

Physical equivalence between nonlinear gravity theories and a general-relativistic self-gravitating scalar field

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We argue that in a nonlinear gravity theory (the Lagrangian being an arbitrary function of the curvature scalar R), which according to well-known results is dynamically equivalent to a self-gravitating scalar field in general relativity, the true physical variables are exactly those which describe the equivalent general-relativistic model (these variables are known as the Einstein frame). Whenever such variables cannot be defined, there are strong indications that the original theory is unphysical, in the sense that Minkowski space is unstable due to the existence of negative-energy solutions close to it. To this aim we first clarify the global net of relationships between the nonlinear gravity theories, scalar-tensor theories, and general relativity, showing that in a sense these are "canonically conjugated" to each other. We stress that the isomorphisms are in most cases local; in the regions where these are defined, we explicitly show how to map, in the presence of matter, the Jordan frame to the Einstein one and vice versa. We study energetics for asymptotically flat solutions for those Lagrangians which admit conformal rescaling to the Einstein frame in the vicinity of flat space. This is based on the second-order dynamics obtained, without changing the metric, by the use of a Helmholtz Lagrangian. We prove for a large class of these Lagrangians that the ADM energy is positive for solutions close to flat space, and this is determined by the lowest-order terms $R + aR^2$ in the Lagrangian. The proof of this positive-energy theorem relies on the existence of the Einstein frame, since in the (Helmholtz-)Jordan frame the dominant energy condition does not hold and the field variables are unrelated to the total energy of the system. This is why we regard the Jordan frame as unphysical, while the Einstein frame is physical and reveals the physical contents of the theory. The latter should hence be viewed as *physically* equivalent to a self-interacting general-relativistic scalar field.

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I. INTRODUCTION

Metric theories of gravitation, which are based on a Lagrangian density depending in a nonlinear way on the scalar curvature [such theories somehow, improperly, are usually called "nonlinear gravity" (NLG) models], share a general property, which has been extensively described in several works: acting on the metric by a suitable conformal transformation, the field equations can be recast into Einstein ones for the rescaled metric, interacting with a scalar field. It is therefore claimed that any NLG theory is equivalent to general relativity (with the scalar field).

A similar phenomenon occurs in Jordan-Brans-Dicke theory and its generalizations, the scalar-tensor gravity theories (STG). The original pair of variables (metric + scalar field) can be replaced by a new pair in which the metric has been conformally rescaled, and in the new variables the field equations become those of general relativity: the scalar field, which is not affected by the transformation, turns out to be minimally coupled to the rescaled metric. The original set of variables is commonly called the *Jordan conformal frame*, while the transformed set, whose dynamics is described by Einstein equations, is called the *Einstein conformal frame*.

A problem thus arises of whether the tensor representing the *physical* metric structure of space-time is the one belonging to the Jordan frame or to the Einstein frame.

This problem can be traced back to Pauli in the early 1950s (quoted in [1]). In a system consisting of metric gravity and a scalar field there is an ambiguity: the metric tensor can be conformally rescaled by an arbitrary (positive) function of the scalar. Thus, in addition to the original (Jordan) and Einstein frames, there exists an infinite number of conformally related frames, each consisting of a pair $(F(\varphi)g_{\mu\nu}, \varphi)$ with a different F . One asks, are the metrics in the frames $(g_{\mu\nu}, \varphi)$ and $(F(\varphi)g_{\mu\nu}, \varphi)$ also *physically* equivalent? The same question would clearly arise also for a more general change of variables $g'_{\mu\nu} = g'_{\mu\nu}(g_{\alpha\beta}, \varphi)$, $\varphi' = \varphi'(g_{\alpha\beta}, \varphi)$.

NLG and STG theories are in fact deeply connected, as described in this paper, and can actually be viewed as different versions of the same model. We now define the notion of Jordan frame and Einstein frame for NLG theories; the main purpose of this paper is to analyze the problem of determining which "frame"¹ is the *physical*

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¹We use here the word "frame" to denote a choice of dynamical variables, rather than a choice of a reference frame in space-time; this terminology is commonly used in the literature on the subject. Sometimes the term "gauge" (also abused in this context) is used instead of "frame" [10].

one. By “physical frame” we mean a set of field variables which are (at least in principle) measurable and satisfy all general requirements of classical field theory, e.g., give rise to positive-definite energy density (we are aware of the fact that the term “physical” is frequently abused in the literature).

We assume, for simplicity, that the space-time is four dimensional, although all the calculations can be actually carried out in higher dimension without significant modifications. The signature is $(-+++)$, and we set $\hbar = c = 8\pi G = 1$. Let us consider the Lagrangian of a vacuum NLG theory,

$$L = f(R)\sqrt{-g}, \quad (1.1)$$

which generates the fourth-order equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = 0, \quad (1.2)$$

where $f'(R) \equiv \frac{df}{dR}$ and $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$. In the vacuum case, the Jordan frame includes only the metric tensor $g_{\mu\nu}$. According to a well-known procedure [2–4], we introduce a pair of new variables $(\tilde{g}_{\mu\nu}, p)$, related to $g_{\mu\nu}$ (and to its derivatives) by

$$p = f'(R), \quad \tilde{g}_{\mu\nu} = pg_{\mu\nu}. \quad (1.3)$$

(This transformation was rediscovered several times and generalized to the case of Lagrangians depending also on $\square^k R$ [5, 6].) The scalar p is dimensionless, and to ensure the regularity of the conformal rescaling it is usually assumed that $p > 0$ (we will retake this point below). Then the two metrics $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ have the same signature. Let $r(p)$ be a solution of the equation $f'[r(p)] - p = 0$ (it may be not unique, and we will consider this problem later); the field equations for the new variables become

$$\begin{aligned} \tilde{G}_{\mu\nu} = p^{-2} \left\{ \frac{3}{2} \left(p_{,\mu} p_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} p_{,\alpha} p_{,\beta} \right) \right. \\ \left. + \frac{1}{2} \{f[r(p)] - p r(p)\} \tilde{g}_{\mu\nu} \right\}, \end{aligned} \quad (1.4a)$$

whereby \tilde{G} is the Einstein tensor of the rescaled metric $\tilde{g}_{\mu\nu}$, and

$$\tilde{\square} p = p^{-1} \left\{ \tilde{g}^{\mu\nu} p_{,\mu} p_{,\nu} + \frac{1}{3} \{2f[r(p)] - p r(p)\} \right\}. \quad (1.4b)$$

These (second-order) equations can be derived from the Lagrangian [3]

$$\begin{aligned} \tilde{L} = \tilde{R} \sqrt{-\tilde{g}} - \left\{ \frac{3}{2} p^{-2} \tilde{g}^{\mu\nu} p_{,\mu} p_{,\nu} \right. \\ \left. + p^{-1} r(p) - p^{-2} f[r(p)] \right\} \sqrt{-\tilde{g}}. \end{aligned} \quad (1.5)$$

It is customary to redefine the scalar field by setting $p = e^{\sqrt{2/3}\phi}$; the Lagrangian then becomes

$$\tilde{L} = \left[\tilde{R} - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) \right] \sqrt{-\tilde{g}}. \quad (1.6)$$

One sees that ϕ is minimally coupled to $\tilde{g}_{\mu\nu}$ and is an ordinary massive self-interacting scalar field with a potential

$$V(\phi) = \frac{1}{2} e^{-\sqrt{2/3}\phi} r[p(\phi)] - \frac{1}{2} e^{-2\sqrt{2/3}\phi} f(r[p(\phi)]), \quad (1.7)$$

which is determined by the original Lagrangian (1.1). The variables $(\tilde{g}_{\mu\nu}, \phi)$ provide the Einstein frame for the NLG theory.

In the Jordan frame, gravity is entirely described by the metric tensor $g_{\mu\nu}$. In the Einstein frame, the scalar field ϕ acts as a source for the metric tensor $\tilde{g}_{\mu\nu}$ and formally plays the role of an external “matterfield;” however, the original theory did not include any matter; hence, we are led to regard the scalar field occurring in the Einstein frame (which corresponds to the additional degree of freedom due to the higher order of the field equations in Jordan frame) as a “nonmetric” aspect of the gravitational interaction. From this viewpoint, the NLG theory, although *mathematically* equivalent to general relativity (GR), is physically different because in the Einstein frame, where the equations coincide with those of GR, gravity is no longer represented by the metric tensor alone. However, another viewpoint is possible: one could assume that the fourth-order picture in the Jordan frame represents an “already unified” model including a nongravitational degree of freedom (a minimally coupled nonlinear scalar field), and that the gravitational interaction is described only by the rescaled metric in the Einstein frame (see Appendix C).

This clearly indicates that the problem of the physical nature of the fields occurring in both frames should be addressed *prior* to any other consideration on the physical significance of the equivalence between NLG and GR. From the outline presented above, one might be led to the following conclusion: if one starts from a vacuum NLG theory, there is no way to decide *a priori* which frame should be taken as the physical one; the choice of the physical metric is an additional datum, which affects the physical interpretation of the model but is essentially independent of the formal structure of the theory. On the other hand, if we formulate a gravitational theory including matter fields, the ambiguity is broken by the coupling of the metric tensor with such matter fields. In fact, the two metric tensors $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ interact in a different way with each external field (we will discuss this point in greater detail below); therefore, one should be able to single out the physical metric by requiring that matter fields be minimally coupled with it and that neutral test particles fall along its geodesic lines.

This viewpoint, more or less explicitly formulated,

seems to be shared by many authors. However, a criterion based on the coupling of matter with gravity would be effective only if the total Lagrangian were determined on independent grounds, e.g., by string theory. As a matter of fact, this is not so: the theories we consider in this paper are commonly viewed as fundamental ones and constructed by adding usual interaction Lagrangians to a purely gravitational one. The point, which some authors seem to overlook, is that adding minimal-coupling terms to the Lagrangian (1.1) already entails that the Jordan frame is assumed to be the physical one. In this situation, claiming that the Einstein frame is unphysical, because the coupling between matter and the rescaled metric is not the usual one, is a *petitio principii*. It would be equally reasonable to add standard interaction terms, minimally coupled to $\tilde{g}_{\mu\nu}$, to the Lagrangian (1.6): the resulting coupling with $g_{\mu\nu}$ would then turn out to be unphysical. In other terms, the frequently repeated argument [7] in favor of the Jordan frame, namely, that physical measurements are made in this frame (“atomic frame”), since atomic masses are physical constants there, can be equally well used in favor of the Einstein frame because the argument is a direct consequence of the arbitrary (at this level of reasoning) choice of the full action for a gravitational theory. Thus, the ambiguity is still present.

Indeed, this ambiguity is faithfully represented in the literature. The authors dealing with nonlinear or scalar-tensor theories of gravity in the context of cosmology or of high-energy physics can be broadly divided into four groups. According to the authors of the first group, the Jordan frame is the physical one and the Einstein frame merely serves as a practical computational tool² [7–9]. (Barrow and Maeda [7] remark that the Einstein frame is computationally advantageous only in a vacuum theory. If matter, e.g., a perfect fluid, is included, the conformal transformation usually does not lead to simpler equations, since in the Einstein frame the scalar field is coupled to the fluid.) The authors in the second group regard the Einstein frame as being physical, either because of its resemblance to general relativity³ [10,11], or since the standard formalism for quantizing the scalar field fluctuations in the linear approximation does not apply in the Jordan frame (or at least is suspect there) [12,13] or because the massless spin-2 graviton in Jordan-Brans-Dicke theory is described by the Einstein-frame metric [14], or finally because this is implied by the dimensional reduction of a higher-dimensional action [15]. The third group consists of the authors claiming that the two frames *are* physically equivalent, at least at the classical level, since conformal transformations do not change the mass ratios of elementary particles and therefore do

not alter physics⁴ (“physics cannot distinguish between conformal frames”) [16]. The last, rather inhomogeneous group, involves authors who either use both conformal frames without addressing the problem of which of them (if any) is physical or work exclusively in the original Jordan frame without making any reference to the existence of the rescaled metric [17].

In our opinion, the strongest argument that exists in the literature in favor of one of the conformal frames (if the universe is exactly four dimensional) is the one based on quantization of field fluctuations. “It appears as if the quantization and conformal transformation are two mutually noncommutable procedures” [13]. Here we show that classical relativistic field theory provides another argument, which points to the same conformal frame.

The main arguments presented in this paper can be outlined in the following statements.

(i) In contrast with claims raised in the previous literature, nonlinear and scalar-tensor theories of gravity can be equally well formulated, as far as *formal consistency* is considered, in terms of either one of the frames; from this viewpoint, either frame can in principle be assumed to be the physical one.

(ii) However, the physics described by nonlinear and scalar-tensor theories of gravity is not conformally invariant and using different conformal frames one finds inequivalent effects. Therefore, choosing a correct (i.e., physical) frame is an indispensable part of theoretical investigation [18,19].

(iii) The physical metric should be singled out already in the vacuum theory; the coupling of a given metric to matter fields is, in fact, *determined* by the physical significance ascribed to it, i.e., by its relation to the physical metric.

(iv) In particular, if the physical metric is assumed to be the rescaled (Einstein-frame) one, the original nonlinear Lagrangian should include interaction with ordinary matter in such a way that the corresponding coupling with the Einstein-frame metric turns out to be minimal. This problem has, to our knowledge, never been addressed in the previous literature. We provide here a systematic and unique way to obtain Lagrangians, which reduce to any given vacuum nonlinear Lagrangian in the absence of matter, and reproduce upon a suitable rescaling any (with some restrictions) minimally coupled matter Lagrangian.

(v) Since consistency arguments do not allow one to es-

²However, Kalara *et al.* [7], while investigating the power-law inflation in the Einstein frame, interpret the solutions and fit their parameters as if this frame were the physical one.

³Gibbons and Maeda [10] admit that there might be an argument (or merely a someone’s prejudice) stating that the “physically correct” frame is different from the Einstein one.

