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Two-dimensional Poincaré gauge gravity with matter

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Two-dimensional Poincaré gauge gravity is known to be completely integrable in vacuum. Interaction with matter fields usually destroys the integrability. We give a coordinate- and gauge-invariant formulation of the matter-gravity dynamics for the general quadratic Poincaré gauge model. The particular cases of massless spinor and scalar field sources are analyzed in detail. Exact general solutions for the chiral boson and fermion matter field configurations are constructed. These describe the gravitational field of a black hole type and are similar to the vacuum solutions discovered earlier. Nonchiral solutions are investigated with the help of numerical methods.

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I. INTRODUCTION

Two-dimensional models of gravity have attracted considerable attention recently [1–4]. Specifically, the string-motivated theories [5,6] were discussed in connection with the lower-dimensional black hole physics. The gauge approach to gravity, which was developed previously in four dimensions [7–10], underlies now the attempts of constructing string theories with dynamical geometry [11–13]. At the same time, two-dimensional gauge gravity is of interest in itself [14–16], since it offers a simple system with which one can study difficult nonperturbative quantization problems [17].

Recently we have demonstrated [16] the complete integrability of the two-dimensional teleparallelism and the general Poincaré gravity model in vacuum. The coupling to gauge, scalar, and spinor matter fields was shown to destroy the integrability, in general. However, some *static type* exact solutions were obtained. These turned out to be of the same black hole structure as the vacuum solutions. In this paper we extend and develop the earlier results, presenting a general coordinate- and gauge-invariant formulation of the gravity-matter dynamics in two dimensions. A peculiar but common feature of the standard matter sources (gauge, scalar, and spinor fields) in two dimensions is that all of them have a vanishing spin current. Hence the material energy-momentum current is symmetric and covariantly conserved with respect to the Riemannian connection. Thus, quite generally the Lorentz connection is explicitly decoupled from the two-dimensional matter. We restrict ourselves only to these standard matter fields, which are otherwise described by a general Lagrangian. The absence of the spin-connection coupling considerably facilitates integration of the matter-gravity equations of motion. Although in [16] we proved the integrability of the gauge gravity with an arbitrary gravitational Lagrangian, here we discuss a highly interesting particular *quadratic* model which is described by a general action containing squares of torsion and curvature. As before [16], the basic idea behind the integration of the gravitational field equations is to treat the torsion one-form and its dual as a coframe basis, and to choose the local coordinates on a two-dimensional manifold as directly related to geometrical and material variables. Previous attempts in integrating quadratic gauge gravity with scalar and spinor matter sources were reported in [18,19].

Using the general invariant formulation of the problem, we prove its internal consistency and demonstrate that the only degenerate torsion solutions are purely Riemannian de Sitter (constant curvature) spacetimes with constant matter field configurations. As a particular application of the formalism developed, we consider the coupling of massless spinor and scalar fileds to the two-dimensional gauge gravity. The chiral solutions are constructed for both matter sources in the most general case. Relevant gravitational field configurations are again of the black-hole type, although in general these are nonstatic and nonstationary. The nonchiral configurations are more complicated; these are investigated numerically.

The structure of the paper is as follows. In Sec. II the central notion of invariant energy one-forms is introduced, and we construct the invariant formulation of the dynamics of quadratic gravity and arbitrary matter sources in two dimensions. We give the complete description of the torsion degenerate solution; this reduces generically to the de Sitter geometry without torsion. The consistency of the formalism is checked explicitly. In Sec. III the massless spinor and scalar field models are formulated in two dimensions. The matter equations of motion are shown to be integrable on an arbitrary Riemann-Cartan two-manifold, admitting the chiral solutions. These chiral fermion and boson matter field configurations are considered in Sec. IV as the sources of the gauge gravitational field. The invariant gravity field equations are integrated analytically. Finally, in Sec. V the nonchiral solutions are discussed and the numerical

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analysis of the gravitational field dynamics reveals again the black-hole-type behavior of geometry.

II. GENERAL INVARIANT FORMULATION OF MATTER AND GRAVITY DYNAMICS

A. Preliminaries: torsion as a two-dimensional coframe

The two-dimensional Riemann-Cartan geometry has rather remarkable properties. These are briefly outlined below. The notation and conventions coincide with that of Ref. [16].

In the Poincaré gauge approach, coframe one-form ϑ^α and a linear connection one-form $\Gamma^{\alpha\beta}$ are considered as the (respectively, translational and the Lorentz) gauge potentials of the gravitational field. The corresponding field strengths are given by the torsion two-form $T^\alpha = D\vartheta^\alpha$ and the curvature two-form $R^{\alpha\beta}$. In two dimensions torsion is irreducible and contains only the vector piece $T^\alpha = -t^\alpha\eta$, where η is the volume two-form and the vector-valued zero-form of torsion is defined via the Hodge dualization:

$$t^\alpha := *T^\alpha. \tag{2.1}$$

As in the purely Riemannian case, the Riemann-Cartan curvature two-form has only one irreducible component, and it can be expressed in terms of the curvature scalar $R = e_\alpha \rfloor e_\beta \rfloor R^{\alpha\beta}$.

The case when the torsion square is not identically zero, $t^2 := t_\alpha t^\alpha \neq 0$, will be called a *nondegenerate Riemann-Cartan geometry.* Introducing [16] the new basic object of the Riemann-Cartan geometry, the *torsion one-form*,

$$T := e_\alpha \rfloor T^\alpha, \tag{2.2}$$

we can write a coframe as

$$\vartheta^\alpha = -\frac{1}{t^2}\left(T\eta^{\alpha\beta}t_\beta + *Tt^\alpha\right). \tag{2.3}$$

Thus, the torsion one-form and its dual $*T$ represent in fact a specific coframe, and all the geometrical objects can be expanded with respect to this basis. When $t^2 \neq 0$, this coframe is nondegenerate, hence the terminology. The volume two-form can be calculated, in the nondegenerate case, as an exterior square of the torsion one-form

$$\eta := \frac{1}{2}\eta_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta = \frac{1}{t^2} * T \wedge T. \tag{2.4}$$

The use of the torsion one-form as a coframe in the two-dimensional Riemann-Cartan spacetime turns out to be extremely convenient, and, in fact, underlies all the subsequent discussion.

B. Invariant formulation of Poincaré gravity field equations

In [16] the general Poincaré gauge theory in two dimensions was formulated. Here we analyze the two-dimensional Poincaré gravity with the general Lagrangian *quadratic* in torsion and curvature:

$$V = -\left(\frac{a}{2}T_\alpha{}^*T^\alpha + \frac{1}{2}R^{\alpha\beta}\eta_{\alpha\beta} + \frac{b}{2}R_{\alpha\beta}{}^*R^{\alpha\beta}\right) + \Lambda\eta. \tag{2.5}$$

Here a, b, and Λ are the coupling constants. Variation of the total action $\int(V + L)$ (where L is the matter field Ψ Lagrangian two-form) with respect to the coframe and connection yields the gravitational field equations [16]

$$Dt_\alpha = \frac{1}{a}(-\widetilde{\mathcal{V}}\eta_\alpha + \Sigma_\alpha), \tag{2.6}$$

$$b\,dR = at^\alpha\eta_\alpha, \tag{2.7}$$

where

$$\widetilde{\mathcal{V}} = \frac{a}{2}t^2 - \frac{b}{4}R^2 - \Lambda \tag{2.8}$$

is the so-called modified Lagrangian function. Notice that the specific feature of the standard matter (scalar, spinor, Abelian, and non-Abelian gauge fields) in two dimensions is that the spin current is zero, and only the canonical energy-momentum one-form Σ_α enters the equations of motion. These equations should be solved together with the matter field equations $\delta L/\delta\Psi = 0$.

Although Eqs. (2.6) and (2.7) have a transparent covariant form, it will be more convenient to rewrite them in terms of simpler variables which are *explicitly gauge and coordinate invariant.* With this aim in mind, we introduce, instead of the vector-valued energy-momentum one-form Σ_α, two scalar-valued one-forms:

$$S := t^\alpha\Sigma_\alpha, \quad S^\star := t_\alpha\eta^{\alpha\beta}\Sigma_\beta. \tag{2.9}$$

The notation with the right * is borrowed from the Lie-dual operator introduced in [16]; note that here it is understood as applied to Σ_α and not to another factor. The two *energy variables* are independent, in the general case. However, a useful relation is discovered with the use of the Hodge dualization,

$$S^\star + *S = *(\vartheta^\alpha \wedge \Sigma_\alpha) * T. \tag{2.10}$$

Recall, we are considering the matter with vanishing spin current (all the normal matter in two dimensions satisfy this condition), which was used in the derivation of (2.10). The two-form $\vartheta^\alpha \wedge \Sigma_\alpha$ describes the trace of the energy-momentum tensor. The latter vanishes for massless conformally invariant matter, and in this case the energy properties of matter are described by a single one-form S (and $S^\star = - * S$).

