Improved Hamiltonians for lattice gauge theory with fermions

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Several improved Hamiltonians for lattice gauge theory with fermions are proposed. A test of the Schwinger model and some applications to QCD are presented. The results for the mass spectrum indicate that the use of the improved Hamiltonians gives better results.

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I. INTRODUCTION

Although the lattice formulation has widely been accepted as one of the most promising nonperturbative schemes for gauge field theories based on first principles, the forms of the lattice actions or Hamiltonians are not uniquely determined, and systematic errors are present even when the theory is formulated.

One of the major sources of these errors comes from the finite lattice spacing a. The lattice regularization makes the standard action or Hamiltonian differ from the continuum one by $O(a^n)$. In the continuum limit $a \to 0$ or equivalently $1/g^2 \to \infty$ in an asymptotically free theory, these differences in principle disappear and the action or Hamiltonian becomes the continuum one. Practically, both numerical and analytical calculations can only be carried out up to the intermediate coupling region (in the numerical simulation, one of the reasons for this is the finite volume; in our Hamiltonian method, this is due to the finite-link approximations as will be discussed in Sec. III and Sec. IV in detail). Then systematic errors due to finite a will show up. Another problem is in the theory with Wilson fermions for solving the species doubling, which differs from the continuum one already by order of $O(a)$, explicitly breaks chiral symmetry and induces the mixing of the corresponding lattice operators with those of different chiralities or even of lower dimension. Some examples of mixing are in the chiral order operator

$$\bar{\psi}\psi^{\rm con} = C^{\bar{\psi}\psi}\bar{\psi}\psi + C^I I, \tag{1.1}$$

and in four-fermion operators relevant for the weak interactions of the form

$$\begin{aligned}\bar{s}\gamma_k(1+\gamma_5)d\bar{u}\gamma_k(1+\gamma_5)u^{\rm con} = {} & Z\bar{s}\gamma_k(1+\gamma_5)d\bar{u}\gamma_k(1+\gamma_5)u \\ & + \sum_{n=S,V,T,A,P} Z_n\bar{s}\Gamma_n d\bar{u}\Gamma_n u + Z_3\bar{s}d \\ & + Z_3'\bar{s}d + Z_5\bar{s}\sigma_{j_1j_2}F_{j_1j_2}d,\end{aligned} \tag{1.2}$$

where "con" means continuum, I is the identity matrix, ψ is the fermion field, u, d, and s are those with different flavors, $F_{j_1j_2}$ is the lattice definition of the gauge field strength tensor, $C^{\bar{\psi}\psi}, C^I, Z, Z_i, Z_3, Z_3'$ and Z_5 are the mixing coefficients, and S, V, T, A and P stand for the scalar, vector, tensor, axial vector and pseudoscalar index of the quark bilinear respectively. In (3+1)-dimensional QCD, the mixing coefficients of the additional operators cannot be reliably calculated in lattice perturbation theory if the bare coupling cannot be reached weakly enough, which would again result in large uncertainty or lattice artifacts in the computation of hadronic matrix elements.

One possible way to tackle these problems is to use the improved theory. Inspired by the success of the Symanzik's improvement program in pure gauge theory, Hamber and Wu proposed an improved action [1] for Wilson fermions by adding next-to-nearest neighbor interaction terms so that the $O(a)$ effects due to the Wilson term become higher order. Similar [2,3] and different [4] proposals on this subject were offered too. Recently, there has been increasing attention paid to this direction [5–11], and a noticeable reduction of the finite spacing effects and improvement in the chiral behavior of the hadronic matrix elements have been observed.

In the analytical investigation of the theory's local structure, the Hamiltonian formulation is of most inter-

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est. Motivated by this, we propose in parallel, several improved Hamiltonians for lattice gauge theory (LGT) with fermions in Sec. II, test our improved approach to the Schwinger model in Sec. III, and discuss their applications to QCD in Sec. IV. The main results are summarized in Sec. V.