⁴Garay and Garcia-Bellido in [16] introduce a concept of physical frame which is different from ours. According to them the physical frame is one “in which observable particles have constant masses, since in this frame particles follow geodesics of the metric.” Then, by the assumption on the form of the full action, the Jordan frame coincides with the physical one. We notice at this point that in general relativity particles (with constant masses) usually do not move along geodesics: consider for instance a perfect fluid with pressure. Geodesic motion is rather exceptional, it occurs in the case of pressureless dust and for noninteracting test particles.

establish which set of variables is physical, we investigate a different criterion. This criterion is purely classical and is provided by the positivity of energy for small fluctuations of *every* dynamical variable around the ground state solution.

(vi) The notion of gravitational energy in NLG theories is inapplicable to this aim, due to the lack of a positivity theorem for higher-order gravitational theories. To circumvent this problem, one should compare two *second-order* versions of the same theory, namely, the second-order dynamics obtained in the Einstein frame and the “Hamiltonian” formulation of the theory constructed, *without* rescaling the metric, through a Legendre transformation.

(vii) We show that the positive energy theorem for nonlinear gravity, proven by Strominger [20] for a quadratic case, holds for a larger class of Lagrangians for which the Einstein frame can be defined around flat space. Actually, whether or not the Arnowitt-Deser-Misner (ADM) energy is positive depends on the potential (1.7). The existence of the Einstein frame is in any case essential for assessing classical stability of Minkowski space and positivity of energy for nearby solutions. In the Jordan frame, the dominant energy condition never holds. For these reasons, the Einstein frame is the most natural candidate for the role of physical frame in NLG theories.

The paper is organized as follows. In Sec. II, we discuss in full detail the statements (i), (ii), and (iii) above, using examples taken from the literature. We provide there a global and complete picture of relationships between NGL and STG theories and general-relativistic models of a scalar field, while previously only separate connections were known. Section III is the heart of the paper. We discuss there items (v), (vi), and (vii). We investigate there a large class of nonlinear Lagrangians, for which flat space (in the Jordan frame) is a solution and the Einstein frame exists for it. While the ADM energy can be defined for any asymptotically flat solution, one is unable to establish its sign in the Jordan frame. In the Einstein frame, the relation between the interior of the system and its total energy takes on the standard form known from general relativity. This fact convinces us that the Jordan frame is unphysical. Section IV contains conclusions. All technical parts of the paper are moved to the appendices and the main body of the paper can be read without consulting them. In particular, Appendix B provides theoretical background for some results applied in Secs. II and III. The “inverse problem of nonlinear gravity” (finding the purely metric nonlinear Lagrangian, which generates a prescribed potential for the scalar field) is presented in Appendix C.

II. CONSERVATION LAWS AND INTERACTION OF GRAVITY WITH MATTER

Some authors have tried to solve the problem of determining the physical metric by showing that only one possible choice allows one to obtain a divergenceless energy-momentum tensor for any self-gravitating matter. Brans used this argument to claim that only the Jordan frame is physical [18], while in a recent paper [21] Cotsakis was

led by a more detailed investigation of this point to the conclusion that the Einstein frame is the physical one. Unfortunately, an error invalidates Cotsakis’ proof.⁵ Here we show that studying the conservation laws for matter does not allow one to find out which frame is physical: the equations of motion for matter and gravity form consistent and closed systems for both Jordan and Einstein frames.

Our basic assumption is that the gravitational interaction of matter should be described by minimal coupling with the *physical* metric tensor field. In other words, the physical metric of space-time should be identified prior to the construction of the Lagrangian for a gravitating system. Different identifications of the physical frame will give rise to (mathematically and physically) inequivalent Lagrangians. This assumption is not commonly accepted; e.g., the authors of [13] argue that the Einstein frame is physical, but nevertheless they assume that matter minimally couples to the metric in the (unphysical) Jordan frame. A similar viewpoint is taken by Alonso *et al.* in [11].

The assumption is based on the postulate that the great advantage of Einstein’s general relativity, i.e., the universal validity of the minimal-coupling principle, should be retained in NLG and scalar-tensor theories of gravity. The principle is partially abrogated in the so-called “extended Jordan-Brans-Dicke theory,” where the ordinary (“visible”) and dark (“invisible”) matter minimally couple to different (conformally related) metric fields (Damour, Gibbons, and Gundach in [7]). Therefore, although observationally viable, we do not take this theory into account in this paper, on theoretical grounds.

In the first part of this section, we discuss the implication of our assumption in the context of NLG theory, considering separately the two cases: either the Jordan-frame or the Einstein-frame metric is regarded as physical. We show that the two cases describe different physical models of gravitational interaction of matter fields, and we show that *each* of the two models can be consistently formulated in *both* frames.

The scalar-tensor gravity (STG) theories share with NLG theories the feature that they can be reformulated as a general-relativistic model for a self-gravitating scalar field.⁶ As a matter of fact, any NLG theory can be also expressed in terms of a scalar-tensor theory with nontrivial cosmological function (without changing the metric), as is explained in detail in Appendix B. Few authors seem

⁵The crucial step of the proof consists in taking the divergence of the fourth-order equation [Eq. (2.2) below]: Cotsakis claims that it does not vanish in general. Actually its vanishing follows from a generalized Bianchi identity valid for any gravitational Lagrangian depending on the full curvature tensor and its derivatives.

⁶In principle, we should discriminate between the dynamical equivalence of Lagrangians and equality of the action integrals; this difference is significant in quantum theory, while in classical field theory it is irrelevant, as physical meaning is given only to solutions of the field equations.

to be aware of this relation, while many authors establish a connection between NLG and STG theories via the equivalence to general relativity. NLG and STG theories thus appear strictly intertwined in the literature, sometimes causing confusion. There is indeed a physical difference between STG and NLG theories: in the former it is *a priori* postulated that matter couples minimally to the Jordan-frame metric. The dynamical equivalence clearly shows that, despite appearances, there is no deeper difference between the two classes of gravity theories. Nevertheless, for clarity reasons, we also discuss here interactions with matter and conservation laws for STG theories. First we investigate the consequences of our assumption for the case of NLG theory.

**A. NLG theory, case I:
the original metric $g_{\mu\nu}$ is physical**

According to the assumption, the Lagrangian for gravity and matterfields (collectively denoted by Ψ) reads⁷

$$L = [f(R) + 2\ell_{\text{mat}}(\Psi, g)]\sqrt{-g}. \quad (2.1)$$

The gravitational field equations are then

$$\frac{\delta}{\delta g^{\mu\nu}}[\sqrt{-g}f(R)] = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = T_{\mu\nu}(\Psi, g), \quad (2.2)$$

where, as usual, $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}\ell_{\text{mat}})$. It follows from Noether theorem (which in this case is equivalent to a generalized Bianchi identity) that $\nabla_\nu T^{\mu\nu} = 0$. Using the general procedure described in Appendix B and upon conformal rescaling (1.3) of the metric one gets the Einstein-frame Lagrangian dynamically equivalent to (2.1) [3]:

$$\tilde{L}(\tilde{g}, \phi, \Psi) = \left[\tilde{R}(\tilde{g}) - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) + 2e^{-2\sqrt{2/3}\phi} \ell_{\text{mat}}(\Psi, e^{-\sqrt{2/3}\phi} \tilde{g}) \right] \sqrt{-\tilde{g}}. \quad (2.3)$$

Here, as before, $\phi \equiv \sqrt{\frac{3}{2}} \ln p$ and the potential $V(\phi)$ is given in (1.7). The field equations for the metric and the scalar in the Einstein frame can be obtained either by transforming Eq. (2.2) or directly from the Lagrangian (2.3). As the latter contains interaction terms between ϕ and Ψ , a variational stress tensor for the matter field Ψ alone cannot be derived unambiguously from (2.3). Therefore, it is advisable to formulate the gravitational equations in both frames in terms of the same tensor $T_{\mu\nu}$ already defined in terms of the physical metric. The metric field equations are then

$$\tilde{G}_{\mu\nu} = t_{\mu\nu}(\phi, \tilde{g}) + e^{-\sqrt{2/3}\phi} T_{\mu\nu}(\Psi, e^{-\sqrt{2/3}\phi} \tilde{g}), \quad (2.4)$$

where

$$t_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi) \tilde{g}_{\mu\nu} \quad (2.5)$$

plays the role of the effective stress-energy tensor for ϕ ; the equation of motion for ϕ is

$$\tilde{\square} \phi = \frac{dV}{d\phi} + \frac{1}{\sqrt{6}} e^{-\sqrt{2/3}\phi} \tilde{g}^{\mu\nu} T_{\mu\nu}(\Psi, e^{-\sqrt{2/3}\phi} \tilde{g}), \quad (2.6)$$

with

$$\frac{dV}{d\phi} = \frac{1}{\sqrt{6}p} \left[\frac{2}{p} f[r(p)] - r(p) \right]_{p=p(\phi)=e\sqrt{2/3}\phi}. \quad (2.7)$$

The Bianchi identity now implies

$$\tilde{\nabla}^\nu \left[t_{\mu\nu}(\phi, \tilde{g}) + e^{-\sqrt{2/3}\phi} T_{\mu\nu}(\Psi, e^{-\sqrt{2/3}\phi} \tilde{g}) \right] = 0, \quad (2.8)$$

and the two stress tensors are not separately conserved, since the scalar field interacts with matter, as is explicitly seen in Lagrangian (2.3).

In this picture the scalar field ϕ influences the motion of any gravitating matter, except for the particular case in which the interaction Lagrangian ℓ_{mat} is conformally invariant (see Appendix A). Hence, this theory is *not* physically equivalent to general relativity: in fact, gravity is completely represented by a metric tensor only in the Jordan frame (where it obeys fourth-order equations), while in the Einstein frame, where equations of motion for $\tilde{g}_{\mu\nu}$ are formally Einstein ones, there is a non-geometric gravitational degree of freedom, represented by the scalar ϕ , which is universally coupled to matter. Nevertheless, conservation laws consistent with the assumption that the original metric $g_{\mu\nu}$ is the physical one can indeed be found.

**B. NGL theory, case II:
the conformally rescaled metric $\tilde{g}_{\mu\nu}$ is physical**

The original metric $g_{\mu\nu}$ plays now the role of a variable providing an already unified fourth-order version of a theory including the gravitational metric plus a nonlinear scalar field (these theories form a basis for inflationary cosmological models). The vacuum Lagrangian (1.1) should first be transformed into the corresponding Einstein-frame Lagrangian (1.6), and *then* a minimal-coupling Lagrangian for matter can be added:

$$\tilde{L} = \left[\tilde{R} - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2V(\phi) + 2\ell_{\text{mat}}(\Psi, \tilde{g}) \right] \sqrt{-\tilde{g}}. \quad (2.9)$$

The matter Lagrangian ℓ_{mat} is ϕ -independent, since there is no physical motivation to assume that matter interacts with the scalar. As previously, ℓ_{mat} is constructed by defining on physical grounds the form of the Lagrangian for Ψ in Minkowski space, then replacing the flat metric by the metric, which is viewed as physical. Therefore $\ell_{\text{mat}}(\Psi, g)$ in case I and $\ell_{\text{mat}}(\Psi, \tilde{g})$ of case II are the same functions of their respective arguments; clearly, for conformally related metrics, $\ell_{\text{mat}}(\Psi, g) \neq \ell_{\text{mat}}(\Psi, \tilde{g})$.

The field equations resulting from (2.9) now read

⁷The coefficient in front of ℓ_{mat} is chosen on the assumption that the linear term in the Taylor expansion of $f(R)$ has coefficient 1.

$$\tilde{G}_{\mu\nu} = t_{\mu\nu}(\phi, \tilde{g}) + T_{\mu\nu}(\Psi, \tilde{g}) , \quad (2.10)$$

$$\tilde{\square} \phi = \frac{dV}{d\phi} , \quad (2.11)$$

$$\frac{\delta}{\delta\Psi}(\ell_{\text{mat}}\sqrt{-\tilde{g}}) = 0 , \quad (2.12)$$

where $t_{\mu\nu}$ is given in (2.5), with the potential as in (1.7) and (2.7).

Here $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta}{\delta\tilde{g}^{\mu\nu}}(\ell_{\text{mat}}\sqrt{-\tilde{g}})$, and due to the absence of any interaction between ϕ and Ψ not only the total stress-energy tensor $t_{\mu\nu} + T_{\mu\nu}$ is conserved, but the stress tensors of each of the fields are also separately conserved, $\tilde{\nabla}^\nu t_{\mu\nu} = \tilde{\nabla}^\nu T_{\mu\nu} = 0$.

To recast the theory with matter interaction in terms of the original metric $g_{\mu\nu}$, one should *not* use the relations $p(\phi) = f'(R)$ and $r[p(\phi)] = R(g)$: these ones hold only in the absence of matter. In fact, let us reexpress the field equations (2.10) and (2.11) in terms of the variables $(p, g_{\mu\nu})$, with $p = e^{\sqrt{2/3}\phi}$. These read

$$G_{\mu\nu}(g) = p^{-1}(\nabla_\mu \nabla_\nu p - g_{\mu\nu} \square p) - pV(p)g_{\mu\nu} + T_{\mu\nu}(\Psi, pg) , \quad (2.13)$$

$$\square p = \frac{2p^3}{3} \frac{dV}{dp} . \quad (2.14)$$

Taking the trace of (2.13), eliminating $\square p$ with the aid of (2.14) and using the explicit form of the potential, $V = \frac{1}{2p}r(p) - \frac{1}{2p^2}f[r(p)]$, one arrives at the equation

$$R(g) - r(p) + g^{\mu\nu}T_{\mu\nu}(\Psi, pg) = 0 . \quad (2.15)$$

Recall that $r(p)$ is defined by solving the equation $f'[r(p)] = p$. Unless the matter is absent or its stress tensor is traceless, $p = f'(R)$ is inconsistent with (2.15) and does not provide the correct conformal rescaling to the Jordan frame. Provided that $T_{\mu\nu}$ does not contain covariant derivatives of Ψ , (2.15) can be viewed as an *algebraic*⁸ equation for the scalar field p . Solving this equation for p provides the correct relation, $p = P(R; g, \Psi)$, which allows one to reexpress the scalar field p in terms of the curvature scalar $R(g)$ and obtain higher-order equations of motion in terms of g and Ψ only. To avoid confusion, from now on we shall distinguish between the original unphysical frame, in which the vacuum Lagrangian takes the form $L = f(R)\sqrt{-g}$, which we call the *vacuum* Jordan conformal frame (VJCF), and the (also unphysical) frame into which the scalar field ϕ can be reabsorbed, in the presence of the matter term in (2.9), which shall be referred as the *matter* Jordan conformal frame (MJCF). Except for conformally invariant material systems, the two Jordan frames are different. A deeper understanding of the reason why the two frames do not coincide is provided by the Legendre-transformation method. The explicit construction of the MJCF, the corresponding non-

linear Lagrangian and the resulting field equations are given in Appendix B. In MJCF, being the already unified frame, matter is nonminimally coupled to the metric, and it is a generic feature that in the absence of gravitation, in flat space-time, the nonlinear Lagrangian does not reduce to the standard form for a given species of matter. This fact significantly influences conservation laws. As is shown in Appendix B, it is possible to separate a “purely gravitational part” in the gravitational field equations in MJCF; this part satisfies the generalized Bianchi identity. Four matter conservation laws then follow from the equations in the same way as in general relativity. These involve a number of terms mixing the curvature scalar with the matter variables and consequently do not resemble at all the elegant conservation laws $T_{\mu\nu}{}^{;\nu} = 0$ of Einstein’s theory. One can only learn from them that in this frame the matter world lines explicitly depend on curvature except for massless particles (photons); see Appendix A.