Contracting Eq. (2.6) with t^α, η^α, and ϑ^α, one finds three equations: respectively,

$$d(t^2) = \frac{2}{a}(\widetilde{\mathcal{V}}T + S), \tag{2.11}$$

$$dT = 0, \tag{2.12}$$

$$d * T = \left(t^2 - \frac{2}{a}\widetilde{\mathcal{V}}\right)\eta + \frac{1}{a}\vartheta^\alpha \wedge \Sigma_\alpha. \tag{2.13}$$

Equation (2.12) is trivially satisfied in view of the field Eq. (2.7), so this is a good check of consistency.

Furthermore, contracting (2.6) with Σ^α and with its Lie dual, one obtains

$$dS = T \wedge S, \tag{2.14}$$

$$dS^\star = T \wedge S^\star + \left(\frac{1}{a}\widetilde{\mathcal{V}} - t^2\right)\vartheta^\alpha \wedge \Sigma_\alpha + \frac{1}{a}\eta^{\alpha\beta}\Sigma_\alpha \wedge \Sigma_\beta. \tag{2.15}$$

In derivation of the last equation the relation (2.10) was used.

It is worthwhile to compute the differential of the modified Lagrangian:

$$d\widetilde{\mathcal{V}} = \frac{a}{2}\left[d(t^2) + RT\right]. \tag{2.16}$$

Finally, one can consider the contraction of (2.6) with the Lie dual of the torsion vector, $\eta^{\alpha\beta}t_\beta$. This gives the explicit form of the Lorentz connection. Recall that in two dimensions, the Lorentz connection one-form has only one component which is expressed most conveniently in terms of its Lie dual, $\Gamma^\star := \frac{1}{2}\eta_{\alpha\beta}\Gamma^{\alpha\beta}$. We invert to obtain

$$\Gamma^{\alpha\beta} = -\eta^{\alpha\beta}\Gamma^\star. \tag{2.17}$$

After some algebra, one finds that the above-mentioned contraction yields the Lie dual connection form explicitly:

$$t^2(\Gamma^\star + du) = \frac{1}{a}(\widetilde{\mathcal{V}} * T + S^\star). \tag{2.18}$$

Here the auxiliary variable u is defined by the differential relation

$$t^2 du := \eta^{\alpha\beta}t_\alpha dt_\beta. \tag{2.19}$$

This variable is unphysical and represents a pure gauge degree of freedom for the local Lorentz group. As a whole the right-hand side of (2.18) is gauge invariant.

The complete set of Eqs. (2.11)–(2.16), (2.18) represents the same information which is contained in the original gravitational field equations (2.6), (2.7) and in the equations of motion of matter. A further analysis of the integration of the coupled gravity-matter field equations will be carried out in this invariant formulation.

C. Degenerate torsion solutions

The two cases of the two-dimensional Poincaré gravity should be treated separately: degenerate Riemann-Cartan geometry with the null vector torsion, $t^2 = 0$, and the nondegenerate case with nontrivial square of torsion. In this section we discuss the degenerate solutions of the quadratic Poincaré gravity. Let us formulate the answer for arbitrary matter sources.

When $t^2 = 0$, the gravitational (2.6), (2.7), and matter field equations have the following solution: The torsion one-form is either self- or anti-self-dual,

$$T = \pm * T, \tag{2.20}$$

while the curvature is determined from the equation

$$\frac{b}{2}R^2 + 2\Lambda - *(\vartheta^\alpha \wedge \Sigma_\alpha) = 0, \tag{2.21}$$

and the matter p-form Ψ, besides the equation of motion,

$$\frac{\delta L}{\delta\Psi} = \frac{\partial L}{\partial\Psi} - (-1)^p D\frac{\partial L}{\partial D\Psi} = 0, \tag{2.22}$$

satisfies the constraints

$$*(T \wedge D\Psi)\left(T \wedge \frac{\partial L}{\partial D\Psi}\right) = 0, \tag{2.23}$$

$$*(D\Psi) \wedge \frac{\partial L}{\partial D\Psi} = 0. \tag{2.24}$$

These results (2.21)–(2.24) are direct consequences of Eqs. (2.11) and (2.13), in which one must put $t^2 = 0$, and the general expression for the energy-momentum one-form [20,21]:

$$\Sigma_\alpha = (e_\alpha \rfloor L) - (e_\alpha \rfloor D\Psi)\left(\frac{\partial L}{\partial D\Psi}\right). \tag{2.25}$$

In general, one cannot tell more without specifying the matter Lagrangian L and using the matter equations of motion. However, in one physically important case we can move one step further. This is the case of massless conformally invariant matter. Then the energy-momentum trace vanishes, $\vartheta^\alpha \wedge \Sigma_\alpha = 0$, and the degenerate solution reduces to the torsionless de Sitter geometry:

$$T^\alpha = 0, \quad R = \text{const}, \quad \Psi = \text{const}, \tag{2.26}$$

where the constant value of the curvature is determined by Eq. (2.21).

The same turns out to be true also for some conformal noninvariant matter, e.g., for massive scalar field with arbitrary self-interaction [22]. In the rest of the paper we will always consider the nondegenerate case with $t^2 \neq 0$.

D. Consistency check of the invariant formulation

As is clearly suggested by the field equation (2.7), the Riemann-Cartan curvature R can be conveniently treated as one of the local coordinates on a two-dimensional manifold. However, one has then to check the consistency of the whole scheme through the explicit calculation of the curvature constructed from the local Lorentz connection obtained from the field equations. In earlier studies this was done for vacuum solutions [16]. In this section we will demonstrate consistency in general, for arbitrary matter sources. Let us consider the nondegenerate case with $t^2 \neq 0$. Equation (2.18) gives the general solution for the Lorentz connection:

$$\Gamma^\star + du = \frac{1}{at^2}(\widetilde{\mathcal{V}} * T + S^\star). \tag{2.27}$$

One can straightforwardly prove, by taking the exterior differential of the left- and right-hand sides of this equation, that $d\Gamma^\star = -\frac{1}{2}R\eta$. A useful identity holds for the square of energy one-forms:

$$\eta^{\alpha\beta}\Sigma_\alpha \wedge \Sigma_\beta = \frac{2}{t^2} S \wedge S^\star. \tag{2.28}$$

With the help of this relation and Eqs. (2.11)–(2.16) one finds

$$d(\widetilde{\mathcal{V}} * T + S^\star) = \frac{1}{t^2} d(t^2) \wedge (\widetilde{\mathcal{V}} * T + S^\star) + \frac{a}{2} RT \wedge *T, \tag{2.29}$$

and the consistency proof is completed after taking (2.4) into account.

III. MASSLESS MATTER FIELD SOURCES

A. Massless fermions

It is probably worth mentioning that there are two types of fermion models in two dimensions. Although one can clearly establish a correspondence between them, formally the Lagrangians are different. Dirac spinors in two dimensions have two (complex) components,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.1}$$

and the spinor space at any point of the space-time manifold is related to the tangent space at this point via the spin-tensor objects: the Dirac and the Pauli matrices. These were described in [16]. To recall, everything is defined by a single object, a matix-valued Dirac one-form

$$\gamma = \gamma_\alpha \vartheta^\alpha, \tag{3.2}$$

which satisfies

$$\gamma \otimes \gamma = g, \qquad \gamma \wedge \gamma = -2\gamma_5 \eta, \tag{3.3}$$

where γ_5 is determined by the Hodge dual,

$$*\gamma = \gamma_5 \gamma. \tag{3.4}$$

We will use the following explicit realization of the Dirac one-form:

$$\gamma := \begin{pmatrix} 0 & -(\vartheta^{\hat{0}} - \vartheta^{\hat{1}}) \\ (\vartheta^{\hat{0}} + \vartheta^{\hat{1}}) & 0 \end{pmatrix} \tag{3.5}$$

or, equivalently, the Dirac matrices

$$\gamma^{\hat{0}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^{\hat{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.6}$$

The Dirac matrices γ^α satisfy the standard relations $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}$.

It is straightforward to see that the gauge- and coordinate-invariant Lagrangian two-form for the massless Dirac spinor field is not unique, and one can choose it either in the form

$$L = \frac{i}{2}(\bar{\psi}\gamma \wedge d\psi + d\bar{\psi} \wedge \gamma\psi), \tag{3.7}$$

or in the form

$$L' = \frac{i}{2}(\bar{\psi}\gamma \wedge *d\psi + *d\bar{\psi} \wedge \gamma\psi). \tag{3.8}$$

To recall, in two dimensions there is no interaction of spinors with the Lorentz connection, and hence the above Lagrangians contain ordinary exterior differentials and not the covariant ones. Nevertheless, the theory is invariant under local Lorentz rotations. Two Lagrangians L and L' provide two different models of spinor dynamics in two dimensions. Although it is easy to prove that the equations of motion for the spinor field ψ are the same for (3.7) and (3.8), their energy-momentum forms are different. Thus the two models are distinguished by the gravitational interaction. We will analyze in detail the model defined by (3.7), and will comment on the possible difference for (3.8) later.