II. IMPROVED HAMILTONIANS

The conventional Hamiltonian of LGT with Wilson fermions is

$$
\begin{aligned}
H &= H_f + H_g, \\
H_f &= H_m + H_k + H_r, \\
H_m &= m \sum_x \bar{\psi}(x)\psi(x), \\
H_k &= \frac{1}{2a} \sum_{x,k} \bar{\psi}(x)\gamma_k U(x,k)\psi(x+k), \\
H_r &= \frac{r}{2a} \sum_{x,k} [\bar{\psi}(x)\psi(x) - \bar{\psi}(x)U(x,k)\psi(x+k)], \\
H_g &= \frac{g^2}{2a} \sum_{y,j} E_j^\alpha(y) E_j^\alpha(y) - \frac{G}{ag^2} \sum_p \mathrm{Tr}(U_p + U_p^\dagger - 2),
\end{aligned}
\tag{2.1}
$$

where $U(x,k)$ is the gauge link variable at site x in the direction k, with $k = \pm j$, and $\gamma_{-j} = -\gamma_j$, H_m, H_k, H_r, and H_g are, respectively, the mass term, kinetic term, Wilson term and pure gauge energy, and G is some constant depending on the gauge group. In [12,13], some improved Hamiltonians for the pure gauge sector H_g were proposed. The advantages and existing problems have been discussed in [14,15].

In this paper, we concentrate on the improvement of the fermionic Hamiltonian. As discussed in Sec. I, the $O(a^n)$ terms in H_f lead to systematic effects if a or g^2 is not small enough. We propose some improved Hamiltonians to reduce these effects. The first one is

$$
\begin{aligned}
H_f^{\text{improved}} &= H_m + H_k^{\text{improved}} + H_r^{\text{improved}}, \\
H_k^{\text{improved}} &= \frac{b_1}{2a} \sum_{x,k} \bar{\psi}(x)\gamma_k U(x,k)\psi(x+k) \\
&\quad + \frac{b_2}{2a} \sum_{x,k} \bar{\psi}(x)\gamma_k U(x,2k)\psi(x+2k), \\
H_r^{\text{improved}} &= \frac{r}{2a} \sum_{x,k} \bar{\psi}(x)\psi(x) - c_1 \frac{r}{2a} \\
&\quad \times \sum_{x,k} \bar{\psi}(x)U(x,k)\psi(x+k) \\
&\quad - c_2 \frac{r}{2a} \sum_{x,k} \bar{\psi}(x)U(x,2k)\psi(x+2k),
\end{aligned}
\tag{2.2}
$$

where $U(x,2k) = U(x,k)U(x+k,k)$ and the coefficients b_1, b_2, c_1, and c_2 are determined by requiring that H_f^{improved} recovers the continuum one in the $a \to 0$ limit. Here we choose

$$
b_1 = \frac{4}{3}, \quad b_2 = -\frac{1}{6}, \quad c_1 = \frac{4}{3}, \quad c_2 = -\frac{1}{3}, \tag{2.3}
$$

so that the p^3 term of H_k^{improved}, p^0 and p^2 terms of H_r^{improved} in the momentum space are eliminated at the tree level, and the improved theory is expected to reduce the finite spacing effects. Similar to the action of Hamber and Wu [1], our improved Hamiltonian is the mixture of the nearest neighbor and next-to-nearest neighbor interactions.

A unitary transformation $e^{-iS_{\text{free}}} H_f^{\text{improved}} e^{iS_{\text{free}}}$, like that in [16], which diagonalizes exactly H_f^{improved} with free fermions, gives the vacuum state of H_f^{improved} as

$$
|\Omega_{\text{free}}\rangle = \exp(iS_{\text{free}})|0\rangle = \exp\left(i \sum_p \theta_p S_p\right)|0\rangle, \tag{2.4}
$$

with $|0\rangle$ being the bare vacuum and

$$
\begin{aligned}
S_p &= -\frac{1}{A_p} \sum_j \psi^\dagger(p)\gamma_j\psi(p) \frac{\sin pja}{a}, \\
A_p &= \left\{ \sum_j \left(\frac{\sin pja}{a} \right)^2 \right\}^{1/2}, \\
\tan 2\theta_p &= \frac{A_p(b_1 + b_2 \cos pja)}{m + \frac{r}{a}(1 - c_1 \cos paj - c_2 \cos 2paj)}.
\end{aligned}
\tag{2.5}
$$