In the Einstein frame, on the other hand, the strong equivalence principle holds; the only possible difference between this version of gravity and general relativity may arise from the physical interpretation given to the scalar field, whose entire role is confined to influence the metric field. In fact, ϕ is assumed to describe a nongeometric spin-0 component of gravitation [22,23].

C. Frames and conservation laws in STG theories

Next we proceed to scalar-tensor theories of gravity (see [1,24] for recent reviews). These are conceptually different from NLG theories because these are not purely metric gravitational models, as the gravitational field is a doublet consisting of a spin-2 field and a (nongeometric) spin-0 field, the Brans-Dicke scalar.

The action of a generic STG theory is (we use conventions of Will’s book [25])

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi} \left[\varphi R - \frac{\omega(\varphi)}{\varphi} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} + 2\varphi \lambda(\varphi) \right] + \ell_{\text{mat}}(\Psi, g) \right\} . \quad (2.16)$$

The “cosmological function” $\lambda(\varphi)$ is often omitted, and accordingly we consider here the models in which $\lambda(\varphi) \equiv 0$. If the coupling function $\omega(\varphi)$ is constant, the action is that of the Thiry-Jordan-Fierz-Brans-Dicke theory (see [1] and [25] for references). The field variables $g_{\mu\nu}$ and φ form the Jordan conformal frame (it is here that the concept originated). By assumption, ordinary matter minimally couples to the metric $g_{\mu\nu}$ (and does not couple to the scalar gravity); thus, by this assumption the Jordan frame is physical, i.e., describes measurable space-time intervals. Proceeding as in the case of NLG theories one introduces a scalar variable

$$p = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial R} L(g, \varphi, \Psi) = \frac{1}{16\pi} \varphi ; \quad (2.17)$$

in this case it is not a function of the curvature but it

⁸If f is not a rational function, (2.15) may be a transcendental equation, but in any case it is not differential with respect to p .

coincides (up to a constant factor) with the already existing scalar field. With the aid of p one defines a new conformally related metric $\tilde{g}_{\mu\nu} = pg_{\mu\nu}$, the transformation being already known since 1962 as *Dicke transformation* [26]. In terms of the new variables $(\tilde{g}_{\mu\nu}, p)$ the action is (up to a full divergence term)

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \tilde{R} - [\omega(\phi) + \frac{3}{2}] p^{-2} \tilde{g}^{\mu\nu} p_{,\mu} p_{,\nu} + p^{-2} \ell_{\text{mat}}(\Psi, p^{-1} \tilde{g}_{\mu\nu}) \right\}, \quad (2.18)$$

and after a redefinition of the Brans-Dicke scalar,

$$d\phi \equiv [\omega(\phi) + \frac{3}{2}]^{1/2} \frac{d\varphi}{\varphi}, \quad \omega > -\frac{3}{2}, \quad (2.19)$$

it takes the standard form of the action for the linear massless scalar field minimally coupled to the metric:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \left(\frac{16\pi}{\varphi(\phi)} \right)^2 \ell_{\text{mat}} \left(\Psi, \frac{16\pi}{\varphi(\phi)} \tilde{g}_{\mu\nu} \right) \right]. \quad (2.20)$$

In terms of the Einstein-frame variables $(\tilde{g}_{\mu\nu}, \phi)$ the matter part of the action describes an interaction between ordinary matter and the scalar gravity. Clearly, it is due to the use of the unphysical (by assumption) variables for gravity. As a consequence the variational (with respect to $\tilde{g}_{\mu\nu}$) “energy-momentum” tensor for Ψ that can be defined by the Lagrangian $\varphi^{-2} \ell_{\text{mat}}(\Psi, \varphi^{-1} \tilde{g}_{\mu\nu})$ is different from the matter stress tensor defined in the Jordan frame and transformed to the Einstein frame; as already mentioned, this notion is rather ambiguous and of little use.

To study conservation laws for a STG theory one should first correctly identify the energy-momentum tensor for the spin-zero gravity. First, the purely metric gravitational Lagrangian L_g should be separated out. To this end, one formally views the Brans-Dicke scalar as a test field in a given fixed background $g_{\mu\nu}$. The basic assumption of scalar-tensor gravity theories is that the average value of the field φ determines the present value of the gravitational constant, $\langle \varphi \rangle = \frac{1}{G}$. Therefore, the situation where the scalar gravity is “switched off” does not correspond to $\varphi \equiv 0$, but rather to assuming that φ is constant and equal to the present value of $\frac{1}{G}$ ($= 8\pi$ in our units); notice that $\varphi = 8\pi$ is actually a solution of the field equation only if $R = 0$. Setting $\varphi \equiv 8\pi$ and $\ell_{\text{mat}} \equiv 0$ reduces the Lagrangian in (2.16) to its purely metric part, $L_g = \frac{1}{2} R \sqrt{-g}$. This is the Einstein-Hilbert Lagrangian of general relativity, as should be expected. The scalar-gravity Lagrangian is thus defined (in the absence of matter) as $L_\varphi = L - L_g$: i.e.,

$$L_\varphi = \frac{1}{16\pi} \left[(\varphi - 8\pi) R - \frac{\omega(\varphi)}{\varphi} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} \right] \sqrt{-g}. \quad (2.21)$$

In the presence of minimally coupled matter the full Lagrangian in (2.16) is thus decomposed as $L = L_g + L_\varphi + \ell_{\text{mat}} \sqrt{-g}$. The term L_φ generates the variational energy-

momentum tensor

$$\tau_{\mu\nu} = \frac{1}{8\pi} \left[(8\pi - \varphi) G_{\mu\nu} + \nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \square \varphi + \frac{\omega(\varphi)}{\varphi} (\varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} \varphi^{,\alpha} \varphi_{,\alpha}) \right] \quad (2.22)$$

and the equation of motion

$$\square \varphi + \frac{1}{2\omega} \varphi^{,\alpha} \varphi_{,\alpha} \left(\frac{d\omega}{d\varphi} - \frac{\omega}{\varphi} \right) + \frac{\varphi R}{2\omega} = 0. \quad (2.23)$$

The invariance of the action integrals S_m and S_φ under spacetime translations generates eight conservation laws (Noether’s theorem), $\nabla^\nu \tau_{\mu\nu} = 0$ and $\nabla^\nu T_{\mu\nu} = 0$ [the former can be directly verified using (2.23)]. $\tau_{\mu\nu}$ explicitly depends on curvature. In Eqs. (2.22) and (2.23) $g_{\mu\nu}$ is an external metric field. When the back reaction of φ on the metric is accounted for, an ambiguity arises. Varying the action (2.16) with respect to $g_{\mu\nu}$ one arrives at field equations for spin-two gravity,

$$G_{\mu\nu}(g) = \tau_{\mu\nu}(\varphi, g) + T_{\mu\nu}(\Psi, g), \quad (2.24)$$

and these allow one to eliminate the Einstein tensor from the expression for $\tau_{\mu\nu}$ and the curvature scalar from (2.23). After these eliminations the field equations take the standard form, which is usually applied in SGT theories:

$$G_{\mu\nu} = \frac{1}{\varphi} (\nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \square \varphi) + \frac{\omega}{\varphi^2} (\varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} \varphi^{,\alpha} \varphi_{,\alpha}) + \frac{8\pi}{\varphi} T_{\mu\nu}(\Psi, g) \equiv \theta_{\mu\nu}(\varphi, g), \quad (2.25)$$

$$\square \varphi = \frac{1}{2\omega + 3} \left(8\pi T_\alpha{}^\alpha - \frac{d\omega}{d\varphi} \varphi^{,\alpha} \varphi_{,\alpha} \right). \quad (2.26)$$

Equation (2.25) defines the effective stress-energy tensor $\theta_{\mu\nu}$ for φ , acting as one of two sources for the metric field; this tensor is curvature independent. Clearly, also $\tau_{\mu\nu}$ is a source of metric gravity, according to (2.24). Thus, two stress-energy tensors are assigned to scalar gravity. Such ambiguity arises whenever spin-two gravity is generated by two different sources, since Einstein field equations alone determine only the total stress tensor, which can be expanded in various ways into contributions from each source separately. The effective stress tensor is *not* conserved and the relation

$$\theta_{\mu\nu} = \tau_{\mu\nu} + \left(1 - \frac{8\pi}{\varphi} \right) T_{\mu\nu} \quad (2.27)$$

yields

$$\theta^{\mu\nu}{}_{;\nu} = \frac{8\pi}{\varphi^2} T^{\mu\nu} \varphi_{,\nu}. \quad (2.28)$$

It is worth stressing that $\theta_{\mu\nu}$ appears when the Brans-Dicke scalar acts as a source of metric gravity, while the variational definition of $\tau_{\mu\nu}$ is always valid, and the latter tensor should be viewed as the correct expression for a conserved stress-energy tensor (any possible ambiguities

in the construction of L_φ are irrelevant in this respect). Whether or not $\tau_{\mu\nu}$ provides a good physical notion of energy is a separate problem, which will be addressed in Sec. III.

In studying conservation laws for matter in the Einstein frame we restrict ourselves to the case of Thiry-Jordan-Brans-Dicke theory for simplicity; i.e., we set $\omega = \text{const}$. Transforming (2.25) and (2.26) to the Einstein frame one finds the following field equations for $\tilde{g}_{\mu\nu}$ and for the redefined scalar $\phi = \frac{1}{\gamma} \ln \varphi$, with $\gamma = (\omega + \frac{3}{2})^{-1/2}$:

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + 8\pi e^{-\gamma\phi}T_{\mu\nu} \\ &\equiv t_{\mu\nu}(\phi, \tilde{g}) + 8\pi e^{-\gamma\phi}T_{\mu\nu}(\Psi, g), \end{aligned} \quad (2.29)$$

$$\tilde{\square}\phi = 4\pi\gamma e^{-\gamma\phi}T_{\mu\nu}\tilde{g}^{\mu\nu}. \quad (2.30)$$

The notation $T_{\mu\nu}(\Psi, g)$ recalls that the matter stress-energy tensor is defined in the Jordan frame. By taking the divergence of (2.29) and using (2.30) one arrives at four matter conservation laws:

$$\tilde{\nabla}^\nu T_{\mu\nu} - \gamma\tilde{g}^{\alpha\beta}(T_{\mu\alpha}\phi_{,\beta} - \frac{1}{2}T_{\alpha\beta}\phi_{,\mu}) = 0. \quad (2.31)$$

Finally, for the sake of completeness,⁹ we comment on the conformally invariant scalar field model [27]. This theory differs from STG theories in the physical interpretation given to the scalar field. The latter is in fact commonly viewed as a special kind of matter rather than being a spin-0 component of the gravitational interaction. Bekenstein [28] has shown 20 years ago that the conformally invariant¹⁰ scalar field is equivalent to the massless linear field under a suitable conformal map, but this result is surprisingly little known to relativists.¹¹ The form of the full action in the Jordan frame,

$$\begin{aligned} S = \int d^4x \sqrt{-g} \left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\chi_{,\mu}\chi_{,\nu} - \frac{1}{12}\chi^2 R \right. \\ \left. + \ell_{\text{mat}}(\Psi, g) \right], \end{aligned} \quad (2.32)$$

clearly shows that this frame is assumed to be physical. It is natural to identify the Lagrangians for the metric and the scalar as $L_g = \frac{1}{2}\sqrt{-g}R$ and $L_\chi = -\sqrt{-g}(\frac{1}{2}g^{\mu\nu}\chi_{,\mu}\chi_{,\nu} + \frac{1}{12}\chi^2 R)$, then the latter generates the variational stress-energy tensor for the scalar, $\tau_{\mu\nu}(\chi, g)$. Instead of deriving it and the field equations from the Lagrangian, one views (2.32) as a version of STG theory: upon identifying $\varphi = 8\pi - \frac{4\pi}{3}\chi^2$, we recover the action (2.16) with $\omega(\varphi) = \frac{3}{2}\frac{\varphi}{8\pi - \varphi}$. Then the explicit form of $\tau_{\mu\nu}(\chi, g)$ and the field equations follow from (2.22)–(2.24). As described for the preceding case, the Einstein tensor and

the curvature tensor can be eliminated, giving rise to the “effective stress-energy” tensor $\theta_{\mu\nu}$ and to field equations analogous to (2.25) and (2.26). In the Jordan frame the Noether theorem implies $T^{\mu\nu}{}_{;\nu} = \tau^{\mu\nu}{}_{;\nu} = 0$, while (2.27) and (2.28) show that the effective stress-energy tensor $\theta_{\mu\nu}$ is not conserved in the presence of matter:¹²

$$\theta^{\mu\nu}{}_{;\nu} = -12\chi(6 - \chi^2)^{-2}T^{\mu\nu}\chi_{,\nu}, \quad \chi^2 < 6. \quad (2.33)$$

The Einstein-frame Lagrangian is again (2.20), with $\phi = \sqrt{\frac{2}{3}} \ln \frac{\sqrt{6+\chi}}{\sqrt{6-\chi}}$ or $\phi = \sqrt{\frac{2}{3}} \ln \frac{\sqrt{6-\chi}}{\sqrt{6+\chi}}$, respectively, if $\chi^2 < 6$ or $\chi^2 > 6$. Thus the conformally invariant field χ is nothing but yet another conformal image of the self-gravitating massless linear scalar ϕ . In general, a conformal map $\tilde{g}_{\mu\nu} = F(\phi)g_{\mu\nu}$ with arbitrary nonvanishing F transforms the Einstein-Hilbert Lagrangian for the scalar ϕ minimally coupled to $\tilde{g}_{\mu\nu}$ into

$$L(g, \phi) = \sqrt{-g} \left[F(\phi)R - \left(F - \frac{3}{2F}F'^2 \right) g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} \right], \quad (2.34)$$

which can be transformed by further conformal maps and field redefinitions into any version of STG theories (Alonso *et al.* in Ref. [11]). The interrelations between the various gravity theories are depicted in Fig. 1.

