67], this condition serves as a selection criterion of physically consistent models [12,13]. Moreover, together with Eqs. (8.9) and (8.12) the distribution function can generate the quantum scale of inflation [14] consistent with observations (see below).

These conclusions are restricted to the one-loop approximation. But general principles of unitarity and covariance, which we have just verified in this approximation, allow us to conjecture that beyond one loop, and even nonperturbatively, the basic algorithm (7.28) and (7.29) will still be valid with $\mathbf{\Gamma}_{1-\mathrm{loop}}$ replaced by the full effective action $\mathbf{\Gamma}$ calculated at the instanton solution of the exact effective equations. Therefore, the condition (8.15) will still hold with the *exact nonperturbative* anomalous scaling $\boldsymbol{Z}$ replacing its simple one-loop expression (8.13).

IX. DISCUSSION

Thus, within the $\hbar$ expansion the theory of tunneling geometries generalizes to complex solutions which, in their turn, can be described entirely in terms of the Euclidean and Lorentzian spacetimes with real metrics and matter fields. In the calculation of the quantum distribution function, they naturally lead to the notion of the gravitational instanton. Despite the underlying complexity, this distribution function features unitarity and, thus, demonstrates a subtle interplay between unitarity, analyticity, and covariance, encoded in a partition function of gravitational instantons weighted by their Euclidean effective action.

The technique of this paper essentially relies on the convexity properties of the physical (reduced) Euclidean gravitational action. In quantum cosmology of closed worlds this presents an immediate difficulty in the conformal sector which, in contrast to asymptotically flat spacetimes [41,42], seems to be ascribed to physical degrees of freedom. If there is no way to consistently declare the conformal mode the unphysical one, one is left with the only option briefly discussed in Sec. V – to shift this variable into a complex plane both in *the integration contour* of the path integral [35] and in *the argument* of the wave function or its quantum distribution $\rho(\varphi, t)$. This begins with isolating this mode in the sector of collective variables φ and modifying the corresponding perturbation theory in the imaginary part of the classical extremal for φ. This modification is not critical for the high-energy behavior of $\rho_{2M}(\varphi)$, $H(\varphi) \to \infty$, because the suppression of the over-Planckian scales originates not from the Gaussian nature of $\rho_{2M}(\varphi)$, but from the anomalous scaling of $\mathbf{\Gamma}_{1-\mathrm{loop}}(\varphi)$, leading to the power falloff. In this limit $\mathrm{Im}\,\varphi = O\,[H^{-1}(\varphi)\,]$, and one can use the ordinary perturbation theory in $\mathrm{Im}\,\varphi$ unrelated to the asymptotic bound (4.29). Futher discussion of this problem goes beyond the scope of this paper. We only mention that, ultimately, the correct treatment of the conformal mode might result in merging (at the technical level) the predictions of the no-boundary and tunneling wave functions [68] recently considered by the authors in Ref. [14] on the ground of the presented technique.

As discussed in the Introduction, one of the motivations for a consistent theory of tunneling geometries is the necessity to justify the semiclassical expansion and obtain the quantum scale of inflation at the sub-Planckian (GUT) energy scale. Although the paradigm of tunneling wave function versus the no-boundary one bears undoubtful advantages in the low-energy domain [4,9,10], the implementation of this program for both of these semiclassical wave functions was not successful. Their distribution functions are extremely flat and unnormalizable at $\varphi \to \infty$, and the only maximum for a semiclassical $\rho_{2M}(\varphi)$ found for the no-boundary state generates insufficient number of inflationary e-foldings [70], violating the necessary bound $N \geq 60$. It is remarkable, that the application of the technique, developed above, drastically changes the situation.

First applied in the authors' work [12], this technique yields the selection criterion (8.15) justifying the semiclassical expansion and, as a by-product, suggesting the supersymmetric nature of particle physics models [13]. Moreover, as shown in Ref. [14], the asymptotic behaviors (8.9) and (8.12) lead to a sharp probability peak in the distribution of chaotic inflationary cosmologies driven by a scalar field with large negative constant of nonminimal interaction [14]. For a tunneling quantum state, the sub-Planckian parameters of this peak (the mean value of the Hubble constant $H \simeq 10^{-5} m_P$, its quantum width $\Delta H/H \simeq 10^{-5}$, and the number of inflationary e-foldings $N \simeq 60$) turn out to be in good correspondence with the observational status of inflation theory [11]. Therefore, this technique generates the quantum scale of inflation at the neeeded GUT scale and serves as a quantum gravitational ground for the inflation model of Bardeen, Bond, and Salopek [69]. This model, in which the inflaton field coincides with the Brans-Dicke scalar, plays an important role in the theory of the early universe, for it provides a very efficient resolution of the known difficulties in the formation of the observable large-scale structure. Thus, the above theory of tunneling geometries can provide us with a numerically sound link between quantum cosmology, inflation theory, and the particle physics of the early universe.

ACKNOWLEDGMENTS

The authors benefited from helpful discussions with G. Gibbons, J.B. Hartle, Don N. Page, and R.M. Wald and are also grateful to A. Umnikov for his help in the preparation of the pictures for this paper. One of the authors (A.O.B.) is grateful for the support of this work provided by CITA and NSERC Grants at the University of Alberta. A.Yu.K. is grateful to Russian Foundation for Fundamental Researches for support under Grant No. 94-02-03850-a. The research described in this publication was also made possible in part by Grant No. N MAE000 from the International Science Foundation.

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unitary map between the Dirac-Wheeler-DeWitt quantization and reduced phase space quantization. This property is discussed in much detail in the series of papers [22–24] both at the level of operatorial and path-integral quantizations.

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