From the vacuum energy or dispersion relation,

$$
E_{\Omega_{\text{free}}} = -N_c N_f \sum_j \left\{ \left[m + \frac{r}{a}(1 - c_1 \cos paj - c_2 \cos 2paj) \right]^2 + [A_p(b_1 + b_2 \cos pja)]^2 \right\}^{1/2}, \tag{2.6}
$$

one sees that there is no species doubling if $r \neq 0$.

The same arguments can be applied to LGT with Kogut-Susskind fermions. The improved Hamiltonian is

$$
H_f^{\text{improved}} = H_m + H_k^{\text{improved}},
$$

$$H_k^{\text{improved}} = \frac{b_1}{2a}\sum_{x,j}\eta_j(x)[\bar{\psi}(x)U(x,j)\psi(x+j) - \bar{\psi}(x)U(x,-j)\psi(x-j)] + \frac{b_2}{2a}\sum_{x,j}\eta_j(x)[\bar{\psi}(x)U(x,3j)\psi(x+3j) - \bar{\psi}(x)U(x,-3j)\psi(x+3j)], \tag{2.7}$$

where $\eta_j(x)$ is the Kawamoto-Smit phase factor in the Hamiltonian version and the coefficients are given by

$$b_1 = \frac{3}{2}, \quad b_2 = -\frac{1}{6}. \tag{2.8}$$

It is easily shown that the p^3 term in the momentum space is canceled.

To reduce the higher-order $O(a^n)$ effects, Eq. (2.2) can be generalized to be

$$H_k^{\text{improved}} = \frac{b_1}{2a}\sum_{x,k}\bar{\psi}(x)\gamma_k U(x,k)\psi(x+k) + \frac{b_2}{2a}\sum_{x,k}\bar{\psi}(x)\gamma_k U(x,2k)\psi(x+2k) + \frac{b_3}{2a}\sum_{x,k}\bar{\psi}(x)\gamma_k U(x,3k)\psi(x+3k),$$

$$H_r^{\text{improved}} = \frac{r}{2a}\sum_{x,k}\bar{\psi}(x)\psi(x) - c_1\frac{r}{2a}\sum_{x,k}\bar{\psi}(x)U(x,k)\psi(x+k) - c_2\frac{r}{2a}\sum_{x,k}\bar{\psi}(x)U(x,2k)\psi(x+2k) - c_3\frac{r}{2a}\sum_{x,k}\bar{\psi}(x)U(x,3k)\psi(x+3k), \tag{2.9}$$

where the coefficients can be obtained by solving the equations

$$b_1 + 2b_2 + 3b_3 = 1,$$
$$b_1 + 2^3 b_2 + 3^3 b_3 = 0,$$
$$b_1 + 2^5 b_2 + 3^5 b_3 = 0,$$
$$c_1 + c_2 + c_3 = 1,$$
$$c_1 + 2^2 c_2 + 3^2 c_3 = 0,$$
$$c_1 + 2^4 c_2 + 3^4 c_3 = 0. \tag{2.10}$$

The results are

$$b_1 = \frac{3}{2}, \quad b_2 = -\frac{3}{10}, \quad b_3 = \frac{1}{30}, \quad c_1 = \frac{3}{2}, \quad c_2 = -\frac{3}{5}, \quad c_3 = \frac{1}{10}. \tag{2.11}$$

At this level, it is necessary to improve H_g as well.