We conclude this section by emphasizing that the gravitational field equations for NLG in the Jordan conformal frame, (2.2) and (2.9), differ from each other in the matter part, and similarly Eqs. (2.3) and (2.8) in the Einstein frame have different matter source term. Thus, it is clear that matter dynamics depends on whether the physical metric is the one forming either the Jordan or the Einstein (or in any other) frame. Transforming from the assumed physical frame to the other (unphysical) one results in a number of bizarre terms depending on ordinary matter and/or the spin-0 gravity; nevertheless, in each case one finds consistent conservation laws. In particular, we stress that the divergence of the total energy-momentum tensor (the sum of all the terms in the gravitational field equations depending on fields different from the metric one) does not provide a criterion for establishing which metric is physically acceptable. Accordingly, in the next section we will revert to a vacuum NLG theory (no ordinary matter), and study the distinguished role played by energy in gravitational physics; this will provide a motivation for regarding the Jordan metric as unphysical.

III. FIELD REDEFINITIONS, PHYSICAL VARIABLES, AND POSITIVITY OF ENERGY

Having shown that, from the formal viewpoint, NLG and STG theories can be formulated in either of the two frames without giving rise to inconsistencies, and having

⁹ Actually one can also consider multiscalar-tensor theories of gravity; see [1].

¹⁰ The field equation $\square\chi - \frac{1}{6}R\chi = 0$ is invariant under an arbitrary conformal map $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$ associated with $\chi \mapsto \Omega^{-1}\chi$ (i.e., χ is scaled as a particle mass).

¹¹ Only recently has appeared a work [29] applying the theorem.

¹² Madsen [30] makes an incorrect statement on the subject.

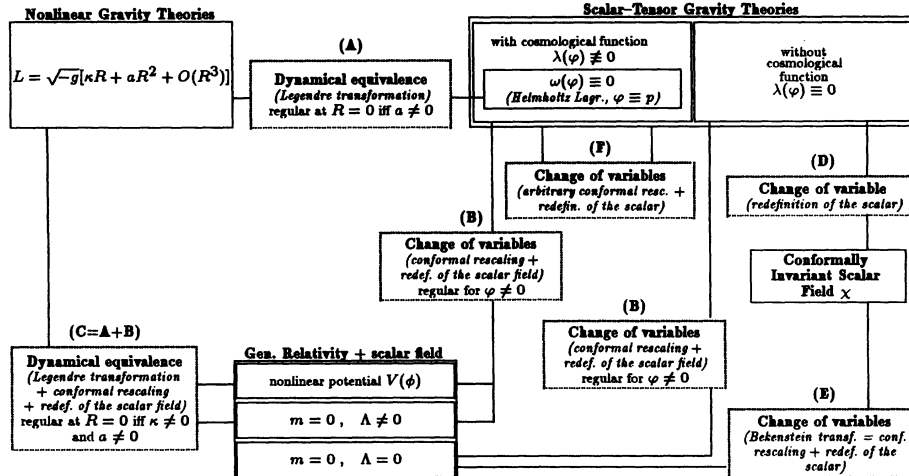


FIG. 1. Relations among NLG theories [the Lagrangian (1.1) is expanded in power series around $R = 0$], STG theories and general relativity. Connection (A) is described in Appendix B, (B1) and (B2); (B) is represented by (2.19) and (2.20); (C) is outlined in Sec. I, (1.3) and (1.6); (D) and (E) are discussed at the end of Sec. II; (F) represents mappings between STG theories with different $\omega(\varphi)$ [and different $\lambda(\varphi)$, if any], being a combination of an arbitrary conformal rescaling and a suitable field redefinition.

clarified the theoretical implications of the choice of the physical metric, let us revert to our original question: can one choose the physical metric on arbitrary grounds?

Brans' response [18] is negative: JBD theory is *not* "nothing but" general relativity plus a scalar, and the metric tensor in the Jordan conformal frame has a direct operational meaning—test particles move on geodesic world lines in this geometry. This is, however, a free assumption, and we have seen in Sec. II that there is *no formal* method that could determine which frame is physical. The problem is further obscured by the fact that there is no experimental evidence on interaction of the scalar with known matter. It might, therefore, seem (and implicit suggestions are sometimes heard) that a self-gravitating scalar field can be arbitrarily coupled to the space-time metric. Presumably this is the origin of the view that physics cannot distinguish between conformal frames [16], the mere fact that the conformal mapping does not affect the particle mass ratios being clearly insufficient for proving it.

Let us make the terminology more precise. Different formulations of a theory in different variables (frames) will be referred to as various versions of the same theory. This includes not only mere transformations of variables, but also transitions to *dynamically equivalent* frames. A theory (expressed in any version) is *physical* if there exists a maximally symmetric ground-state solution that is classically stable. Classical stability means that the ground-state solution is stable against small oscillations—there are no growing perturbation modes with imaginary frequencies. A viable physical theory can be semiclassically unstable: the ground-state solution is separated by a finite barrier from a more stable (i.e., lower-energy) state and can decay into it by semiclassical barrier penetration [31]. The ground-state solution may not exist, e.g., in Liouville field theory [32], but in gravitational physics the existence of the ground-state

solution (Minkowski or de Sitter space) hardly needs justification.

In most versions of a physical theory it is difficult to establish whether the ground-state solution is stable or not and to extract its physical contents. Field variables (i.e., frames) are *physical* if they are operationally measurable and if field fluctuations around the ground-state solution, expressed in terms of these variables, have *positive* energy density. Since the energy density cannot be defined for the metric field, the definition is directly applied to all other fields; the metric tensor influences the energy density of any matter and thereby the physical metric is indirectly determined.

Energy density is sensitive to transformations of variables, particularly to conformal mappings. In terms of unphysical variables the energy density is indefinite and although the total energy is formally conserved, it loses most of its practical use. The ground-state solution has (by definition) a total energy equal to zero, and when the theory is formulated in the physical variables (the "physical version") the solution represents the minimum of energy. Thus, stability is closely related to positivity of energy and instead of searching for growing perturbation modes one can study the total energy for nearby solutions [33].

These definitions apply to a relativistic classical field theory (and not to Newton gravity) and are satisfied by all known unquantized matter. "For reasons of stability we expect all reasonable (though not quantum!) field theories to have positive energy density, and we expect all (classical and quantum) field theories to have positive mass-energy" [34]. The weak energy condition is violated in some quantum states [35], while for all unquantized matter the dominant energy condition holds [36]. Whenever the condition is violated one obtains physically meaningless results [37].

To show how this postulate works, consider a "ghost"

complex scalar field in Minkowski space, minimally coupled to electromagnetism:

$$L = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + D_\mu \psi (D^\mu \psi)^* , \quad (3.1)$$

where $D_\mu \psi \equiv \partial_\mu \psi + ieA_\mu \psi$. Mathematically the theory is acceptable in the sense that the Cauchy problem for the field equations

$$\partial_\nu F^{\mu\nu} - 8\pi \text{Re}(ie\psi^* D^\mu \psi) = 0 , \quad D_\mu D^\mu \psi = 0 \quad (3.2)$$

is well posed. Physically, however, the theory is untenable, since the full energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) - \frac{1}{2} [D_\mu \psi (D_\nu \psi)^* + D_\nu \psi (D_\mu \psi)^* - g_{\mu\nu} D_\alpha \psi (D^\alpha \psi)^*] \quad (3.3)$$

is indefinite. The candidate ground-state solution, $\psi = F_{\mu\nu} = 0$, has total energy $E = 0$ and is unstable, since any other solution with $F_{\mu\nu} = 0$ and $\text{Im}(\psi) = 0$ has negative energy. The “ground state” decays via a self-excitation process, where energy is pumped out from the scalar to the electromagnetic field and radiated away to infinity. Such a system, which can emit an infinite amount of radiation (while total energy is conserved) is clearly unphysical.

We shall show that such effects do not occur for NLG theories if there is exact equivalence between the Jordan and Einstein frames. On the other hand, if the equivalence breaks down for some Lagrangians or solutions, the ground-state solution (Minkowski space) is likely to be unstable. Before doing so, a few remarks on the problem of the physical (in)equivalence of the frames and the notion of energy are in order.

Let us first recall the case of classical Hamiltonian mechanics. In phase space, particle positions and momenta do not form a privileged system of coordinates (any choice of canonical coordinates is equivalent, from the mathematical viewpoint), and the distinction between positions and momenta is usually lost after a canonical transformation (e.g., $P_i = q^i$ and $Q^i = -p_i$). On the other hand, to construct operationally $H(q, p)$ for a given mechanical system one usually discriminates positions and momenta, and kinetic and potential energy. Once H has been constructed in terms of physical variables, one has the freedom of making arbitrary canonical transformations. If the Hamiltonian is *not* empirically determined, e.g., in the case of abstract systems defined by the action of a symmetry group on a Poisson manifold, one should introduce some *additional* criterion to single out the total energy among all conserved quantities. For instance, one may require that the energy be bounded from below and depend quadratically on half of the coordinates. Whenever this criterion is effective, it allows us not only to identify the total energy but also to determine which variables play the physical role of momenta. Whether or not the latter are uniquely determined depends upon the system under question.

To determine the physical variables and the Hamiltonian (or Lagrangian) for the system, it is, in general,

insufficient to study the system alone: one should take into account its actual and possible interactions with its surroundings. The greater variety of interactions, the greater confidence that the dynamical variables describing the system, and its Hamiltonian are correctly defined. One usually pays less attention to this aspect, since in theoretical physics the physical variables are already given from empirical data and form a starting point for theoretical consideration: one is then interested in finding out the largest group of transformations for a given system, rather than in restricting the class of allowable frames.

The very possibility that the system can interact with external agents means that the theory describing it is “open,” in the sense that the surrounding is not included in the Lagrangian. On the contrary, in a “closed” theory the system constitutes the whole universe, and no external agent can make an experiment on it; in this situation, any set of variables describing the system is equally physical and mathematical equivalence of frames means the physical one [19]. In a closed theory, in fact, energy is merely a first integral of motion without the distinguished features it has in an open theory. Each accepted physical theory is open in this sense (there are some implicit trends in quantum gravity to view it as a closed theory [38]) and for instance in classical electrodynamics and quantum mechanics one has no doubts which variables have direct physical meaning and which are merely a convenient mathematical tool for solving a particular problem.

In the case of gravity, in order to identify the physical variables and to formulate a physical version of the theory, one should experimentally study gravitational interactions of various forms of matter: the motion of light and of charges and neutral test particles, behavior of clocks and rigid rods, etc. The point is that the presently available empirical data are *too scarce* to this aim.

A theoretical criterion for pure gravity or for a system consisting of gravity and a scalar field is provided by energy, owing to its unique status in theories of gravity. In no other theories is energy effectively a charge. In any theory of gravity (including string-generated ones) the ADM energy should provide a good notion of energy [39] and the positive-energy theorem (see, e.g., [40]) should hold.

For a NLG theory one should compare energetics in the Jordan and Einstein frames. There is no generally accepted definition of gravitational energy for a higher-derivative theory, and in search for the physical metric one should not compare (as is usually done) the fourth-order version of the theory (1.2) with the second-order one. It turns out that the Einstein frame does *not* provide the *unique* second-order version of the theory, and it was shown in [41] that any NLG theory (as well as theories with Lagrangians depending on Ricci and Weyl curvatures) can be recast in a version revealing a *formal* equivalence with general relativity without changing the metric. [Yet the conformal rescaling of the metric is necessary to have a version of the theory with Einstein-Hilbert Lagrangian (1.6).] This is accomplished with the aid of a Helmholtz Lagrangian [41,42] by applying a Legendre

transformation. The Helmholtz Lagrangian contains the original unrescaled metric (the Jordan frame), and to get second-order equations of motion it is necessary to introduce a new independent field variable, a scalar field p . Thus, the field variables are now $(g_{\mu\nu}, p)$, and to distinguish the frame from the original Jordan frame (consisting of $g_{\mu\nu}$ alone) it is referred to as Helmholtz Jordan conformal frame (HJCF).

It should be stressed that the procedure is not an *ad hoc* trick and the scalar is (up to possible redefinitions) a canonical momentum conjugated to the metric $g_{\mu\nu}$, and represents an additional degree of freedom existing already in a NLG theory, in comparison to (vacuum) general relativity. The HJCF exists if $f''(R) \neq 0$ (the R -regularity condition), i.e., if the theory is a truly nonlinear one.

The formalism [41] is outlined in Appendix B. The Helmholtz Lagrangian, which is dynamically equivalent to (1.1), is

$$L_H = p[R(g) - r(p)]\sqrt{-g} + f[r(p)]\sqrt{-g} \quad (3.4)$$

where, as previously, $r(p)$ is a solution of the equation $f'[r(p)] = p$. The resulting field equations (B3), after some manipulations, take on the form

$$\begin{aligned} G_{\mu\nu} &= p^{-1}\nabla_\mu\nabla_\nu p - \frac{1}{6}\{p^{-1}f[r(p)] + r(p)\}g_{\mu\nu} \\ &\equiv \theta_{\mu\nu}(g, p) \end{aligned} \quad (3.5)$$

and

$$\square p = \frac{2}{3}f[r(p)] - \frac{1}{3}p \cdot r(p); \quad (3.6)$$

these are equivalent to (2.14) and (2.15) in the absence of matter. Equations (3.5) have the form of Einstein field equations with the effective energy-momentum tensor $\theta_{\mu\nu}$ of the scalar as a source.¹³ Although the scalar is a new independent variable, the number of degrees of freedom remains unchanged: the Lagrangian (1.1) describes a system with three degrees of freedom [22,23], while it is obvious from the field equations (3.5) and (3.6) that L_H represents two degrees of freedom for $g_{\mu\nu}$ and one for p .