The (Dirac) field equation obtained from the variation of L with respect to $\bar{\psi}$ reads

$$\gamma \wedge d\psi - \frac{1}{2}(d\gamma)\psi = 0. \tag{3.9}$$

The exceptional degenerate case was analyzed above, hence we assume that $t^2 \neq 0$, and the one-forms T and $*T$ can be treated as the coframe basis in a two-dimensional Riemann-Cartan spacetime. As in the preceding section, we prefer to work with explicitly gauge-invariant objects. It is convenient then to consider instead of the spinor (3.1) two Lorentz-invariant complex functions:

$$\varphi_1 := \sqrt{(t^{\hat{0}} + t^{\hat{1}})}\psi_1, \quad \varphi_2 := \sqrt{(t^{\hat{0}} - t^{\hat{1}})}\psi_2. \tag{3.10}$$

The Dirac equation (3.9) for these variables yields [using (2.3)]

$$(T - *T) \wedge d\varphi_1 - \frac{1}{2} t^2 d\left(\frac{T - *T}{t^2}\right)\varphi_1 = 0, \tag{3.11}$$

$$(T + *T) \wedge d\varphi_2 - \frac{1}{2} t^2 d\left(\frac{T + *T}{t^2}\right)\varphi_2 = 0. \tag{3.12}$$

For nontrivial spinor fields these equations are immediately transformed into

$$d\left[\frac{\varphi_1^2}{t^2}(T - *T)\right] = 0, \qquad d\left[\frac{\varphi_2^2}{t^2}(T + *T)\right] = 0. \tag{3.13}$$

Combining (3.11) and (3.12) with their complex conjugates, one finds similar real equations,

$$d\left[\frac{|\varphi_1|^2}{t^2}(T - *T)\right] = 0, \qquad d\left[\frac{|\varphi_1|^2}{t^2}(T + *T)\right] = 0. \tag{3.14}$$

Lemma Poincaré tells us that locally there exist real functions; we denote them x and y, such that

$$\frac{|\varphi_1|^2}{t^2}(T - *T) = dx, \qquad \frac{|\varphi_2|^2}{t^2}(T + *T) = dy. \quad (3.15)$$

Introducing the phases of spinor components explicitly,

$$\varphi_1 = |\varphi_1|e^{i\alpha}, \quad \varphi_2 = |\varphi_2|e^{i\beta}, \quad (3.16)$$

we find, using (3.13), that these phases depend only on one of the above variables:

$$\alpha = \alpha(x), \quad \beta = \beta(y). \quad (3.17)$$

In principle, we have described the general exact solution of the massless Dirac equation in an arbitrary two-dimensional Riemann-Cartan spacetime. This is the same for both the fermion models, (3.7) and (3.8). Let us now turn to the analysis of the gravitational field equations. We will use the explicitly invariant formulation of the Sec. II.

The energy one-forms read now

$$S = \frac{i}{2}\Big(\varphi_1^* d\varphi_1 - (d\varphi_1^*)\varphi_1 + \varphi_2^* d\varphi_2 - (d\varphi_2^*)\varphi_2\Big)$$
$$= -|\varphi_1|^2 d\alpha - |\varphi_2|^2 d\beta, \quad (3.18)$$

$$S^\star = \frac{i}{2}\Big(\varphi_1^* d\varphi_1 - (d\varphi_1^*)\varphi_1 - \varphi_2^* d\varphi_2 + (d\varphi_2^*)\varphi_2\Big)$$
$$= -|\varphi_1|^2 d\alpha + |\varphi_2|^2 d\beta. \quad (3.19)$$

It is easy to see, that for the spinor model (3.8) the terms with φ_2 and β will have a different sign.

Inserting (3.15)–(3.17) into (3.18), we can write finally

$$S = A_1(T - *T) + A_2(T + *T), \quad (3.20)$$

where we denote

$$A_1 := -\frac{d\alpha}{dx}\frac{|\varphi_1|^4}{t^2}, \quad A_2 := -\frac{d\beta}{dy}\frac{|\varphi_2|^4}{t^2}. \quad (3.21)$$

The one-form $S^\star = - * S$, well in accordance with (2.10).

B. Massless bosons

Let us now consider a gravitationally coupled massless scalar field ϕ with the Lagrangian two-form

$$L = -\frac{1}{2} d\phi \wedge {}^*d\phi. \quad (3.22)$$

The sources of the gravitational field are obtained as the variational derivatives of (3.22) with respect to the frame one-form ϑ^α and the Lorentz connection one-form $\Gamma^{\alpha\beta}$. One easily obtains

$$\Sigma_\alpha = \frac{1}{2}[*(d\phi \wedge \vartheta_\alpha)d\phi - *(d\phi \wedge \eta_\alpha) * d\phi], \quad \tau_{\alpha\beta} = 0. \quad (3.23)$$

Variation with respect to ϕ yields the field equation of the scalar matter: i.e., the Klein-Gordon equation

$$*d * d\phi = 0. \quad (3.24)$$

Since we are considering only the nondegenerate case with $t^2 \neq 0$, we can use the torsion coframe basis, and write, in the most general case,

$$d\phi = \Phi_1(T - *T) + \Phi_2(T + *T), \quad (3.25)$$

with some functions $\Phi_{1,2}$. Substituting this into the Klein-Gordon equation (3.24), one finds that locally there exists such a scalar function z that

$$-\Phi_1(T - *T) + \Phi_2(T + *T) = dz. \quad (3.26)$$

This describes a general solution of the Klein-Gordon equation. A subcase comprises the field configurations, which are characterized by

$$\Phi_1(T - *T) = d\widetilde{x}, \qquad \Phi_2(T + *T) = d\widetilde{y}, \quad (3.27)$$

in complete analogy with the fermions, (3.15). The new functions certainly satisfy $\widetilde{x} + \widetilde{y} = z$.

It is straightforward to calculate the energy one-forms for the boson fields. From (3.23),

$$S = t^\alpha \Sigma_\alpha = \frac{1}{2}[*(d\phi \wedge T)d\phi - *(d\phi \wedge *T) * d\phi], \quad (3.28)$$

and, as for the fermions, $S^\star = - * S$, since $\vartheta^\alpha \wedge \Sigma_\alpha = 0$. Substituting (3.25) into (3.28), we obtain similar to (3.20),

$$S = A_1(T - *T) + A_2(T + *T), \quad (3.29)$$

where now

$$A_1 := -t^2\Phi_1^2, \quad A_2 := -t^2\Phi_2^2. \quad (3.30)$$

We are in a position now to analyze the solutions of the gravitational field equations.

IV. CHIRAL SOLUTION

Both the massless Dirac equation and the massless Klein-Gordon equation admit chiral solutions. For fermions this effectively means that only one component of spinor field is nontrivial. For bosons chirality can be formulated in the sense of self- or anti-self-duality of the velocity one-form $d\phi$. And in both cases the field equations describe right- or left-moving configurations. In this section we describe the corresponding gravitational field. Let us assume that $\varphi_2 = \psi_2 = 0$ for fermion field, and $\Phi_2 = 0$ for boson one. Then $A_2 = 0$ both in (3.21) and (3.30), and the energy one-form S is anti-self-dual (hence $S^\star = S$).

Equation (2.7) and the integrals (3.15) and (3.27),

$$T - *T = \frac{t^2}{|\varphi_1|^2}dx \quad \text{(spinor)},$$
$$\Phi_1(T - *T) = d\widetilde{x} \quad \text{(scalar)}, \quad (4.1)$$

suggest a natural interpretation of the variables R and x (R and $\widetilde{x}$) as two local spacetime coordinates. Clearly, x and $\widetilde{x}$ are different in each case, but we can unify the two problems easily without a risk of confusion. This is described as follows. Equations (4.1) and (2.7) establish the explicit form of the torsion coframe,

$$*T = -\left(\frac{b}{a}dR + Bdx\right), \quad T = -\frac{b}{a}dR, \tag{4.2}$$

and hence, of the volume two-form,

$$\eta = \frac{1}{t^2} * T \wedge T = \frac{bB}{at^2}dx \wedge dR, \tag{4.3}$$

and, finally of the spacetime interval,

$$\begin{aligned} ds^2 = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta &= \frac{1}{t^2}(*T \otimes *T - T \otimes T) \\ &= \frac{1}{t^2}\left[\left(Bdx + \frac{b}{a}dR\right)^2 - \frac{b^2}{a^2}dR^2\right]. \end{aligned} \tag{4.4}$$

Here the unified notation is introduced, with

$$B := \frac{t^2}{|\varphi_1|^2}. \tag{4.5}$$

for fermions, while for bosons this is function helping to relate the two "coordinates," namely, $d\widetilde{x} = \Phi_1 B dx$.

The solution will be complete, provided we find the functions t^2 and B explicitly. For this we should use the invariant gravitational field equations (2.11)–(2.14) which now read

$$d(t^2) = \frac{2}{a}(\widetilde{\mathcal{V}}T + S), \tag{4.6}$$

$$dT = 0, \tag{4.7}$$

$$d(*T) = \left(t^2 - \frac{2}{a}\widetilde{\mathcal{V}}\right)\eta, \tag{4.8}$$

$$dS = T \wedge S. \tag{4.9}$$

Equation (2.15) degenerates into (2.14).