Hamiltonians of the form (called "clover" Hamiltonian)

$$H_f^{\text{improved}} = H_m + H_k + H_r^{\text{improved}},$$
$$H_r^{\text{improved}} = \frac{r}{2a}\sum_{x,k}\bar{\psi}(x)\psi(x) - c_1\frac{r}{2a}\sum_{x,k}\bar{\psi}(x)U(x,k)\psi(x+k) - ic_2\frac{rg}{2a}\sum_{x,j_1,j_2}\bar{\psi}(x)\sigma_{j_1 j_2}F_{j_1 j_2}(x)\psi(x) \tag{2.12}$$

have aroused some interest [4–10] because the additional term, representing the Pauli interaction of fermions with the field strength tensor, contains only the nearest neighbor interactions. The field strength tensor can be defined on the lattice as

$$F_{jk}(x) = \frac{1}{8ia^2}\sum_{n=1}^{4}[U_{p_n}(x) - U_{p_n}^{\dagger}(x)], \tag{2.13}$$

where the summation is over four plaquettes in the jk plane with a corner at x. A similar improvement condition leads to

$$c_1 = 1, \quad c_2 = \frac{1}{2}. \tag{2.14}$$

By redefining the fermion field

$$\psi \to \psi' = \left(1 - \frac{ra}{2}\not{D}\right)\psi, \tag{2.15}$$

where

$$\not{D}_j\psi(x) = \frac{1}{2a}\gamma_j[U(x,j)\psi(x+j) - U(x,-j)\psi(x-j)], \tag{2.16}$$

making a small a expansion

$$U(x,k) = 1 + igaA_k(x) + \cdots, \tag{2.17}$$

and using the identity

$$[D_{j_1}D_{j_2} - D_{j_2}D_{j_1}]\psi = igF_{j_1j_2}\psi, \tag{2.18}$$

it can be shown that Eq. (2.2) is equivalent to Eq. (2.12) up to $O(a)$.

With the above improved Hamiltonians (2.2) or (2.12) while coefficients are fixed at the tree level, the $O(a)$ errors in the Wilson term are corrected. Furthermore, the lattice Feynman-rule argument [5,6] indicates that such an improved theory obtained by the tree level calculations is beyond the tree level, leading to the cancellations of terms of order $(g^2)^n a\ln^n a$ in the matrix elements which deviate from the continuum theory. To further eliminate terms of order $(g^2)^n a\ln^{n-1} a$ in the matrix elements, one has to include the g^2 dependence of the coefficients consistently order by order. Again in [5], it is argued that these corrections, which would be less important in the weaker coupling regime, are finite and comparatively small. In fact, the numerical simulations in [7–9,11] were carried out by using the improved actions determined at the tree level.

Concerning the coefficients in the improved theory, it should be mentioned that there has been another interesting way [10] of estimations based on the mean field ansatz (evaluations of the effective coupling constant) proposed by Lepage and Mackenzie [17]. This is a good direction, but it is still under debate in the lattice community whether it has better control of the systematic errors of order $O(a)$ than the order by order improved theory mentioned above.

How does the modified theory described by the Hamiltonian (2.2), which is free of errors of the discrete lattice at least up to order $O(a)$, improve the physical values from lattice nonperturbative calculations? The results for the mass spectrum to be presented in the following sections will indeed give a positive answer. For the spectrum computations in this paper, we use the improved Hamiltonian (2.2) with coefficients determined at the tree level. In the QCD calculations to be presented in Sec. IV, we will only use the improved theory with naive fermions, where chiral symmetry is explicitly maintained. Although the Wilson term is absent in the improved Hamiltonian (2.2) with $r = 0$, it does appear in the transformed Hamiltonian, which is favorable for a reasonable ρ-ω splitting. This may again imply that such an improvement is beyond the tree level, and there could be an interplay between the variational unitary transformation approach and the determination of the coefficients.

III. THE IMPROVED SCHWINGER MODEL

The exactly solvable massless Schwinger model is a good laboratory for the improved theory. In 1+1 dimensions, there is no plaquette term in H_g so that the calculations are much simpler.