Assuming that the Lagrangian $f(R)\sqrt{-g}$ does not contain the cosmological term, $f(0) = 0$, one finds that Minkowski space is a possible candidate ground-state solution to¹⁴ (1.2). Then $g_{\mu\nu} = \eta_{\mu\nu}$ and $p = p_o$ with $r(p_o) = R(\eta) = 0$ are the solution in the HJCF. Is this solution stable? To assess it one considers solutions $(g_{\mu\nu}, p)$ to (3.5) and (3.6), which approach $(\eta_{\mu\nu}, p_o)$ at spatial infinity at sufficient rate. Then the total energy

of these solutions is given by the ADM surface integral at spatial infinity and the candidate ground-state solution is stable (classically and semiclassically) if $E_{ADM} \geq 0$ and vanishes only for $(\eta_{\mu\nu}, p_o)$. In general relativity the positive-energy theorem holds provided the right-hand side of Einstein equations satisfies the dominant energy condition for *any* source of gravity. The effective stress-energy tensor $\theta_{\mu\nu}$ for the spin-0 gravity does not satisfy the condition. The kinetic part of $\theta_{\mu\nu}$ is quite bizarre: it contains second-order derivatives and is a homogeneous function of order zero in the field. The latter peculiarity can be removed by a field redefinition, while the former cannot. In fact, setting $p = F(\varphi)$, with $F' > 0$, one has $\nabla_\mu\nabla_\nu p = F'\nabla_\mu\nabla_\nu\varphi + F''\partial_\mu\varphi\partial_\nu\varphi$ and the term survives for any F . By setting $p = e^\varphi$ the effective stress tensor becomes

$$\begin{aligned} \theta_{\mu\nu}(g, \varphi) &= \nabla_\mu\nabla_\nu\varphi + \partial_\mu\varphi\partial_\nu\varphi \\ &- \frac{1}{6}\{e^{-\varphi}f[r(e^\varphi)] + r(e^\varphi)\}g_{\mu\nu}. \end{aligned} \quad (3.7)$$

The presence of a linear term in the effective energy-momentum tensor for any long-range field is undesired because it causes difficulties in determining the total energy. The general procedure to couple a field to gravity consists in regarding it at first as a test field on a fixed space-time background (which we choose, for simplicity, to be Ricci flat), whereby the energy-momentum tensor can be gauged by any identically conserved tensor $\sigma_{\mu\nu}$. In the case of the scalar φ , the additional term $\sigma_{\mu\nu} = \nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\square\varphi$ influences the total energy as measured at infinity. To avoid such ambiguity it is generally accepted that the energy-momentum tensor for any field should be quadratic in the field derivatives and contain no linear terms. In the case of Eq. (3.5), this argument is not crucial because $\theta_{\mu\nu}$ is uniquely determined by the original nonlinear Lagrangian and no ambiguity can arise. A definitive argument to regard the linear term in $\theta_{\mu\nu}$ as unphysical is provided by the fact that due to its presence the energy density can be of either sign and be transported faster than light. The appearance of such linear terms signals that either the theory is unphysical at all (as is the case of Einstein-Gauss-Bonnet theory with compactified extra dimensions [43]) or merely that the field variables are unphysical and one should transform to another frame. The same term also flaws the energy-momentum tensors (both variational and effective ones) for the Brans-Dicke scalar and the conformally invariant scalar field in the Jordan frame.

In terms of the HJCF field variables one cannot prove that solutions near the flat space have positive energy; on the contrary, the indefiniteness of $\theta_{\mu\nu}$ deceptively suggests that negative-energy solutions exist and Minkowski space is classically unstable. To establish whether it is actually stable or not one makes the transformation to the Einstein frame. While the HJCF exists under the R -regularity condition $f''(R) \neq 0$, the Einstein frame exists if the further condition $f'(R) > 0$ is satisfied at all space-

¹³The fact that an effective matter source arises from the nonlinear part of the Lagrangian was already noticed and applied in [48] in the case of a generic quadratic Lagrangian.

¹⁴We do not consider in this paper other possible ground states (for $\Lambda \neq 0$), such as de Sitter and anti-de Sitter spaces, although our arguments can be suitably extended to deal with them.

time points for a given solution.¹⁵ Since one is interested in solutions which are asymptotically flat at infinity and do not differ too much (in the sense given below) from the flat solution in the interior, one requires $f'(R) > 0$ and $f''(R) \neq 0$ for $R \rightarrow 0$. These conditions hold for¹⁶

$$f(R) = R + aR^2 + \sum_{k=3} c_k R^k \quad \text{with } a \neq 0. \quad (3.8)$$

The existence of the first two terms in the expansion, $R + aR^2$, is of crucial importance for the equivalence of the Jordan and Einstein frames. These ensure the invertibility of the Legendre transformation and yield $p \approx 1$ in the vicinity of flat space. The vicinity consists of all space-times for which R is close to zero; these include spaces of arbitrarily large Riemann curvature if, e.g., $R_{\mu\nu} = 0$. The quadratic Lagrangian therefore carries most features of any NLG theory for which the equivalence holds. On the other hand, if (3.8) does not hold, the background solution, with respect to which the ADM energy is defined, has no counterpart in the Einstein frame, and therefore there is no hope that information on the total energy can be obtained by ordinary general-relativistic techniques.

Let us assume that (3.8) holds. Consider now a spacelike three-surface Σ embedded in an asymptotically flat spacetime $(M, g_{\mu\nu})$. The asymptotic flatness in the HJCF is defined as in general relativity, since the field equations are of second order; the weakest assumptions [44] are (in obvious notation)

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1/2-\epsilon}), \quad g_{\mu\nu,\alpha} = O(r^{-3/2-\epsilon}), \quad \epsilon > 0. \quad (3.9)$$

Then the leading-order contribution to R is $O(r^{-5/2-\epsilon})$. The HJCF dynamics implies $p = f'(R)$ and the Lagrangian (3.8) yields the asymptotic behavior of the scalar field:

$$p = 1 + O(R) = 1 + O(r^{-5/2-\epsilon}). \quad (3.10)$$

Under these conditions one now proves, contrary to what might be expected from the properties of $\theta_{\mu\nu}$, that the total energy is positive for a NLG theory.

Positive-energy theorem for NLG theories. Let Σ be an asymptotically flat, nonsingular spacelike hypersurface in a spacetime $(M, g_{\mu\nu})$ topologically equivalent to \mathbb{R}^4 . If (i) the Lagrangian is given by (3.8), (ii) $p > 0$ and $f''(R) \neq 0$ everywhere on Σ , (iii) a solution $(g_{\mu\nu}, p)$ to (3.5) and (3.6) satisfies the condition (3.9) [and hence (3.10)] on Σ , and (iv) the potential V (1.7) for the scalar

ϕ in the Einstein frame is non-negative, $V(\phi) \geq 0$, on Σ , then the ADM energy in the Jordan frame is non-negative:

$$E_{\text{ADM}}[g] = \frac{1}{2} \int_{S_{\infty}^2} dS_i (g_{ij,j} - g_{jj,i}) \geq 0. \quad (3.11)$$

Proof. The proof is a direct extension of that given by Strominger [20] for $f = R + a^2 R^2$. Consider the total energy of the conformally related solution $(\tilde{g}_{\mu\nu}, \phi)$ in the Einstein frame. Because of the falloff rate in the HJCF, the metric $\tilde{g}_{\mu\nu} = p g_{\mu\nu}$ is asymptotically flat and the total energy of the solution is finite and given by the ADM integral over a boundary two-surface S_{∞}^2 at infinity, $E_{\text{ADM}}[\tilde{g}] = \frac{1}{2} \int_{S_{\infty}^2} dS_i (\tilde{g}_{ij,j} - \tilde{g}_{jj,i})$. If $V \geq 0$ the dominant energy condition holds and this energy is nonnegative [40]. Now, replacing $\tilde{g}_{\mu\nu}$ by $g_{\mu\nu}$ and applying (3.10) one easily finds that $E_{\text{ADM}}[\tilde{g}] = E_{\text{ADM}}[g]$ and hence that $E_{\text{ADM}}[g] \geq 0$.

There is a subtle conceptual difference between our proof and Strominger's one: the use of the ADM integral for the total energy is not based on the fact that at large distances the dynamics is governed by the lower-derivative terms in (1.2), but rather on the existence of the HJCF. In the latter the integral is defined as in general relativity.

It should be stressed that the global equivalence between the Jordan and Einstein conformal frames holds if $p > 0$ and $f''(R) \neq 0$ everywhere on $(M, g_{\mu\nu})$, while to prove the theorem one needs only to assume that these conditions hold on the initial-data surface Σ , what implies that the frames are equivalent in some neighborhood of Σ . Consider for example the collapse of a cloud of dust. The exact relation between R and the dust energy density ρ depends on the form of the field equations (which metric is physical, Sec. II); in general one expects that R grows when ρ does. On the initial surface Σ the dust is diluted and $R \approx 0$, thereby the frames are equivalent in the vicinity of Σ . Near the singularity ρ is divergent and R is unbounded, p may change sign, and thus in this region of the manifold there may be no mapping to the Einstein frame. Yet the energy, being conserved, is still given by its value on Σ .

In the class of solutions for which the two conformal frames are at least locally equivalent, Minkowski space is the unique one having zero energy. In fact, let M contain a spacelike surface Σ on which the assumptions of the theorem hold. Then, they hold in some neighborhood $\mathcal{U}(\Sigma)$ and the space-time region $(\mathcal{U}(\Sigma), g_{\mu\nu})$ can be mapped onto $(\mathcal{U}(\Sigma), \tilde{g}_{\mu\nu})$. Let $E_{\text{ADM}}[g] = 0$ when evaluated on Σ ; then $\tilde{g}_{\mu\nu}$ is flat in $\mathcal{U}(\Sigma)$ (Minkowski space-time is the unique zero-energy solution in general relativity) and the solution $(\tilde{g}_{\mu\nu} = \eta_{\mu\nu}, \phi = 0)$ is identically mapped back onto $(g_{\mu\nu} = \eta_{\mu\nu}, p = 1)$ in $\mathcal{U}(\Sigma)$. Then the space-time $(M, g_{\mu\nu})$ is flat in the open region $\mathcal{U}(\Sigma)$ and this solution can be analytically extended onto the entire manifold M , thereby the space-time actually is Minkowski space (\mathbb{R}^4 topology is always assumed).

Thus the total energy is positive in the Jordan frame for all solutions to (1.2) containing an open region $\mathcal{U}(\Sigma)$ where these are close to Minkowski space (R close to zero)

¹⁵Whether the Einstein frame can be defined globally or only locally is a delicate problem, which deeply affects the structure of global solutions of NLG models. A detailed discussion of the problem will be given in a forthcoming review article on metric theories of gravity.

¹⁶For technical reasons, we restrict ourselves to the analytic case.

and where $V(\phi) \geq 0$. The flat solution is classically stable against decay into these solutions and *is* a ground-state solution for a given NLG theory.

Now we consider a few examples.

(1) $f(R) = R + aR^2$. The Lagrangian is R regular, since $f'' = 2a \neq 0$ everywhere. If $a > 0$ (this is the case studied by Strominger [20]), using the variable p instead of ϕ one has $V(p) = \frac{1}{8ap^2}(p-1)^2 \geq 0$ everywhere, while $p > 0$ for $R > -\frac{1}{2a}$. For all solutions in the Jordan frame such that there exists a spacelike asymptotically flat surface Σ with $R > -\frac{1}{2a}$ on it, the total energy is non-negative and flat space is classically stable. Strominger has also shown that Minkowski space is the unique solution in the Jordan frame with vanishing energy. If $a < 0$, the conformal equivalence holds in the region $R < |\frac{1}{2a}|$, but $V(p) \leq 0$ everywhere and the theorem does not hold. One may expect that in the vicinity of flat space there are solutions with negative energy and Minkowski space is classically unstable.

(2) $f(R) = \frac{1}{a}(e^{aR} - 1)$. This is again a R -regular Lagrangian. Furthermore, the Jordan and Einstein frames are globally equivalent since $p = e^{aR} > 0$ for any solution. The potential $V(p) = \frac{1}{2ap}(\ln p + \frac{1}{p} - 1)$ is positive for $a > 0$ and negative for $a < 0$ [$V(1) = 0$]. Thus, for $a > 0$, all asymptotically flat solutions have positive total energy and flat space is stable against *any* perturbations, small or large (classical and semiclassical stability). On the contrary, for $a < 0$ the dominant energy condition does not hold, what signals existence of negative-energy states and classical instability of flat space.

(3) $f(R) = \frac{1}{a} \ln(1 + aR)$. In this case one has $f''(R) = -\frac{a}{(1+aR)^2} \neq 0$ for $R \neq -\frac{1}{a}$, and $V(p) = \frac{1}{2ap^2}(\ln p - p + 1)$. For $a > 0$ one finds $V(p) < 0$ for $p \neq 1$ and flat space can classically decay into negative-energy states. For $a < 0$ the conformal equivalence holds for $R < -\frac{1}{a}$ and $V(p) > 0$ for all $p \neq 1$, thus for these solutions the total energy is positive and flat space is classically stable.

In all these examples states close to Minkowski space have positive energy if the coefficient of the R^2 term in the Taylor expansion of f is positive. This is a generic feature of Lagrangians (3.8). For R close to zero the two frames are equivalent and the theorem holds if the potential is positive. For vacuum the relation $r(p) = R(g)$ holds; using it in the expression for $V(p)$, given after equation (2.14), and expanding the potential around $R = 0$ one arrives at $V = \frac{1}{2}aR^2 + O(R^3)$ for solutions. All $R = 0$ solutions represent a local extremum of the potential. Hence, energetics and stability of flat space are determined by the R^2 term in the Lagrangian of a NLG theory.