From (3.18)–(3.20), (3.28), and (3.29) we have, for the energy one-form,

$$S = Adx, \tag{4.10}$$

with

$$A := -|\varphi_1|^2\frac{d\alpha}{dx} \quad \text{(spinor)}, \quad A := -t^2 B\Phi_1^2 \quad \text{(scalar)}. \tag{4.11}$$

Substituting (4.10) and (4.2) into (4.6), (4.8), and (4.9) one finds

$$\frac{\partial t^2}{\partial R} = -\frac{2b}{a^2}\widetilde{\mathcal{V}}, \tag{4.12}$$

$$\frac{\partial t^2}{\partial x} = \frac{2}{a}A, \tag{4.13}$$

$$\frac{1}{B}\frac{\partial B}{\partial R} + \frac{b}{a} - \frac{2b}{a^2t^2}\widetilde{\mathcal{V}} = 0, \tag{4.14}$$

$$\frac{\partial A}{\partial R} + \frac{b}{a}A = 0. \tag{4.15}$$

Equation (4.7) is trivially satisfied.

The system (4.12)–(4.15) is integrated without difficulties, yielding

$$A = f(x)\exp\left(-\frac{b}{a}R\right), \tag{4.16}$$

$$B = B_0(x)t^2\exp\left(\frac{b}{a}R\right), \tag{4.17}$$

$$\begin{aligned} t^2 = \frac{2}{a}&\left(\int f(x)dx\right)\exp\left(-\frac{b}{a}R\right) \\ &+\left(\frac{2\Lambda}{a} + \frac{a}{b} + \frac{b}{2a}R^2 - R\right), \end{aligned} \tag{4.18}$$

with the arbitrary functions $f(x)$ and $B_0(x)$. Without any loss of generality one can put $B_0 = 1$ (this is always possible by redefinition of x). The gravitational field defined by (4.16)–(4.18) has the same form for chiral fermion and boson sources. The function $f(x)$ is however different for each particular source.

For *fermions* combining (4.17) and (4.5) we find

$$|\varphi_1|^2 = \exp\left(-\frac{b}{a}R\right), \tag{4.19}$$

while comparison of (4.16) and (4.11) yields

$$f(x) = -\frac{d\alpha}{dx}. \tag{4.20}$$

Hence, we finally have the solution: chiral fermion field (invariant component) reads

$$\varphi_1 = \exp\left(-\frac{b}{2a}R + i\alpha(x)\right), \tag{4.21}$$

while the gravitational field is described by the spacetime interval (4.4) with the B function given by (4.17), and

$$\begin{aligned} t^2 = \frac{2}{a}&[c_0 - \alpha(x)]\exp\left(-\frac{b}{a}R\right) \\ &+\left(\frac{2\Lambda}{a} + \frac{a}{b} + \frac{b}{2a}R^2 - R\right), \end{aligned} \tag{4.22}$$

where c_0 is an arbitrary integration constant. The local Lorentz connection reads

$$\Gamma^\star = d\widetilde{u} + \frac{1}{2}\left(\frac{a}{b} - R\right)\exp\left(\frac{b}{a}R\right)dx, \tag{4.23}$$

where $\widetilde{u}$ is a pure gauge contribution. Although the solution (4.21)–(4.23) is similar to the one discussed in [19], these two are different. Namely, our solution does not admit a matter-free limit, and contains a "ghost" case when nontrivial fermion configuration has zero energy-momentum (for α =const). These properties are opposite to that of solution given in [19].

For *bosons*, combining (4.16) and (4.17) with (4.11) and (3.25) one finds

$$f(x) = -(\Phi_1 B)^2 = -\left(\frac{d\phi}{dx}\right)^2, \tag{4.24}$$

while the scalar field $\phi(x)$ remains an arbitrary function of x.

V. NONCHIRAL SOLUTION

The nonchiral solutions for the scalar field are studied in [22], so this section is devoted to the massless fermion matter. Let us consider the general case, when both components of a Dirac spinor φ_1 and φ_2 are nontrivial. The equations (3.15) suggest that the variables x, y can be chosen as the two local coordinates on a spacetime manifold. The torsion coframe takes then the form

$$T = \frac{1}{2}(B_1 dx + B_2 dy), \qquad *T = \frac{1}{2}(B_2 dy - B_1 dx), \tag{5.1}$$

where we denoted

$$B_1 := \frac{t^2}{|\varphi_1|^2}, \quad B_2 := \frac{t^2}{|\varphi_2|^2}. \tag{5.2}$$

Denoting also

$$\widetilde{A}_1 := -|\varphi_1|^2 \frac{d\alpha}{dx}, \quad \widetilde{A}_2 := -|\varphi_2|^2 \frac{d\beta}{dx}, \tag{5.3}$$

we write the energy one-forms as

$$S = \widetilde{A}_1 dx + \widetilde{A}_2 dy, \quad S^\star = \widetilde{A}_1 dx - \widetilde{A}_2 dy. \tag{5.4}$$

Now we are in a position to solve the set of invariant gravitational field equations. Equations (4.6)–(4.9) are the same as in the chiral case, while (2.15) is now nontrivial [not reducing to (4.9)] and reads

$$dS^\star = *T \wedge S + \frac{2}{at^2} S \wedge S^\star. \tag{5.5}$$

From (5.2) and (5.3) we see that the functions $t^2, B_{1,2}, \widetilde{A}_{1,2}$ are not completely independent:

$$\frac{\widetilde{A}_1 B_1}{t^2} = -\frac{d\alpha(x)}{dx}, \quad \frac{\widetilde{A}_2 B_2}{t^2} = -\frac{d\beta(y)}{dy}. \tag{5.6}$$

In addition to these algebraic relations, there are some differential ones. Combining (4.9) and (5.5) one finds

$$\begin{aligned} &\frac{1}{\widetilde{A}_2}\frac{\partial \widetilde{A}_2}{\partial x} - \frac{2}{at^2}\widetilde{A}_1 - \frac{1}{2}B_1 = 0, \\ &\frac{1}{\widetilde{A}_1}\frac{\partial \widetilde{A}_1}{\partial y} - \frac{2}{at^2}\widetilde{A}_2 - \frac{1}{2}B_2 = 0. \end{aligned} \tag{5.7}$$

At the same time (4.7) and (4.8) yield

$$\begin{aligned} &\frac{1}{B_2}\frac{\partial B_2}{\partial x} - \frac{1}{at^2}\widetilde{\mathcal{V}} B_1 + \frac{1}{2}B_1 = 0, \\ &\frac{1}{B_1}\frac{\partial B_1}{\partial y} - \frac{1}{at^2}\widetilde{\mathcal{V}} B_2 + \frac{1}{2}B_2 = 0. \end{aligned} \tag{5.8}$$

Equations (5.6)–(5.8) make (4.6) trivially satisfied.

Using the field equation $dR = -\frac{a}{b}T$, we can eliminate everywhere the functions $B_{1,2}$ in favor of the scalar curvature variable:

$$B_1 = -\frac{2b}{a}\frac{\partial R}{\partial x}, \quad B_2 = -\frac{2b}{a}\frac{\partial R}{\partial y}. \tag{5.9}$$

Then Eqs. (5.8) are both equivalent to

$$\frac{\partial^2 R}{\partial x \partial y} + \frac{2b}{a}\left(\frac{1}{at^2}\widetilde{\mathcal{V}} - \frac{1}{2}\right)\frac{\partial R}{\partial x}\frac{\partial R}{\partial y} = 0. \tag{5.10}$$

In the same manner, we can express $\widetilde{A}_{1,2}$ from (5.6) in terms of other functions, and substituting them into (5.7), find a pair of equations

$$\frac{\partial R}{\partial x}\frac{\partial t^2}{\partial x} - \frac{1}{b}\frac{d\alpha}{dx}t^2 + \frac{2b}{a^2}\widetilde{\mathcal{V}}\left(\frac{\partial R}{\partial x}\right)^2 = 0, \tag{5.11}$$

$$\frac{\partial R}{\partial y}\frac{\partial t^2}{\partial y} - \frac{1}{b}\frac{d\beta}{dy}t^2 + \frac{2b}{a^2}\widetilde{\mathcal{V}}\left(\frac{\partial R}{\partial y}\right)^2 = 0. \tag{5.12}$$

These three last equations (5.10)–(5.12) determine the dynamics of two basic geometrical variables, curvature R and torsion t^2. Solutions then describe the behavior of matter and the interval of spacetime. To recall, $\widetilde{\mathcal{V}}$ is given by (2.8).

In general, like in the chiral case, there exists an ambiguity in choosing the functions $\alpha(x)$ and $\beta(y)$. It is not clear what physical conditions should determine these quantities. Hence, we will confine ourselves to a class of special solutions which do not contain this arbitrariness. Namely, let us assume that both curvature and torsion, R and t^2, depend only on one variable which is a linear combination of x, y. Without losing the generality we can choose

$$R = R(\xi), \quad t^2 = t^2(\xi), \qquad \xi := x + y. \tag{5.13}$$

Denoting the derivative with respect to ξ by the prime, and introducing

$$P := R', \tag{5.14}$$

we find, from (5.9),

$$B_1 = B_2 = -\frac{2b}{a}P. \tag{5.15}$$

Hence Eqs. (5.1) yield, for the spacetime metric,

$$ds^2 = \frac{b^2}{a^2}\left(\frac{P^2}{t^2}\right)(-d\xi^2 + d\zeta^2), \tag{5.16}$$

where $\zeta := x - y$. As we see, the sign of t^2 determines whether the variable ξ is "time" (positive t^2) or "space" (negative t^2) coordinate. The physical interpretation changes correspondingly, and solution describes in the former case a homogeneous cosmology, while in the latter case a static black-hole-type configuration. It is convenient to rewrite the invariant equations of motion (5.10)–(5.12) explicitly in terms of the function

$$\mathcal{H} := \frac{P^2}{t^2}, \tag{5.17}$$

which determines the metric properties of spacetime. Under the assumption (5.13) Eqs. (5.11) and (5.12) yield

$$\alpha = cx, \qquad \beta = cy, \tag{5.18}$$

with some constant c. For $c = 0$ the energy one-forms (5.3) are trivial, and hence this is effectively the vacuum solution with a ghost source. We will assume that $c \neq 0$.