In the same way as in [18], the vector meson state $|V\rangle$ of the improved Schwinger model

$$|V\rangle = \sum_{n=0}^{R} A_n[V_n - \langle\Omega|V_n|\Omega\rangle]|\Omega\rangle \tag{3.1}$$

is a superposition of some operators V_n with the given quantum numbers

$$\begin{aligned} V_0 &= i\sum_x \bar\psi(x)\gamma_1\psi(x), \\ V_1 &= i\sum_{x,k} \bar\psi(x)\gamma_1 U(x,k)\psi(x+k), \\ V_2 &= i\sum_{x,k} \bar\psi(x)\gamma_1 U(x,2k)\psi(x+2k), \\ &\cdots, \end{aligned} \tag{3.2}$$

acting on the vacuum state $|\Omega\rangle$:

$$|\Omega\rangle = \exp(iS)|0\rangle,$$

$$\begin{aligned} S &= \theta_1 S_1 + \theta_2 S_2 + \cdots, \\ &= i\theta_1 \sum_{x,k} \psi^\dagger(x)\gamma_k U(x,k)\psi(x+k) \\ &\quad + i\theta_2 \sum_{x,k} \psi^\dagger(x)\gamma_k U(x,2k)\psi(x+2k) + \cdots, \end{aligned} \tag{3.3}$$

where θ_1 and θ_2 are determined by minimizing the vacuum energy. The bare vacuum is $|0\rangle$ defined by

$$\xi(x)|0\rangle = \eta(x)|0\rangle = E_j^\alpha(x)|0\rangle = 0, \tag{3.4}$$

where ξ and $\eta^\dagger$ are, respectively, the up and down components of ψ. The upper bound of the vector mass is the lowest eigenvalue of the equations

$$\begin{aligned} &\sum_{n_1=0}^{R} (H^v_{n_2n_1} - E_v U^v_{n_2n_1})A_{n_1} = 0, \\ &\det|H^v - E_v U^v| = 0, \end{aligned} \tag{3.5}$$

where the coefficients A_{n_1} are determined by solving the equations, and the matrix elements $H^v_{n_2n_1}$ and $U^v_{n_2n_1}$ are defined by

$$\begin{aligned} H^v_{n_2n_1} &= \left\langle V_{n_2}\middle|\left(H_f^{\text{improved}} + \frac{g^2}{2a}\sum_{y,j} E_j(y)E_j(y)\right)\middle|V_{n_1}\right\rangle^N, \\ U^v_{n_2n_1} &= \langle V_{n_2}|V_{n_1}\rangle^N. \end{aligned} \tag{3.6}$$

Here the superscript N means only the matrix elements proportional to the lattice size N are reserved. A detailed description of the method can be found in [18].

Practically, not only the superposition (3.1) has to be truncated to finite R, but also Eq. (3.3) and the matrix elements do to finite orders of θ_1 and θ_2. Therefore, the Hamiltonian cannot be exactly diagonalized and the results are reasonable only for a finite range of $1/g^2$. Might the use of the improved Hamiltonians reduce these systematic errors? For simplicity and illustration, in this section we consider the application of Eq. (2.2), with only the first three operators in Eq. (3.2) and the first term of S in Eq. (3.3) taken into account. For $r = 0$, $M_v a/g$ as a function of $1/g^2$ with different choices of the coefficients is shown in Fig. 1. Those for $r = 0.1$ and 1 are, respectively, plotted in Fig. 2 and Fig. 3, where the exact value (the dashed line) is also included. By dimensional analysis, $M_v a/g$ should be a constant for large $1/g^2$.

For naive fermions ($r = 0$), with the kinetic term in the Hamiltonian improved ($b_1 = 4/3, b_2 = -1/6$), the result (the triangles in Fig. 1) is certainly better than that (the circles) from the unimproved H_k ($b_1 = 1, b_2 = 0$) at least in the region $0 < 1/g^2 \leq 2$. The improved data enter faster the scaling region from $1/g^2 = 1.0$ (the plateau extends to at least $1/g^2 = 2.0$) and are closer to the exact value. Then the improved and unimproved data approach each other. For larger $1/g^2$ (≥ 2.5), the triangle is a little bit higher. This is another systematics which may not be due to the use of the improved Hamiltonian, but due to the finite terms we considered in Eq. (3.1) and Eq. (3.3). As discussed in detail in Ref. [18], when the continuum limit $1/g^2 \to \infty$ is reached, we have to include infinite number of terms in the wave function. Because the motivation of the improved Hamiltonian is to reduce the finite spacing effects in the intermediate ($1/g^2 \approx 1$) or stronger region, its virtue has already been seen at least in this region. It remains to be done for both numerical and analytical methods to extend the calculations to the weaker coupling region.