The lowest-order contribution $R + aR^2$ to the full Lagrangian is crucial for *classical* stability of Minkowski space. Whenever these terms are absent, e.g., for $f = R^k$, $k > 2$, the conformal equivalence with the Einstein frame version is broken at flat space. In this case as well as when the Einstein frame is defined for $R = 0$ but the potential is negative on Σ , the classical decay of flat space may occur because the positive energy theorem at null infinity (see [40]) does not hold. While the ADM energy is conserved, there may be metric perturbations about the

background such that gravitational waves carry away unbound amounts of energy and the Bondi-Sachs mass remaining in the system decreases to minus infinity.

Whenever the ADM mass is positive for solutions close to flat space (i.e., if $a > 0$), the latter need not be *semiclassically* stable. There may exist solutions (in the Jordan frame) with negative energy, which are far from Minkowski space. These are space-times so highly curved (large $|R|$) in the interior, that either the equivalence with the Einstein frame is violated (p changes sign on each asymptotically flat Σ) or the potential becomes negative in a region of any Σ . The occurrence of any of the two possibilities depends on the form of the Lagrangian, e.g., for the exponential one [example (2) above] there are no such solutions, while for a cubic Lagrangian, $R + aR^2 - b^2R^3$, the potential is negative for large p and some values of a . Semiclassical instability of standard ground-state solutions is a generic feature of higher-dimensional theories: it was found both in Kaluza-Klein theory in five [31,45] and in ten [46] dimensions, as well as in string theory [46]. Although disturbing in itself, it is not dangerous. Classical instability is a more severe problem and usually makes the theory untenable. If there is another possible ground state, which is stable; however, it is harmless, as in the case of the uncompactified Einstein-Gauss-Bonnet theory, where d -dimensional anti-de Sitter space is unstable, while Minkowski space is stable [47]. Minkowski space is classically unstable in a gravity theory with Lagrangian $aR + bR^2 + cC_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ [48] and in a four-dimensional theory resulting from Einstein-Gauss-Bonnet theory with compactified extra dimensions [43] (in all these cases there is no *conformal* mapping of the theory onto a model in general relativity).

Equality of the total energy in the two conformal frames is due to the fact that E_{ADM} , being effectively a charge, is evaluated at spatial infinity (where $p \rightarrow 1$) and is rather loosely related to the interior. The only detailed information about the interior of a gravitating system that is needed is whether all local matter energy-flow vectors are timelike or null. This connection is lost in the Jordan frame. Also the Bondi-Sachs mass, when expressed in terms of the Jordan frame, does not satisfy the correct evolution equations, as is mentioned in [1]. These flaws do not reflect the genuine properties of spin-0 and spin-2 gravity, these are merely due to an improper choice of field variables. The effective stress tensor $\theta_{\mu\nu}$ (3.7) does *not* provide the true physical energy momentum of the scalar field. As any redefinition of the scalar cannot help, it is the metric variable, which needs redefinition.

Accordingly, the Jordan frame should be regarded as unphysical not by an arbitrary choice of definition but because in the second-order version of the theory, whereby the spin-0 and spin-2 component of gravity are singled out, these variables are unrelated to the total energy of the system. On the other hand, the Einstein frame satisfies all general requirements of relativistic field theories. Therefore, it is the physical frame, and $\tilde{g}_{\mu\nu}$ determines the space-time intervals in the real world. It provides the physical version of a NLG theory. All other conformally related versions of the theory, including the initial one,

are unphysical in some aspects.¹⁷ This physical frame is *uniquely* determined by the Lagrangian (1.1) and the physical metric for the vacuum theory is $\tilde{g}_{\mu\nu} = f'(R)g_{\mu\nu}$. Physically, this transformation means a transition to the frame where all fields, including spin-0 gravity, satisfy the dominant energy condition. Then it is possible to establish that Minkowski space is a classically stable local minimum of the energy. This holds true, however, if the potential of the scalar field does not attain a local maximum at flat space, which would destabilize the latter. The only remedy is to exclude as unphysical all Lagrangians giving rise to negative potentials near flat space, and this amounts to the requirement $a > 0$ in (3.8).

IV. CONCLUSIONS

Our analysis of the physical features of energy in NLG theories leads to the conclusion that the Einstein-frame metric should be regarded as the physical one. In this perspective, whenever a nonlinear Lagrangian L admits flat space as a stable ground-state solution, it is *physically* equivalent to a scalar field with a potential V (determined by L) minimally coupled to the rescaled metric $\tilde{g}_{\mu\nu}$ in general relativity. Then, any form of matter should minimally couple to $\tilde{g}_{\mu\nu}$ and not to the scalar. The strong equivalence principle is then retained and the scalar, which can (though not necessarily) be viewed as a spin-0 component of universal gravity, appears only as a contribution to the full source in the Einstein field equations.

Scalar fields are now very popular in cosmology. Instead of introducing *ad hoc* a special form of the scalar field potential in order to solve a particular problem, one can start from some nonlinear Lagrangian, which provides the desired potential, provided that one can give a deeper motivation for the former. Our conclusion does not mean that any viable nonlinear gravity theory is identical with general relativity. Although the scalar field has no impact on motion of ordinary matter (in particular, free test particles move on geodesic lines), the space of solutions is different. For instance, for nonconstant scalar field there are no black holes with regular event horizons since a black hole cannot have a scalar hair [49].

It is amusing to notice that 20 years ago Bicknell [2] concluded his study of the phenomenology of purely quadratic theories with the words: “These results eliminate gravitational theories based on quadratic Lagrangians from the realm of viable gravitational theories and point strongly towards the uniqueness of the Einstein equations.” Our results regarding the Lagrangians giving rise to potentials which attain a maximum for flat space ($a < 0$) suggest the same conclusion. However, a rigorous proof of the instability of flat space in that

case is not yet available. The case of Lagrangians for which a conformal mapping onto the Einstein frame does not exist at Minkowski space (e.g., R^3) is more subtle. The presently known techniques of investigating energy and stability fail there, and it remains an open problem whether positivity of energy can be shown for a subclass of these Lagrangians. Another open problem is how to deal with Lagrangians which are not of class C^3 at $R = 0$, e.g., $R + aR^2 + R^{7/3}$.

Finally, it should be emphasized that our method of determining the physical metric for a gravity theory, by studying the energy and stability in the vacuum theory, does not apply when the interaction of matter with gravity is already prescribed by a more fundamental theory. This is the case, e.g., of string-generated gravity, where the string action in the four-dimensional field-theory limit contains a number of fundamental bosonic fields coupled to the metric fields and to the dilaton. The problem of identifying the physical metric appears there too [50] and should be dealt with in a separate way.

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APPENDIX A: MOTION OF DUST AND PHOTONS IN THE JORDAN AND EINSTEIN CONFORMAL FRAMES

We consider here the explicit forms of the conservation laws in the cases I and II for NLG theories and in the Einstein frame for STG theories. We show on explicit examples that the different versions of conservation laws reflect the known geometrical property that null geodesic world lines are preserved under conformal rescaling, while timelike geodesic paths in one frame correspond to non-geodesic paths in the other frame: this fact has a physical counterpart in the different coupling between matter and scalar field in the two frames. The consistency of the conservation laws described in Sec. II with the physical assumptions is thus ensured both in case I and in case II.

1. Case I: the Jordan frame is physical

In Einstein frame the conservation laws for matter are given by (2.8). Inserting the explicit form of the ϕ -field stress tensor $t_{\mu\nu}$ and making use of the equation of motion (2.6) one arrives at four equations

$$\tilde{g}^{\alpha\beta} \left[\tilde{\nabla}_\alpha T_{\mu\beta} + \frac{1}{\sqrt{6}} T_{\alpha\beta} \phi_{,\mu} - \sqrt{\frac{2}{3}} T_{\mu\beta} \phi_{,\alpha} \right] = 0. \quad (\text{A1})$$

¹⁷The need of making all calculations in the Einstein frame for STG theories is stressed in [1], [8], and [9], although these authors assume the Jordan frame as physical.

Clearly, as is seen from the Lagrangians (2.1) and (2.3), only conformally invariant matter does not interact with the scalar (for this matter $\tilde{g}^{\alpha\beta}T_{\alpha\beta} = 0$ and the equation of motion for ϕ becomes Ψ independent). This is the case of electromagnetic field, hence photons travel along null geodesic lines both in the physical and in the rescaled geometries. We show that this result also follows from the conservation laws (A1).

Consider the null electromagnetic field, which is the curved-space classical representation of the photon,

$$F_{\mu\nu} = k_\mu q_\nu - k_\nu q_\mu, \quad \text{with} \quad k^\mu k_\mu = k^\mu q_\mu = 0; \quad (\text{A2})$$

here, k_μ are the covariant components of the wave vector and q_μ is a spacelike covector determined up to $q_\mu \mapsto q_\mu + \lambda k_\mu$. Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is metric independent, the components k_μ and q_μ are independent of the conformal rescaling of the metric. Introducing in Jordan conformal frame a scalar $w = \frac{1}{4\pi} g^{\alpha\beta} q_\alpha q_\beta > 0$ one finds that the stress tensor for the null field is given in this frame by $T_{\mu\nu}(F, g) = w k_\mu k_\nu$. Then $\nabla_\nu T^{\mu\nu} = 0$ imply that $k^\nu \nabla_\nu k^\mu \propto k^\mu$, the wave vector is tangent to null geodesic curves.

In transforming to the Einstein frame, $g_{\mu\nu} \mapsto p g_{\mu\nu}$, $k_\mu \mapsto k_\mu$ and $q_\mu \mapsto q_\mu$, thus $T_{\mu\nu}(F, p^{-1}\tilde{g}) = p\tilde{w} k_\mu k_\nu$, with $\tilde{w} = \frac{1}{4\pi} \tilde{g}^{\alpha\beta} q_\alpha q_\beta$. Equations (A1) for this stress-energy tensor reduce to

$$k^\nu \tilde{\nabla}_\nu k^\mu = \left[\sqrt{\frac{2}{3}} \phi_{,\nu} k^\nu - \tilde{\nabla}_\nu k^\nu - k^\nu \partial_\nu \ln(p\tilde{w}) \right] k^\mu$$

and the wave vector remains tangent to null geodesics but the parameter σ defined by $k^\nu = \frac{dx^\nu}{d\sigma}$ is not the affine parameter along them.

On the other hand, massive particles do interact with the ϕ field. Consider a self-gravitating dust. In Jordan frame its stress tensor is $T_{\mu\nu} = \rho u_\mu u_\nu$ and $\nabla_\nu T^{\mu\nu} = 0$ gives rise to motion along timelike geodesics, $u^\nu \nabla_\nu u^\mu = 0$. Under the conformal rescaling one finds that $u_\mu = p^{\frac{1}{2}} \tilde{u}_\mu$ and $\rho = p^2 \tilde{\rho}$ [28], thus the dust stress tensor defined in the Jordan frame and expressed in terms of the Einstein frame is $T_{\mu\nu} = p \tilde{\rho} \tilde{u}_\mu \tilde{u}_\nu$. While in the Jordan frame the density current is conserved, $\nabla_\nu(\rho u^\nu) = 0$, in Einstein frame the corresponding density current is coupled to the scalar, $\tilde{\nabla}_\nu(\tilde{\rho} \tilde{u}^\nu) = -\frac{1}{\sqrt{6}} \tilde{\rho} \tilde{u}^\nu \phi_{,\nu}$. Using this fact in the conservation laws (A.1) one arrives at the following equations of motion for dust in $\tilde{g}_{\mu\nu}$ metric:

$$\tilde{u}^\nu \tilde{\nabla}_\nu \tilde{u}^\mu = \frac{1}{\sqrt{6}} (\phi^{,\mu} + \phi_{,\nu} \tilde{u}^\mu \tilde{u}^\nu). \quad (\text{A3})$$

In particular, the dust worldlines are geodesic for $\tilde{g}_{\mu\nu}$ only if $\phi_{,\mu} = 0$.

2. Case II: the Einstein frame is physical

Consider a pressureless dust either being a source in the field equations (2.10) or just a cloud of test particles. In the Einstein frame its energy-momentum tensor is divergenceless, and the particles follow timelike geodesic curves. In MJCF the interaction with gravity becomes

highly complicated [Eq. (B19) in Appendix B], and the worldlines are not geodesic.

Massless noninteracting particles move along null geodesic paths in the physical metric $\tilde{g}_{\mu\nu}$ and this property is preserved under any conformal rescaling. This fact can be also explicitly derived from the matter conservation laws valid in MJCF. For conformally invariant matter¹⁸ the field equations (B19) and the generalized Bianchi identity gives rise to the four conservation laws

$$\nabla^\nu [f'(R)T_{\mu\nu}(\Psi, f'(R)g_{\alpha\beta})] = 0. \quad (\text{A4})$$

Thus the proof essentially follows the same lines as in case I.