The functions $\mathcal{H}, P$ determine the configuration of matter source. From (5.2) and (5.9) one finds, explicitly

$$|\varphi_1|^2 = |\varphi_2|^2 = -\frac{a}{2b}\frac{P}{\mathcal{H}}, \tag{5.19}$$

and thus, finally,

$$\varphi_1 = |\varphi_1| \exp(icx), \quad \varphi_2 = |\varphi_2| \exp(icy). \tag{5.20}$$

Assuming for definiteness that the sign of constant c is the same as the sign of a, we can introduce the normalized variables

$$z := \xi\sqrt{\frac{c}{a}}, \quad r := \frac{b}{a}R, \quad p := \frac{b}{\sqrt{ac}}P, \quad h := \frac{b}{2c}\mathcal{H}, \tag{5.21}$$

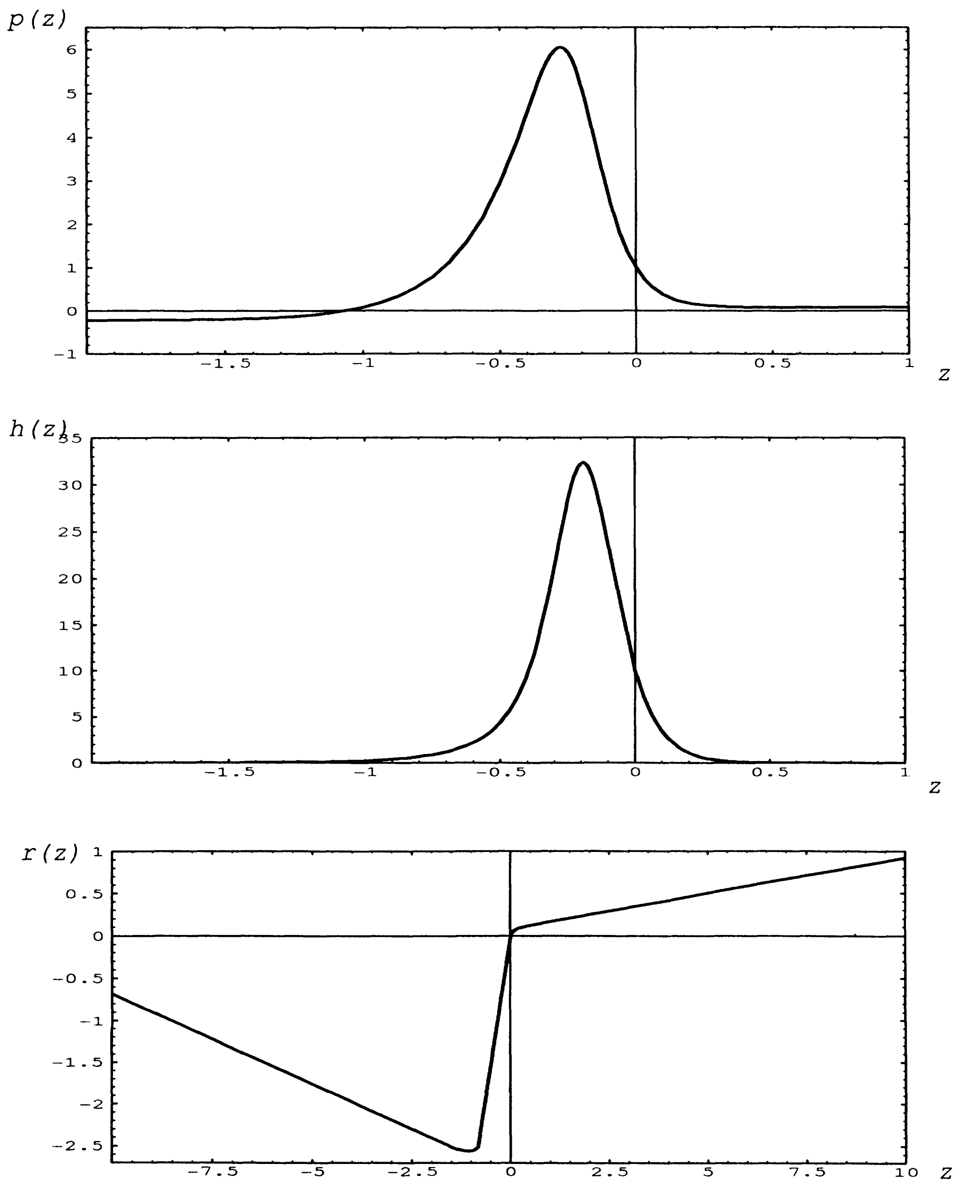

FIG. 1. "Soliton"-type solution for $\lambda = -1$ with stably asymptotically constant p.

and rewrite the system (5.10)–(5.12) as

$$\begin{aligned} \frac{dr}{dz} &= p, \\ \frac{dp}{dz} &= h(r^2+\lambda), \\ \frac{dh}{dz} &= h\left(p - \frac{1}{p} + \frac{h}{p}(r^2+\lambda)\right), \end{aligned} \tag{5.22}$$

where

$$\lambda := \frac{4b}{a^2}\Lambda \tag{5.23}$$

is the only free parameter in the system.

Let us analyze Eqs. (5.22). In the general case, these can be integrated only numerically, however, one can make certain qualitative conclusions without using a computer. The crucial observation is that the local and asymptotic behavior of $p(z)$ in fact determines all the other functions.

To begin with, we notice that we can immediately write one elementary exact solution of (5.22):

$$h = 0, \quad p = p_0 = \text{const}, \quad r = p_0 z. \tag{5.24}$$

This is apparently an unphysical solution, since via (5.19) this describes a matter configuration with constant infinite energy density. However, one should remember that such states of classical matter are not unusual in gravity: these may occur, e.g., in cosmological or black-hole singularities.

Solution (5.24) is stable with respect to small perturbations for $z \longrightarrow \infty$ when