For Wilson fermions, as shown in Fig. 2 and Fig. 3, with the unimproved Hamiltonian ($b_1 = 1, b_2 = 0, c_1 = 1, c_2 = 0$) or only the kinetic term H_k improved ($b_1 = 4/3, b_2 = -1/6, c_1 = 1, c_2 = 0$), the deviations of the value of $M_v a/g$ from the exact one are increased as $1/g^2$ and r. In this situation, the effect of the Wilson term becomes more and more important, and the improvement of the Wilson term ($b_1 = 4/3, b_2 = -1/6, c_1 = 4/3, c_2 = -1/3$) should be taken into account. For $r \ll 1$, the contribution of H_r^{improved} in Eq. (2.2) is so small that the unimproved data (the circles) are under the improved ones (the diamonds) as in Fig. 1. Nevertheless, for intermediate Wilson parameter ($r = 0.1$) as in Fig. 2, the unimproved ones coincide with the improved ones, showing no obvious improvement. This is not surprising, and may give us an insight into the delicate interplay of H_k^{improved} and H_r^{improved}. For a larger Wilson parameter ($r = 1$), the individual terms in H_r^{improved} are large and the importance of the improvement of the Wilson term is evident just in this case. As shown in Fig. 3, the improved ones (the diamonds) are under the unimproved ones and are closer to the exact value, which indicates that the results for the improved Hamiltonian with larger Wilson parameter are better than the unimproved one.

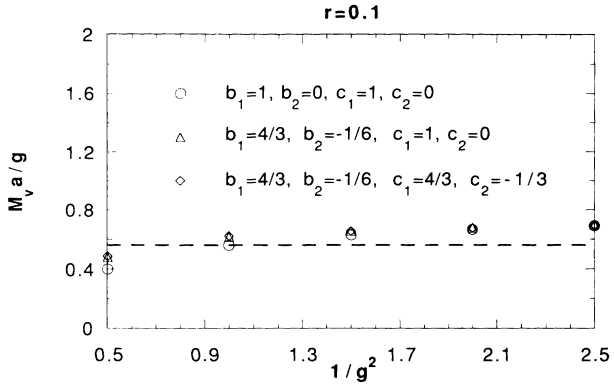

FIG. 2. $M_v a/g$ as a function of $1/g^2$ for $r = 0.1$. The dashed line is the exact value in the continuum model.

IV. IMPROVED QCD

Another interesting topic is the application of the improved theory to lattice QCD in 3+1 dimensions. The

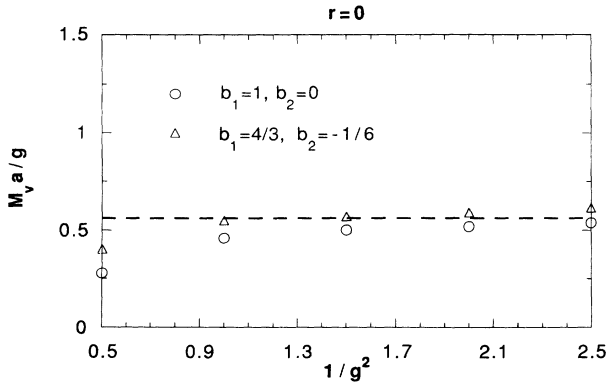

FIG. 1. $M_v a/g$ as a function of $1/g^2$ for $r = 0$. The dashed line is the exact value in the continuum model.