3. Scalar-tensor gravity theories

By assumption, in the Jordan frame dust particles follow timelike geodesic worldlines and photons follow the null ones. In the Einstein frame, we can relate the motion of matter to the conservation laws (2.31) by the same procedure used for NLG theories, case I; in fact, equation (A1) coincides with (2.31) with $\omega \equiv 0$. The calculations for the general case follow essentially the same lines, leading to the conclusion that photon worldlines are geodesic in both frames, while dust particles move along nongeodesic curves for the metric $\tilde{g}_{\mu\nu}$:

$$\tilde{u}^\nu \tilde{\nabla}_\nu \tilde{u}^\mu = \frac{1}{2} \gamma (\phi^{,\mu} + \phi_{,\nu} \tilde{u}^\mu \tilde{u}^\nu). \quad (\text{A5})$$

APPENDIX B: DIRECT AND INVERSE LEGENDRE TRANSFORMATIONS IN THE PRESENCE OF MATTER

The equivalence of any NLG theory (1.1) to general relativity plus the scalar field (1.6) is dynamical in the sense that the spaces of *classical solutions* for each theory are locally isomorphic, the isomorphism being given by (1.3). This is directly achieved by the conformal transformation of the field equations (1.2) [2,5], and is sufficient for various purposes. On the other hand, using this *ad hoc* procedure it is rather difficult to establish the particle contents of NLG theories [22,23] and to recognize the field ϕ as an independent degree of freedom present in the theory; instead one merely views ϕ as a function of the curvature scalar R of the original metric [6]. These goals can be easily achieved using a Lagrangian formalism. A Legendre transformation allows to transform the NLG Lagrangian (1.1) into a dynamically equivalent one, linear in the curvature scalar and including an auxiliary scalar field. This equivalent Lagrangian is named the ‘‘Helmholtz Lagrangian’’ by analogy with classical mechanics. In addition to being elegant and mathematically well grounded, the Legendre trans-

¹⁸In this case the MJCF coincides with the VJCF.

formation and the related Helmholtz Lagrangian turn out to be indispensable tools in dealing with NLG theories: one would hardly guess how matter fields with a prescribed coupling with the Einstein-frame metric interact with the Jordan-frame metric, without relying on the corresponding Lagrangians. As is mentioned in Sec. II, a naive approach leads to inconsistencies. In this appendix we outline of the general setting of the method [3,4,41] (we refer the reader to [51] for a rigorous mathematical exposition), and then present in detail how matter interaction terms should be inserted into the gravitational Lagrangians (both the original nonlinear one and the Helmholtz Lagrangian) when the matter fields are assumed to be minimally coupled to the Einstein-frame metric.

1. Helmholtz Lagrangian

For the nonlinear Lagrangian density (1.1) one introduces the *generalized conjugate momentum*

$$p = \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial R}. \quad (\text{B1})$$

The scalar field p is clearly nothing else but the scalar defined by (1.3). As in Sec. I, let $r(p)$ be a function such that $f'[R]|_{R=r(p)} \equiv p$; such a function exists if $f''(R) \neq 0$, and in this case we say that the nonlinear Lagrangian (1.1) is *R-regular*. The Helmholtz Lagrangian corresponding to (1.1) is then defined as

$$L_H = p[R(g) - r(p)]\sqrt{-g} + f[r(p)]\sqrt{-g}. \quad (\text{B2})$$

The action functional corresponding to (B2) is formally a degenerate case of the STG action (2.16), with $\omega \equiv 0$ and nontrivial cosmological function. A basic feature of the Helmholtz Lagrangian is that it does not contain derivatives of the scalar field p . The Euler-Lagrange equations obtained by varying the metric $g_{\mu\nu}$ and the scalar p *independently* in the action defined by (B2) are

$$R(g) = r(p), \quad (\text{B3a})$$

$$pG_{\mu\nu}(g) = \nabla_\mu \nabla_\nu p - g_{\mu\nu} \square p + \frac{1}{2} \left(f[r(p)] - p r(p) \right) g_{\mu\nu}, \quad (\text{B3b})$$

which are manifestly equivalent to (1.2).

What has been done is nothing but a generalization of a standard method of classical mechanics. Given a first-order Lagrangian $L = L(t, q^i, \dot{q}^i)$ for a system of point particles, the second-order Euler-Lagrange equations $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0$ can be recast into a first-order system, provided the Lagrangian is regular ($\det \left| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0$). This can be done in two ways: either one writes the Euler-Lagrange equations in normal form, $\ddot{q}^i = a^i(t, q^j, \dot{q}^j)$, then one introduces independent velocity variables u^i and writes the equivalent system in the velocity space

$$\dot{q}^i = u^i, \quad (\text{B4})$$

$$\dot{u}^i = a^i(t, q^j, u^j),$$

or one defines the Legendre map $(q^i, u^i) \mapsto (q^i, p_i = \frac{\partial L}{\partial \dot{q}^i})$ and finds the inverse map $u^i = u^i(q^j, p_j)$, which transforms the previous first-order system into the Hamilton equations in the phase space of the system. An interesting point, which is not always emphasized in textbooks, is that both first-order systems arise from a variational principle, defined by a degenerate first-order Lagrangian, the Helmholtz Lagrangian [42]

$$L_H = p_i (\dot{q}^i - u^i) + L(q^j, u^j),$$

which can be regarded either as a function over the velocity space, whereby (q^j, u^j) are the dynamical variables and $p_i = p_i(q^j, u^j)$ or as a function over the phase space, where the variables are (q^j, p_j) and $u^i = u^i(q^j, p_j)$ is the inverse of the Legendre map defined above. In fact, the solutions of (B4) are the extremal curves in the velocity space for the action functional

$$S_L = \int \{ p_i(q^j, u^j) [\dot{q}^i - u^i] + L(q^j, u^j) \} dt,$$

while the Hamilton equations can be derived from the action (in phase space)

$$S_H = \int \{ p_i [\dot{q}^i - u^i(q^j, p_j)] + L[q^j, u^j(q^k, p_k)] \} dt.$$

The Lagrangian (B2) plays the role of the Helmholtz Lagrangian in phase space for any *R-regular* NLG Lagrangian (1.1). As in particle mechanics, it is also possible to define a Helmholtz Lagrangian in the velocity space: this corresponds to regarding directly the “velocity” R as an independent field u (as is done, e.g., in [6], following [52] and without invoking the notion of Helmholtz Lagrangian), and introducing the Lagrangian

$$L_H = f'(u)[R(g) - u]\sqrt{-g} + f[u]\sqrt{-g}. \quad (\text{B2}')$$

The corresponding field equations are nothing but the system (B3), upon the substitution $p \rightarrow f'(u)$, $r(p) \rightarrow u$. For quadratic Lagrangians, which most frequently occur in the literature, the two fields u and p coincide up to a constant factor. More generally, since the *R-regularity* is still necessary to ensure the equivalence of (B2') and (1.1), the choice of either p or u to represent the independent scalar field occurring in the second-order picture is, from the mathematical viewpoint, a mere matter of convenience. The use of the scalar field u allows one to bypass the problem of finding explicitly the function $r(p)$, but then the gravitational equation contains a rather complicated dynamical term for the u field, $\nabla_\mu \nabla_\nu f'(u) - g_{\mu\nu} \square f'(u)$, which depends on the particular choice of $f(R)$ in (1.1). On the other hand, there is a *physical* motivation to formulate the theory in terms of scalar field ϕ , which can be equivalently defined either by $\phi \propto \ln p$ or by $\phi \propto \ln f'(u)$, since this field interacts with the Einstein-frame metric in the standard way (to this

purpose, the problem of inverting f' cannot be avoided). In the sequel of this appendix, however, it will be convenient to write all the equations in terms of the conjugate momentum p .

2. Inverse Legendre transformation

So far, we have recalled how a nonlinear Lagrangian can be recast into a Helmholtz Lagrangian. Now we present the inverse procedure: how to find a nonlinear Lagrangian equivalent to a given Helmholtz Lagrangian. In fact, let

$$L_H(p; g, \Psi) = \sqrt{-g}[pR - H(p; g, \Psi)] \quad (\text{B5})$$

be a generic Helmholtz Lagrangian. The function H plays the role of a Hamiltonian¹⁹ for a system of matter (denoted collectively by Ψ) and gravity $g_{\mu\nu}$; thus it does not depend on derivatives of the gravitational momentum p , while it depends on covariant derivatives of Ψ up to some order (the semicolon separates the field variables which are accompanied by their derivatives from those that are not). The variation of the action $S_H = \int L_H d^4x$ with respect to Ψ yields the equations of motion for matter,

$$\frac{\delta}{\delta\Psi}(\sqrt{-g}H) = 0, \quad (\text{B6})$$

while variation of it with respect to p yields an *algebraic* equation for the canonical momentum:

$$R(g) - \frac{\partial H(p; g, \Psi)}{\partial p} = 0. \quad (\text{B7})$$

Any solution of this “equation of motion” for p is denoted by $P(R; g, \Psi)$. Finally, varying the metric one gets (second-order) gravitational field equations

$$\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}pR) &\equiv pG_{\mu\nu}(g) - \nabla_\mu \nabla_\nu p + g_{\mu\nu} \square p \\ &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}H); \end{aligned} \quad (\text{B8})$$

clearly (B7) and (B8) are a generalization of (B3). One now defines, with the aid of a solution to (B7), a nonlinear Lagrangian for the metric and the matter:

$$L_{\text{NL}} = P(R; g, \Psi)R\sqrt{-g} - H[P(R; g, \Psi); g, \Psi]\sqrt{-g}, \quad (\text{B9})$$

where R stands for $R(g)$ (and is not an independent variable). To show the equivalence of the two Lagrangians, one finds the field equations resulting from the stationarity of the action

$$S_{\text{NL}} = \int L_{\text{NL}} d^4x. \quad (\text{B10})$$

Variation $\delta\Psi$ of the matter fields yields, after some manipulations, the equations of motion for matter:

$$\left. \frac{\delta(\sqrt{-g}H)}{\delta\Psi} \right|_{p=P} = 0, \quad (\text{B11})$$

which are equivalent to (B6), since (B7) holds identically. The latter also plays a crucial role in deriving gravitational field equations. In fact, a variation $\delta g^{\mu\nu}$ in (B10) gives rise to the field equations

$$\begin{aligned} P(R; g, \Psi)G_{\mu\nu}(g) - \nabla_\mu \nabla_\nu P + g_{\mu\nu} \square P \\ - \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}H)}{\delta g^{\mu\nu}} \Big|_{p=P} = 0, \end{aligned} \quad (\text{B12})$$

and these are equivalent to (B8). The equivalence of the field equations generated by L_H and L_{NL} holds at least whenever equation (B7) has a unique solution.

One now uses the general formalism described above to find out a nonlinear Lagrangian, and the corresponding fourth-order field equations in MJCF in case II of Sec. II, i.e., when matter is minimally coupled to the rescaled metric $\tilde{g}_{\mu\nu}$. The inverse transformation from the Einstein frame does not lead back to the original VJCF but to another (conformally related) space-time metric. This makes no surprise since, after adding interaction with matter in the Einstein frame, the inverse transformation is applied to a different system. Let us see it in more detail.

3. Adding matter interaction in the Einstein frame

The starting point is a nonlinear vacuum Lagrangian for pure gravity. Denoting the metric field in this VJCF by $\hat{g}_{\mu\nu}$, one has $L_{\text{vac}} = f(\hat{R})\sqrt{-\hat{g}}$. One makes the standard transformation to the Einstein frame, $p = f'(\hat{R})$ and $\tilde{g}_{\mu\nu} = p\hat{g}_{\mu\nu}$. Solving the latter equation for \hat{R} , one finds $\hat{R} = r(p)$, and this defines the direct and inverse Legendre maps, $p = \frac{df(r)}{dr}$ and $r = r(p)$. The field p , in the “phase space” of the system, becomes an independent dynamical variable, and the relation $p = f'(\hat{R})$ is a consequence of the equations of motion generated by the Helmholtz Lagrangian, or equivalently (after conformal rescaling) by the Einstein-frame Lagrangian (1.5),

$$\tilde{L}_{\text{vac}} = \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{3}{2p^2} \tilde{g}^{\mu\nu} p_{,\mu} p_{,\nu} - 2V(p) \right] \quad (\text{B13})$$

with

$$2V(p) = \frac{r(p)}{p} - \frac{f[r(p)]}{p^2}. \quad (\text{B14})$$

Assuming that the rescaled metric $\tilde{g}_{\mu\nu}$ is physical, one now adds the minimal-coupling term $2\ell_{\text{mat}}(\tilde{g}; \Psi)$ to the Lagrangian (B13). This clearly affects the equations of motion, and, therefore, the relation $p = f'(\hat{R})$, holding in vacuum, fails to be true in the presence of matter.

The present formalism restricts the allowed matter Lagrangians to those that do not depend on derivatives of

¹⁹This “Hamiltonian” has actually nothing to do with the notion of energy and the ADM formalism in general relativity.

the metric. Although stringent in itself, the restriction admits the physically important cases of perfect fluid, pure radiation, fields of spin 0, and usual fields of spin 1. The full Lagrangian is then

$$\tilde{L} = \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{3}{2p^2} \tilde{g}^{\mu\nu} p_{,\mu} p_{,\nu} - 2V(p) + 2\ell_{\text{mat}}(\tilde{g}; \Psi) \right]. \quad (\text{B15})$$

The usual conformal rescaling $\tilde{g}_{\mu\nu} = pg_{\mu\nu}$ transforms (B15) into the following Lagrangian for $g_{\mu\nu}$, p , and the matter:

$$L_H(p; g, \Psi) = \sqrt{-g} \{ pR - 2p^2[V(p) - \ell_{\text{mat}}(pg; \Psi)] \}. \quad (\text{B16})$$

It has the form of a Helmholtz Lagrangian (B5) because the transformation canceled the kinetic term for the scalar. The information about the original vacuum Lagrangian L_{vac} is encoded in the potential $V(p)$. The requirement that L_H does not depend on derivatives of p is met provided ℓ_{mat} contains no covariant derivatives (thus, it may contain at most first derivatives of Ψ). The Hamiltonian in the present case is

$$H(p; g, \Psi) = p r(p) - f[r(p)] - 2p^2 \ell_{\text{mat}}(pg; \Psi) \quad (\text{B17})$$

and the equation for p becomes (2.15), $R(g) - r(p) + g^{\mu\nu} T_{\mu\nu}(\Psi, pg) = 0$; the matter stress-energy tensor is defined, as always, in terms of the physical metric $\tilde{g}_{\mu\nu}$ and expressed in terms of $g_{\mu\nu}$. If $\ell_{\text{mat}} = 0$ or $T_{\mu\nu}$ is traceless (e.g., matter is conformally invariant), then $R(g) = r(p)$ and $p = f'(R)$ will be a solution; otherwise the solution $P(R; g, \Psi) \neq f'(R)$. The Helmholtz Lagrangian (B16) corresponds to a nonlinear Lagrangian (B9) for $g_{\mu\nu}$ and Ψ . The NLG Lagrangian L_{NL} so obtained reduces to the original vacuum one if the matter fields are set to vanish; however, in general, it is not equal to the sum of L_{vac} and a matter term analogous to ℓ_{mat} (see the examples below). In the purely metric picture, the relation between the two metrics is expressed by $\tilde{g}_{\mu\nu} = P(R; g, \Psi)g_{\mu\nu}$, which explicitly depends on matter. For this reason the MJCF metric corresponding to a given physical metric $\tilde{g}_{\mu\nu}$ does not coincide in general with the VJCF metric $\hat{g}_{\mu\nu}$, which is implicitly defined by $\tilde{g}_{\mu\nu} = f'(\hat{R})\hat{g}_{\mu\nu}$. In MJCF the equations of motion for matter, (B11), read

$$\frac{\delta}{\delta\Psi} \left[\sqrt{-g} p^2 \ell_{\text{mat}}(pg; \Psi) \right]_{p=P} = 0. \quad (\text{B18})$$