$$0 < p_0 < 1 \quad \text{or} \quad p_0 < -1. \tag{5.25}$$

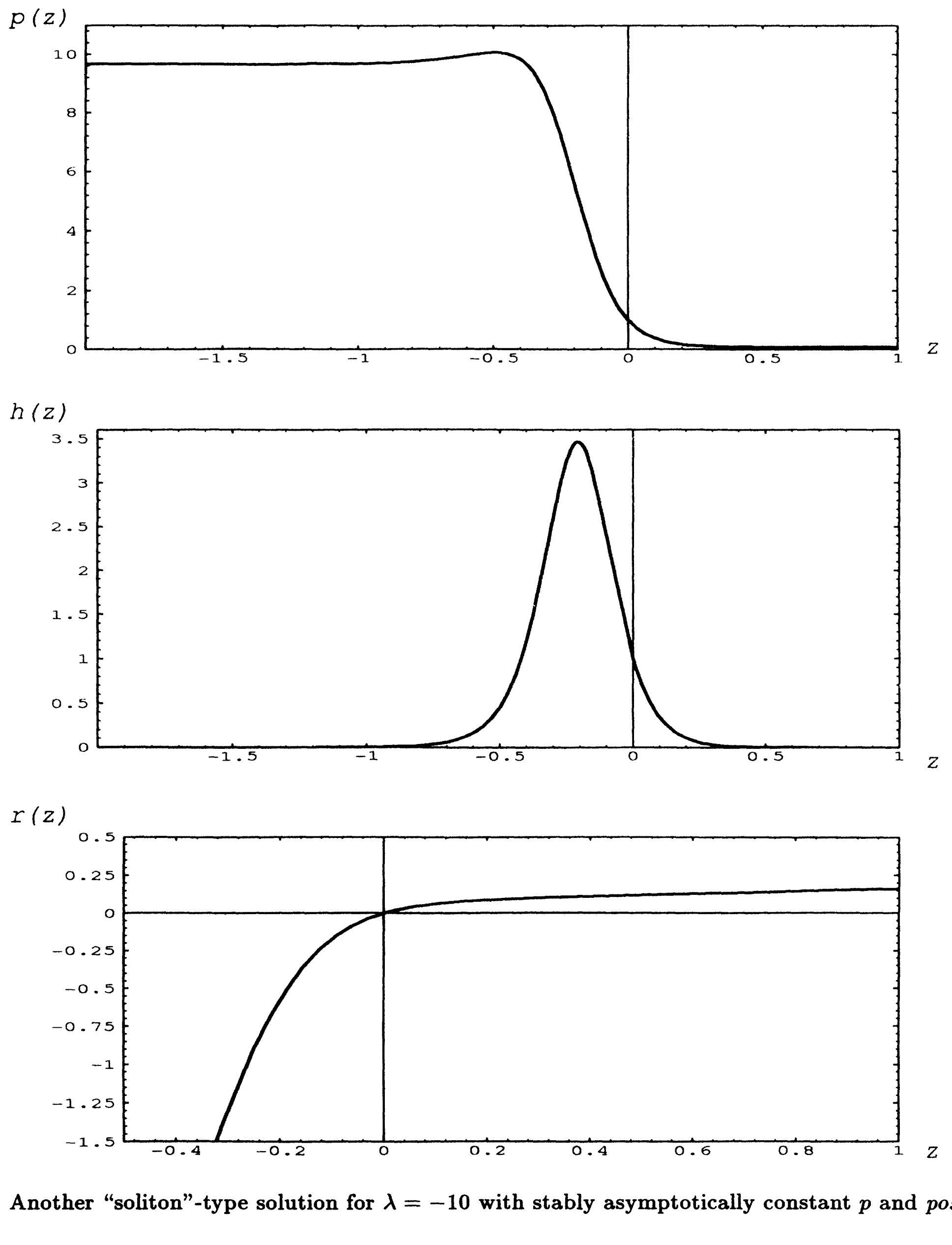

FIG. 2. Another "soliton"-type solution for $\lambda = -10$ with stably asymptotically constant p and *positive* h.

The relevant approximate solution is straightforwardly obtained from the linearized system (5.22),

$$h \approx \epsilon \exp(\kappa z),$$
$$p \approx p_0 + \frac{\epsilon}{\kappa}\exp(\kappa z)\left(\lambda + \frac{p_0^2}{\kappa^2}[(\kappa z - 1)^2 + 1]\right), \qquad (5.26)$$
$$r \approx p_0 z + \frac{\epsilon}{\kappa^2}\exp(\kappa z)\left(\lambda + \frac{p_0^2}{\kappa^2}[(\kappa z - 2)^2 + 2]\right),$$

where $\kappa := (p_0 - \frac{1}{p_0})$, and ϵ is an arbitrary small constant. The same approximate solution can take place also for $z \longrightarrow -\infty$, but for another values of p_0:

$$-1 < p_0 < 0 \quad \text{or} \quad p_0 > 1. \qquad (5.27)$$

Summarizing, the solution (5.24) can be realized as an asymptotic configuration of the system in $z \longrightarrow \pm\infty$, provided the asymptotic value of p belongs to one of the domains (5.25) and (5.27). Notice that this conclusion holds for any value of λ.

Let us now turn to the analysis of possible singularities which can occur at certain finite values of z. Again the behavior of $p(z)$ is crucial. It turns out that this variable can have only a simple pole singularities

$$p(z) \cong -\frac{1}{z - z_0}, \qquad (5.28)$$

at some point z_0. The two other variables have the leading order singularities of the form

$$r \cong -\ln|z - z_0|, \qquad h \cong \left(\frac{1}{(z - z_0)\ln|z - z_0|}\right)^2. \qquad (5.29)$$

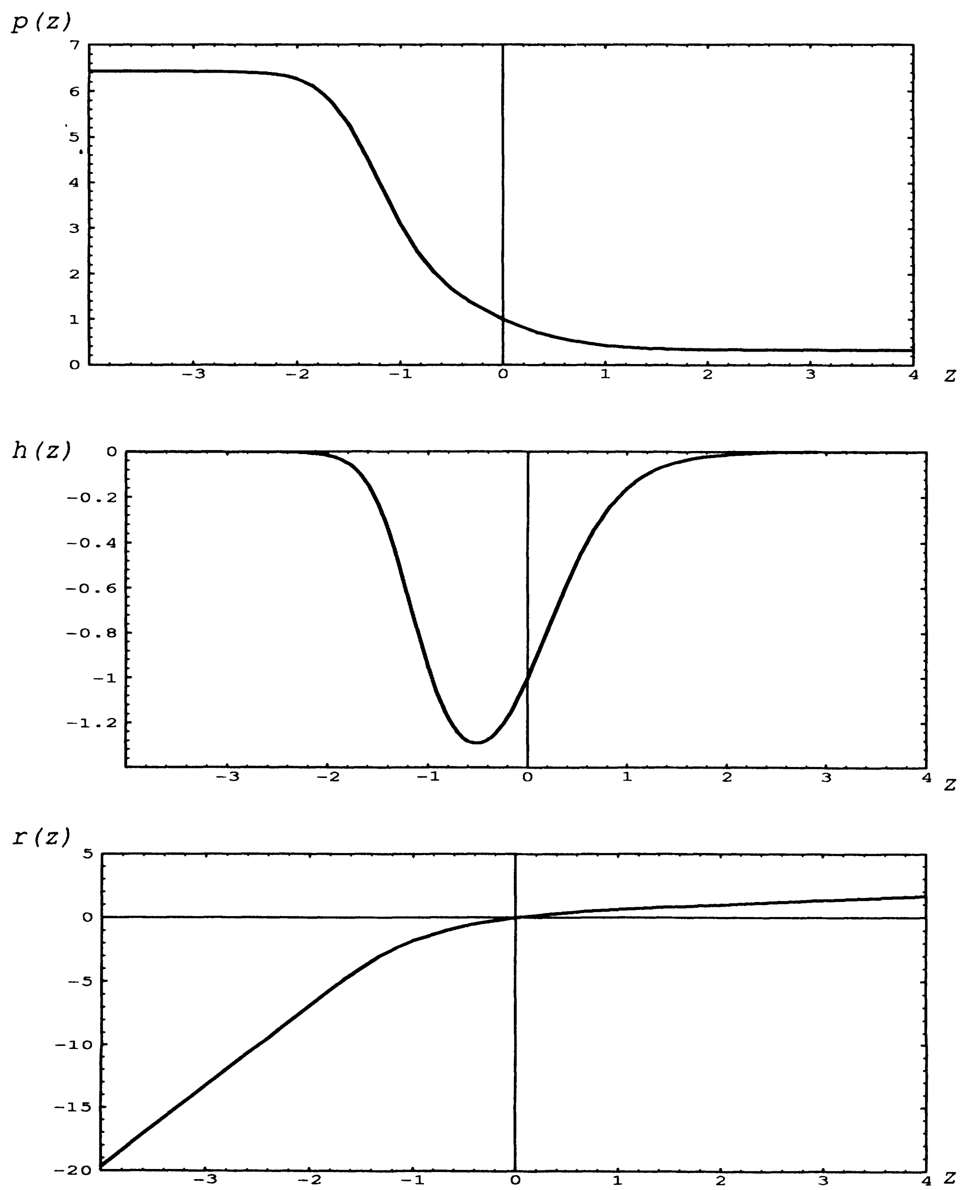

FIG. 3. "Soliton"-type solution for $\lambda = +1$ with stably asymptotically constant p and *negative* h.

This result is again valid for all values of λ.

The easiest way to verify (5.28)–(5.29) is to note that, provided we assume that $p(z)$ is large in a neighborhood of some point $z = z_0$ [it is not necessary to guess a specific ansatz for it, just enough to suppose that $p(z) >> 1$ for z in this domain], one can write an approximate integral of the system (5.22):

$$\ln\frac{p}{h} + r = r_0, \tag{5.30}$$

with some integration constant r_0. Direct calculation using (5.22) yields $\frac{d}{dz}(\ln\frac{p}{h}+r) = \frac{1}{p} \approx 0$ in the neighborhood of z_0. When $p \longrightarrow \infty$, this integral becomes exact. Now we can integrate the system (5.22) to the end, determining the approximate behavior of h, p, r near a singularity. At first, we make use of (5.30) and express $h(z)$ in terms of other variables:

$$h \approx p\exp(r - r_0). \tag{5.31}$$

Substituting this into (5.22), we find

$$p \approx \exp(r - r_0)[(r-1)^2 + \lambda + 1], \tag{5.32}$$

and thus finally the equation $p = \frac{dr}{dz}$ can be formally integrated

$$z - z_0 \approx \int \frac{dr\exp(r_0 - r)}{(r-1)^2 + \lambda + 1}. \tag{5.33}$$

For all values of λ, integration reveals the exponential-integral functions $Ei(r)$ or E_1, and the function $r(z)$ can be determined only in an implicit form by (5.33). For example, when $(\lambda + 1) > 0$, one finds

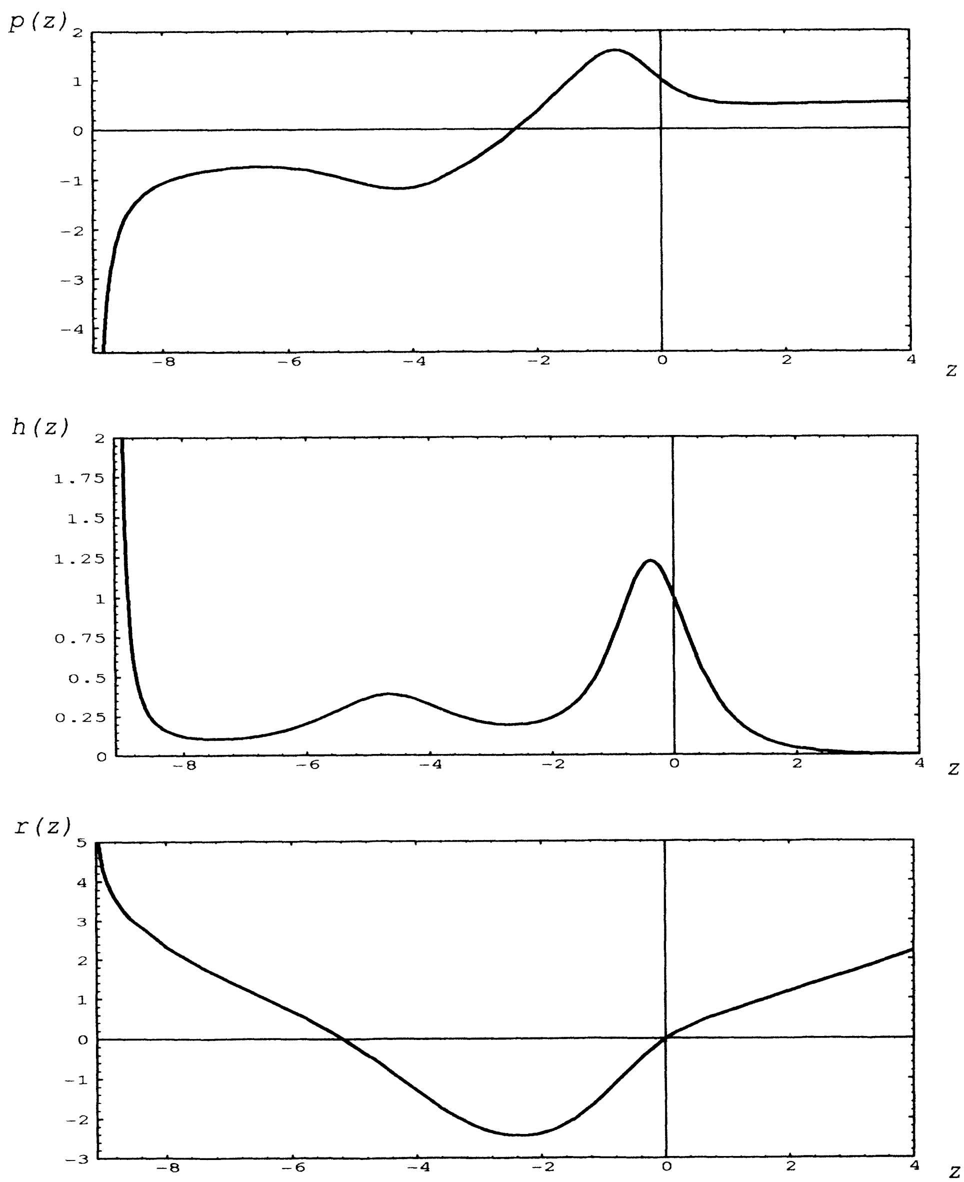

FIG. 4. Singular solution for $\lambda = -1$ (singularity is at $z_0 \approx -9.11471$). For $z \longrightarrow \infty$ function p is stably asymptotically constant.

$$z - z_0 \approx \frac{\exp(r_0 - 1)}{\sqrt{\lambda + 1}} \mathrm{Im} \left[\exp(i\sqrt{\lambda + 1}) \times E_1(r - 1 + i\sqrt{\lambda + 1}) \right]. \quad (5.34)$$

The simple singularity properties (5.28) and (5.29) are now recovered by making use of the asymptotic expansion $E_1(\xi) = \frac{\exp(-\xi)}{\xi}\left[1 - \frac{1}{\xi} + O(\frac{1}{\xi^2})\right]$.

It is probably worthwhile to notice that the behavior of the energy density (5.19) near a singularity is qualitatively the same as in the chiral solution (4.19) (with $r = \frac{b}{a}R \longrightarrow \infty$).

The above qualitative analysis can be supplemented by the explicit numerical integration. Since the system is autonomous, we can choose the zero of a z coordinate freely. It is convenient to integrate, starting from $z = 0$, which coincides with a zero of the curvature r. The "cosmological" parameter λ was varied in a wide range, both for positive and negative values (including the specific value $\lambda = -1$; cf. the singularity analysis). There are three different pictures, described as follows.

(A) The "soliton" type solution, when p starts in $-\infty$ at the asymptotically constant value in one of the stable domains (5.27), and then at $+\infty$ it tends to a constant value in the domains (5.25). The "metric" function $h(z)$ then has a typical solitonic form, while the absolute value of the curvature function $r(z)$ diverges linearly in $z \longrightarrow \pm\infty$. The results of numerical integration are given in Figs. 1–3. Hereafter $(a), (b), (c)$ refer to the graphics of $p(z), h(z), r(z)$, respectively. The matter density (5.19) behaves like in a closed cosmological model: starting with a sort of "big bang" from an infinite value at $z = -\infty$,

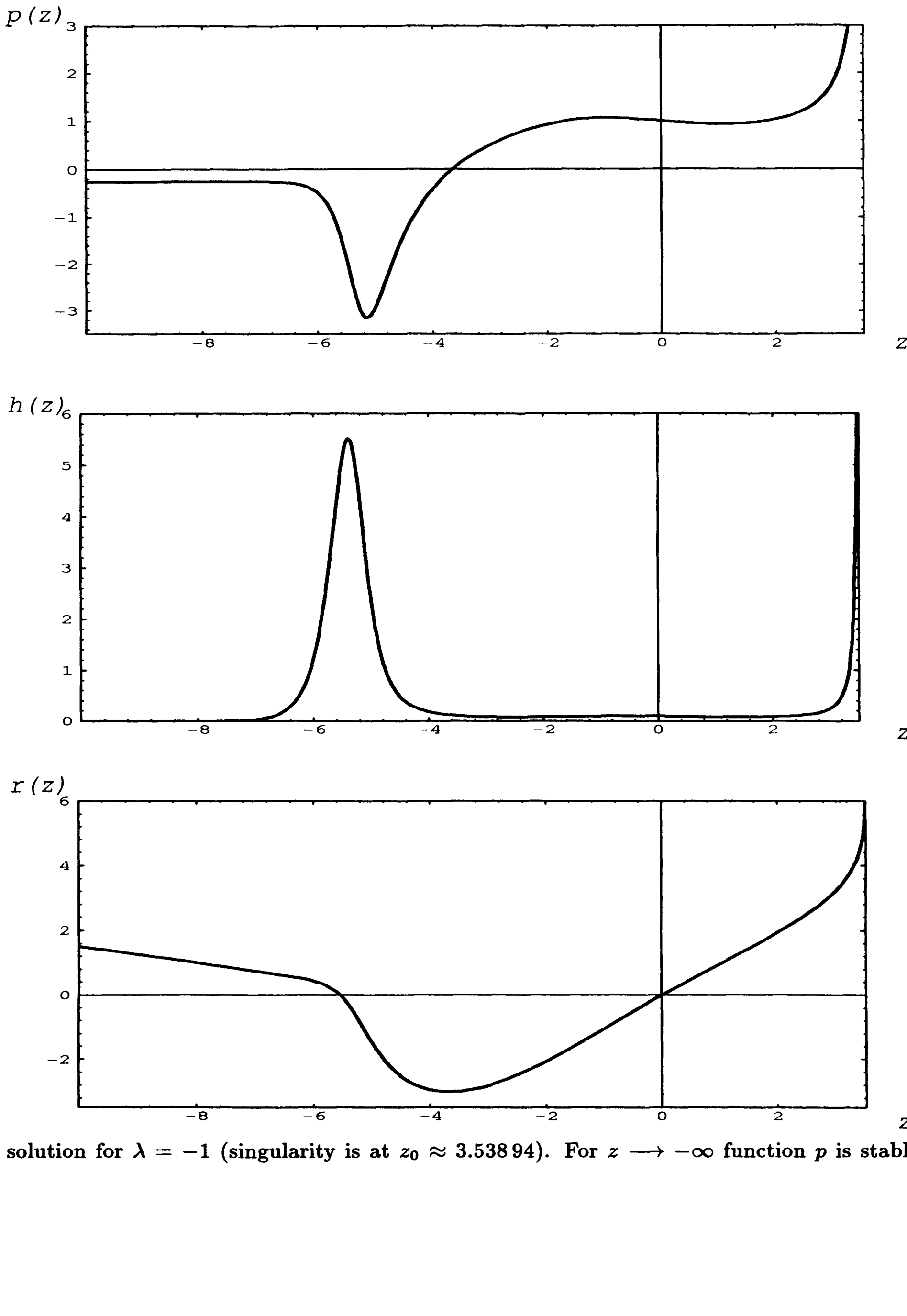

FIG. 5. Singular solution for $\lambda = -1$ (singularity is at $z_0 \approx 3.538\,94$). For $z \longrightarrow -\infty$ function p is stably asymptotically constant.

it then drops to a minimum at the extremum point of h, and finally all ends in a "big crunch" with infinite matter energy density at $z = +\infty$. Notice however that depending on the signs and values of the coupling and integration constants this picture can be interpreted either as a true two-dimensional cosmological evolution, or as a static gravity-matter soliton configuration.