The fourth-order field equations for gravity, (B12), can be put in a number of equivalent forms. Here we recast them in the form that is closest to that for the vacuum case. Upon using $r(p) = R(g) + g^{\mu\nu} T_{\mu\nu}$ one finds that $PG_{\mu\nu} + \frac{1}{2}Pg_{\mu\nu}r(P) = PR_{\mu\nu} + \frac{1}{2}Pg_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}$; then defining a scalar $M(R; g, \Psi)$ as the matter contribution to P , $P = f'(R) + M(R; g, \Psi)$, one arrives at the gravitational field equations

$$\begin{aligned} Q_{\mu\nu} + M(R; g, \Psi)R_{\mu\nu} - \nabla_\mu \nabla_\nu M + g_{\mu\nu} \square M \\ - \frac{1}{2}g_{\mu\nu} \{ f[r(p)] - f(R) \} \\ = P(R; g, \Psi) [T_{\mu\nu}(Pg; \Psi) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}]. \end{aligned} \quad (\text{B19})$$

Here $Q_{\mu\nu} \equiv f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R)$ is the left-hand side of Eq. (1.2) and satisfies the generalized Bianchi identity $\nabla^\nu Q_{\mu\nu} = 0$. One sees that matter and gravitational variables are inextricably intertwined in (B19), thus these equations do not provide conservation laws similar to those in general relativity. Formally, four matter conservation laws arise upon taking the divergence of (B19); then the purely gravitational part, $Q_{\mu\nu}$, disappears and one is left with interaction terms. The resulting equations, however, are too complicated to be of any practical use even in the simplest cases. Below we give two examples of finding L_{NL} for a given form $f(\hat{R})$ of the vacuum Lagrangian and for dust or a scalar field + electromagnetic field as the matter.

4. Example 1: quadratic Lagrangian and charged scalar field

Let the vacuum Lagrangian in VJCF be $L_{\text{vac}} = (a\hat{R}^2 + \hat{R})\sqrt{-\hat{g}}$. Then the vacuum inverse Legendre map is $r(p) = \frac{1}{2a}(p-1)$ and $f[r(p)] = \frac{1}{4a}(p^2-1)$. In the Einstein-frame metric $\tilde{g}_{\mu\nu} = p\hat{g}_{\mu\nu}$ one adds the interaction with a massive complex-valued scalar field ψ minimally coupled to electromagnetic field:

$$\begin{aligned} L = \left[\tilde{R} - \frac{3}{2p^2} \tilde{g}^{\mu\nu} p_{,\mu} p_{,\nu} - \frac{(p-1)^2}{4ap^2} - \tilde{g}^{\mu\nu} D_\mu \psi (D_\nu \psi)^* \right. \\ \left. - m^2 \psi \psi^* - \frac{1}{8\pi} F_{\alpha\mu} F_{\beta\nu} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} \right] \sqrt{-\tilde{g}}. \end{aligned} \quad (\text{B20})$$

Here $D_\mu \psi \equiv \partial_\mu \psi - ieA_\mu \psi$. After conformal rescaling $\tilde{g}_{\mu\nu} = pg_{\mu\nu}$ this Lagrangian becomes (up to a full divergence)

$$\begin{aligned} L_H = \left[pR - \frac{1}{4a}(p-1)^2 - pg^{\mu\nu} D_\mu \psi (D_\nu \psi)^* \right. \\ \left. - m^2 p^2 \psi \psi^* - \frac{1}{8\pi} F_{\alpha\mu} F_{\beta\nu} g^{\mu\nu} g^{\alpha\beta} \right] \sqrt{-g} \end{aligned} \quad (\text{B21})$$

and Eq. (B14) is solved to yield

$$P(R, g; \psi) = 2 \frac{R - g^{\mu\nu} D_\mu \psi (D_\nu \psi)^* + 1/2a}{4m^2 \psi \psi^* + 1/a}. \quad (\text{B22})$$

The nonlinear Lagrangian (B9) in MJCF,

$$\begin{aligned} L_{\text{NL}}(g, \psi, F) = \sqrt{-g} \left\{ \frac{[R - g^{\mu\nu} D_\mu \psi (D_\nu \psi)^* + 1/2a]^2}{4m^2 \psi \psi^* + \frac{1}{a}} \right. \\ \left. - \frac{1}{8\pi} F_{\alpha\mu} F_{\beta\nu} g^{\mu\nu} g^{\alpha\beta} - \frac{1}{4a} \right\}, \end{aligned} \quad (\text{B23})$$

generates the fourth-order gravitational field equations

$$\begin{aligned}
& P(R, g; \psi) R_{\mu\nu} - \nabla_\mu \nabla_\nu P + g_{\mu\nu} \square P - P D_{(\mu} \psi (D_{\nu)} \psi)^* \\
& - \frac{1}{2} g_{\mu\nu} \left\{ \frac{[R - g^{\mu\nu} D_\mu \psi (D_\nu \psi)^* + 1/2a]^2}{4m^2 \psi \psi^* + \frac{1}{a}} - \frac{1}{4a} \right\} \\
& - \frac{1}{4\pi} (F_{\alpha\mu} F_{\beta\nu} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma}) = 0. \quad (B24)
\end{aligned}$$

5. Example 2: logarithmic Lagrangian and dust

Cases where Eq. (B7) for P can be explicitly solved are quite exceptional. Besides the quadratic Lagrangian and a simple matter source, one of these is provided by a logarithmic Lagrangian (an exponential one is not the case) and dust.

In the vacuum Jordan frame (VJCF) one has $L_{\text{vac}} = \frac{1}{a} \ln(1 + a\hat{R}) \sqrt{-\hat{g}}$ [for simplicity, we consider only the sector in which $(1 + a\hat{R}) > 0$] and proceeding as in the preceding case one finds

$$\begin{aligned}
p &= \frac{1}{ar + 1} > 0, \quad r(p) = \frac{1}{a} \left(\frac{1}{p} - 1 \right), \\
f[r(p)] &= -\frac{1}{a} \ln p, \quad V(p) = \frac{1}{2ap} \left[\frac{1}{p} (\ln p + 1) - 1 \right],
\end{aligned}$$

and

$$L_H = \sqrt{-g} \left[pR - \frac{1}{a} (\ln p - p + 1) + 2p^2 \tilde{\rho} \right]. \quad (B25)$$

The corresponding Legendre map is

$$P(R) = \frac{1}{8\tilde{\rho}} \left[\sqrt{\left(R + \frac{1}{a}\right)^2 + \frac{16\tilde{\rho}}{a}} - \left(R + \frac{1}{a}\right) \right]$$

[there is another solution, $P(R) < 0$, which is discarded] and the resulting nonlinear Lagrangian is

$$\begin{aligned}
L_{\text{NL}} &= \frac{\sqrt{-g}}{16\tilde{\rho}} \left\{ \left(R + \frac{1}{a}\right) \left[\sqrt{\left(R + \frac{1}{a}\right)^2 + \frac{16\tilde{\rho}}{a}} - \left(R + \frac{1}{a}\right) \right] \right. \\
& - \frac{16\tilde{\rho}}{a} \ln \left[\sqrt{\left(R + \frac{1}{a}\right)^2 + \frac{16\tilde{\rho}}{a}} - \left(R + \frac{1}{a}\right) \right] \\
& \left. - \frac{8\tilde{\rho}}{a} [2 \ln(8\tilde{\rho}) - 1] \right\}; \quad (B26)
\end{aligned}$$

notice that $L_{\text{NL}} = \frac{1}{a} \ln(1 + aR) \sqrt{-g} + 2\tilde{\rho}(1 + aR)^{-2} \sqrt{-g} + O(\tilde{\rho}^2)$.

APPENDIX C: THE INVERSE PROBLEM OF NONLINEAR GRAVITY

By means of the Helmholtz Lagrangian method and the conformal rescaling it is possible to map a nonlinear vacuum gravity theory with Lagrangian $L_{\text{vac}} = \sqrt{-g} f(R)$ to a dynamically equivalent system consisting of Einstein gravity and a minimally coupled nonlinear scalar field

with a potential determined by the function f . The inverse problem of nonlinear gravity consists in making an inverse transformation. Given a scalar field, which self-interacts via an arbitrary potential $\tilde{U}(\phi)$, is it possible to map the Lagrangian

$$\tilde{L}(\tilde{g}, \phi) = \sqrt{-\tilde{g}} \left[\tilde{R} - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2\tilde{U}(\phi) \right] \quad (C1)$$

in the Einstein frame to an equivalent Lagrangian $L_{\text{NL}} = \sqrt{-g} f(R)$ in the VJCF? With one exception, the answer is "yes," but in most cases the nonlinear Lagrangian cannot be expressed in terms of elementary functions [even if $\tilde{U}(\phi)$ can be]. The procedure is as presented in Appendix B. First one redefines the scalar by setting $\phi = \sqrt{\frac{2}{3}} \ln p$ and then makes the conformal rescaling $\tilde{g}_{\mu\nu} = p g_{\mu\nu}$ to cancel the kinetic term for the field p . After discarding a divergence term, (C1) becomes a Helmholtz Lagrangian,

$$L_H = \sqrt{-g} [pR - 2p^2 U(p)], \quad (C2)$$

where $U(p) \equiv \tilde{U}(\sqrt{\frac{3}{2}} \ln p)$. According to (B7), the inverse Legendre map $p = P(R)$ is a solution of the algebraic (with respect to p) equation

$$R(g) - \frac{d}{dp} [2p^2 U(p)] = 0; \quad (C3)$$

then the function f is determined by the relation $f'(R) = P(R)$.

An alternative method consists in solving an ordinary differential equation directly for f . Comparing (C2) with B16), one sees that $U(p) \equiv V(p)$, where V is given in (B14) with $p = \frac{df(r)}{dr}$, and $r(p)$ its inverse function. Then the equation $U(p) = V(p)$ becomes a differential equation²⁰ for $f(r)$:

$$2(f')^2 U(f') = r f' - f. \quad (C4)$$

Solving this equation, however, is by no means easier than solving (C3). Moreover, to ensure the consistency of the Legendre transformation, a solution to (C4) should also meet the R -regularity condition $\frac{d^2 f}{dr^2} \neq 0$. Differentiating (C4) with respect to r and using $f'' \neq 0$, one arrives at an equivalent equation, which added to (C4) yields

$$2f' \frac{dU}{dp} \Big|_{p=f'} + 2U(f') - \frac{f}{(f')^2} = 0. \quad (C5)$$

Thus f , as a function of p , is given by

$$f[r(p)] = 2p^2 \left[U(p) + p \frac{dU}{dp} \right], \quad (C6)$$

while $r(p)$ is determined from the form of V ,

$$r(p) = 2pU(p) + \frac{1}{p} f[r(p)]. \quad (C7)$$

²⁰In [52], Teyssandier and Tourenç present a system of differential equations, which is equivalent to (C4).

To determine $f(r)$ without any integration, one should compute $f[r(p)]$ from (C6), and then solve (C7) for $p(r)$ [which is the same as solving (C3)] and insert the solution back into $f[r(p)]$.

First we note that the method does not apply to the simplest case of the massless linear field, $\tilde{U}(\phi) \equiv 0$. In fact, in this case (C4) is solved by $f = Cr$, and this solution is excluded because it is not R -regular. Thus, the above-mentioned exception is provided by the Einstein-frame Lagrangian $\tilde{L}(\tilde{g}, \phi) = \sqrt{-\tilde{g}} \left[\tilde{R} - \tilde{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right]$, for which the ϕ -field cannot be eliminated to yield an equivalent purely metric NLG theory. This Lagrangian, on the other hand, can be turned into a STG Lagrangian (2.16) by a suitable conformal rescaling. A constant potential in (C1) is interpreted as a cosmological constant, $\tilde{U}(\phi) = \Lambda$. Then (C6) and (C7) easily yield $f(R) = \frac{1}{8\Lambda} R^2$. This result is also obtained as a particular (“singular”) solution of (C4), which in this case is a Clairaut equation (and has also a general solution, which should be excluded as being linear).

For nonconstant potentials $\tilde{U}(\phi)$ the solutions do exist, but these are practically inaccessible, since one is unable to solve (C3) or (C7). For the most interesting—from the field-theoretical viewpoint—potential, $\tilde{U}(\phi) = \lambda\phi^n$, $\lambda = \text{const}$, $n = 2, 3, 4$, one finds $U(p) = \mu(\ln p)^n$, with $\mu \equiv \left(\frac{3}{2}\right)^{\frac{n}{2}} \lambda$,

$$f[r(p)] = 2\mu p^2 (\ln p + n)(\ln p)^{n-1}, \quad (\text{C8})$$

$$r(p) = 2\mu p (2 \ln p + n)(\ln p)^{n-1}. \quad (\text{C9})$$

The Liouville field theory [32] provides one of the few examples where Eq. (C7) can be explicitly solved. In this case $\tilde{U}(\phi) = Ae^{\alpha\phi}$ (A and α are constants), and one finds $f[r(p)] = 2(1 + \beta)Ap^{\beta+2}$, with $\beta \equiv \sqrt{\frac{3}{2}}\alpha$; $r(p) = 2(2 + \beta)Ap^{\beta+1}$; then

$$\begin{aligned} P(R) &= [2(2 + \beta)A]^{-1/(\beta+1)} R^{1/(\beta+1)}, \\ f(R) &= 2(1 + \beta)A[2(2 + \beta)A]^{(\beta+2)/(\beta+1)} R^{(\beta+2)/(\beta+1)}. \end{aligned} \quad (\text{C10})$$

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