(B) The "semisoliton" solution in which p either starts or ends at an asymptotically constant value, while it diverges at some finite $z = z_0$. The typical results of numerical integration are given in Figs. 4 and 5. Note that the matter density (5.19) is zero in singular points in view of (5.30)–(5.33).

(C) The solution starts and ends at singularities. The relevant pictures are given in Figs. 6 and 7.

It is clear that, depending on the values of the coupling constants, a possible "glueing" of the (B) and (C) pieces can occur at singularities, hence giving rise to a black-hole-type metric function $h(z)$ with a sort of horizon points. Alternatively, since the energy density (5.19) vanishes at singularities, it appears that a matching of "inner" (B) or (C) solution is possible with a vacuum "external" solution. This matching problem will be discussed in details elsewhere.

VI. CONCLUSION

We have suggested an explicitly coordinate- and gauge-invariant formulation of the gauge gravity interacting with an arbitrary matter in two dimensions. This formalism is applied to the study of the integrability problem for the classical nonvacuum field equations. Massless

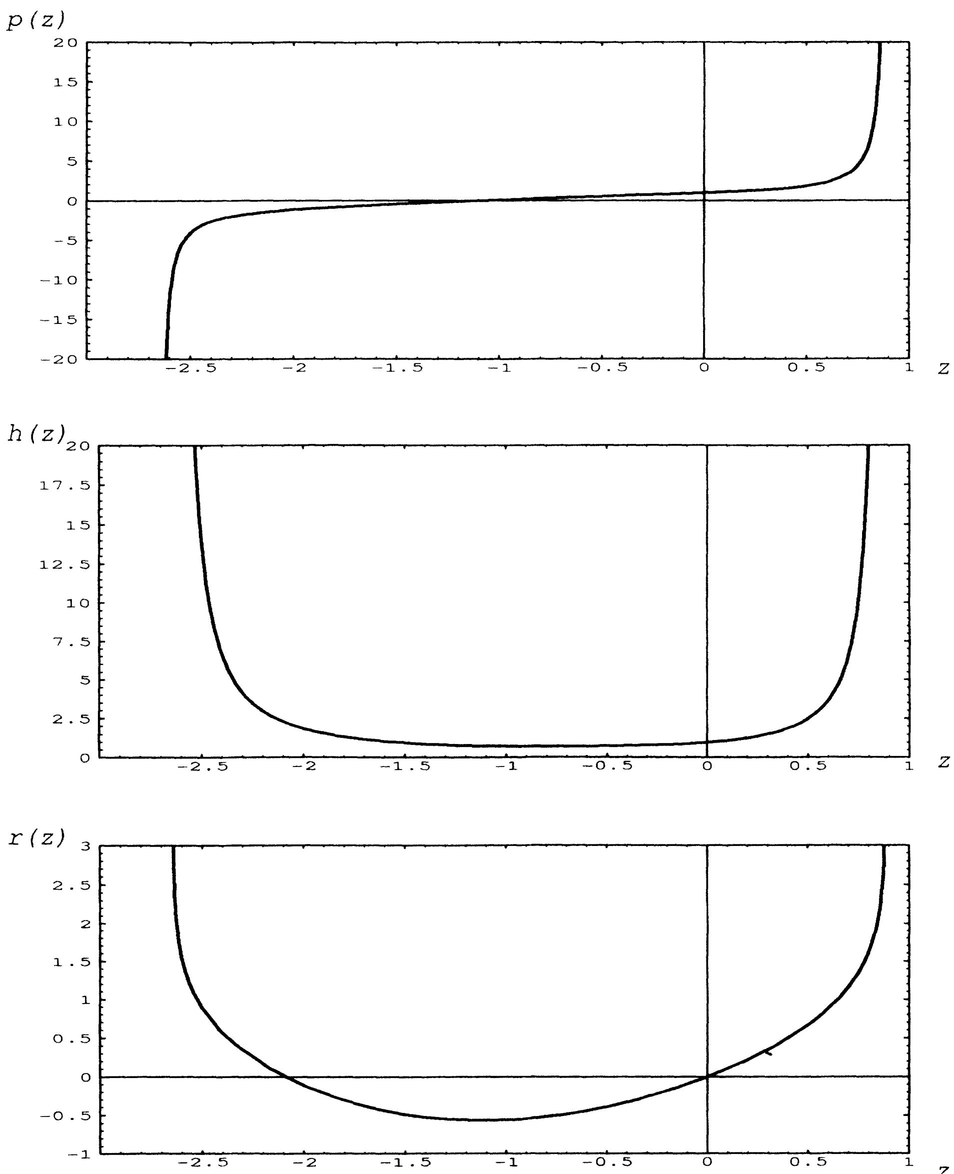

FIG. 6. Solution for $\lambda = 1$ with two singularities (at $z_0 \approx -2.646\,92$ and $z_0 \approx 0.886\,013$).

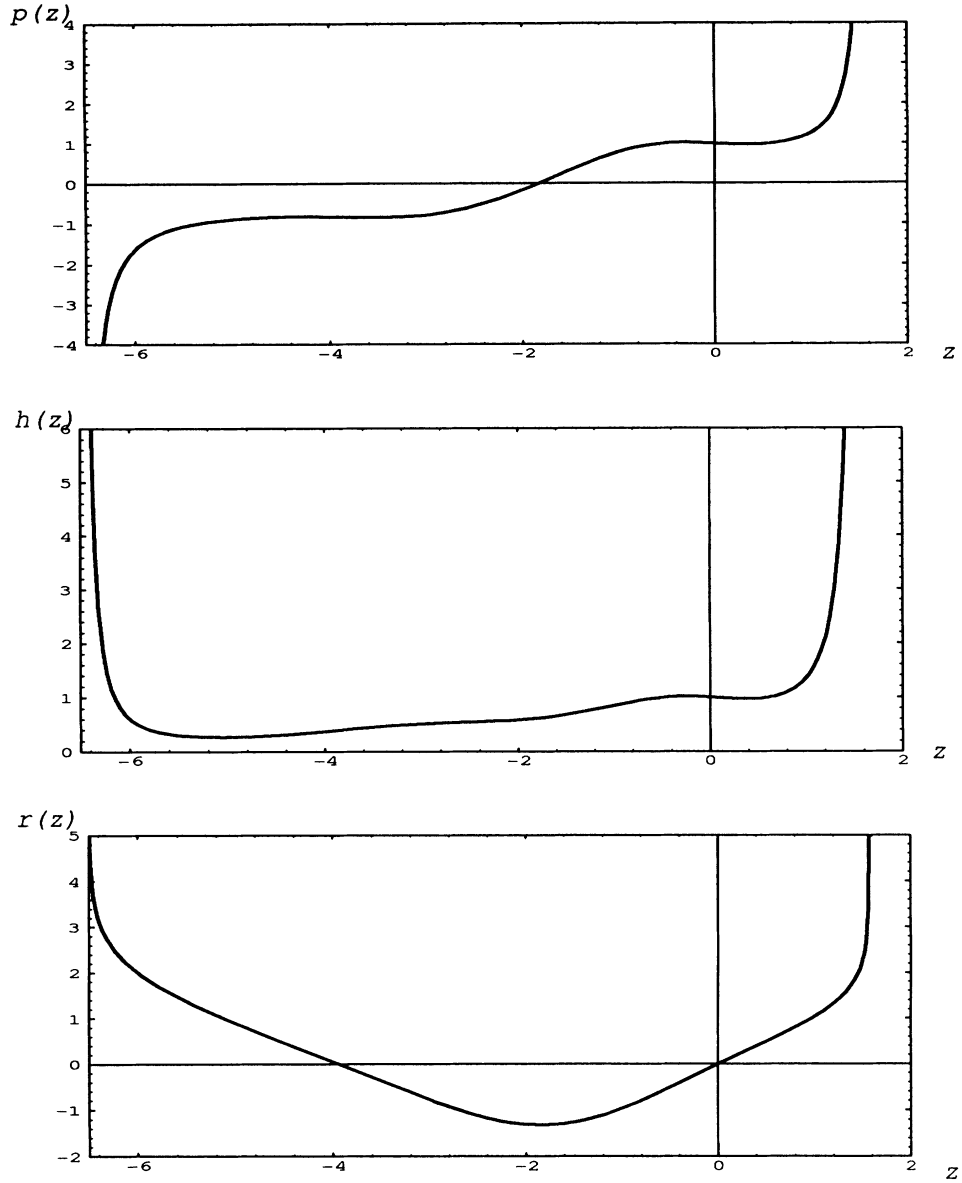

FIG. 7. Solution for $\lambda = -0.1$ with two singularities (at $z_0 \approx -6.500\,05$ and $z_0 \approx 1.569\,18$).

fermion and boson fields have similar properties in two dimensions, which is directly related to the vanishing of their spin current. Chiral and nonchiral exact solutions are constructed for both types of matter sources. The results obtained would have natural applications in the string models with dynamical gravity. It is worthwhile to mention here the recent discussion of canonical quantization of the boson string theory with dynamical gauge gravity [23]. Generalization to the more realistic fermion and superstring models appears to be an extremely interesting problem.

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