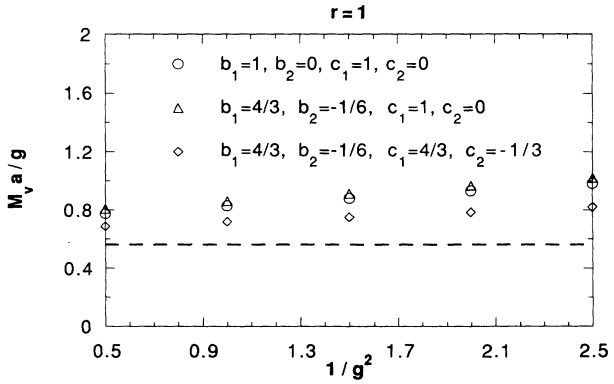

FIG. 3. $M_v a/g$ as a function of $1/g^2$ for $r = 1$. The dashed line is the exact value in the continuum model.

method discussed in Sec. III can be directly employed to calculate the mass spectrum of QCD as well.

LGT with massless naive fermions is considered in this section, mainly because there is explicit chiral symmetry in the Hamiltonian. Of course, the disadvantage is its species doubling problem. As is well known, even in LGT with Kogut-Susskind fermions, this problem still exists. However, it is argued that the flavor doubling affects substantially physical results only through the quark loops, which might play an important role in the thermodynamical quantities such as the chiral condensate. The unitary transformation and the variational method could efficiently suppress the effects of these quark-antiquark pairs. Mathematically, the purpose of this approach is to diagonalize the Hamiltonian. Physically, it is to relate the sea quark picture where there are a lot of quark-antiquark pairs in the vacuum, to the valence (or constituent) quark picture where the hadrons can be described by only a few quark-antiquark pairs in the wave functions. In the valence quark representation, we expect that the spectrum evaluations are not greatly influenced by the flavor doubling, which is indeed confirmed by our results as well as in the Monte Carlo simulation.

Using the unitary transformation and the variational approach to fermions, the vector mass in the Schwinger model with naive fermions [18] agrees well the exact one in the continuum. Using the same approach, in the strong coupling regime we have obtained satisfactory results [15] for three- and four-dimensonal LGT (LGT_3 and LGT_4) with naive fermions, such as the PCAC (partial conservation of axial vector current) relation. It has been observed that a few gauge-link approximations to the operators in Eq. (3.2) and Eq. (3.3) are good enough in lower dimensions, while in higher dimensional theories, however, the validity of the approximations is limited to the strong coupling regime. The reason is that as the coupling g decreases, fermion-antifermion pairs connected by gauge links of longer distances are created in the vacuum and more and more operators of these kind are required to better diagonalize the Hamiltonian and to extend the domain of validity from the strong coupling regime to the weaker coupling one. The secondary effects of the one-gauge link approach are that the ρ-ω mass difference (see below), Δ-N mass difference and $\rho \to \pi\pi$ [20,21] cannot be produced.

A similar problem exists in the strong coupling expansion result [19] obtained by using Kogut-Susskind fermions, where there are no appreciable mass splittings among the s-wave mesons. One solution is the introduction of the Wilson fermions, which leads to mass splitting already in lower orders of the strong coupling expansion series [22], but chiral symmetry is broken explicitly, and a fine-tuning is required to restore this symmetry. A better way out is to use the improved theory to evade this problem.

For $r = 0$, the improved fermionic Hamiltonian is given by Eq. (2.2) with $b_1 = 4/3$ and $b_2 = -1/6$ without the Wilson term. The transformed Hamiltonian $e^{-iS}H^{\text{improved}}e^{iS}$ in the strong coupling regime is

$$
\begin{aligned}
H' = {} & \frac{b_1\theta_1 + b_2\theta_2}{a}\sum_x \bar{\psi}(x)\psi(x) + \frac{b_1\theta_2 + b_2\theta_1}{a}\sum_{x,k}\bar{\psi}(x)U(x,k)\psi(x+k) \\
& - \frac{b_1\theta_1}{a}\sum_{x,k_1\neq k_2}\bar{\psi}(x)\gamma_{k_1}\gamma_{k_2}U(x,k_1,k_2)\psi(x+k_1+k_2) \\
& + \frac{g^2}{2a}\sum_{y,j}([-i\theta_1 S_1, E_j^\alpha]^2 + [-i\theta_2 S_2, E_j^\alpha]^2 + 2[-i\theta_2 S_2, [-i\theta_1 S_1, E_j^\alpha]]) \\
& + \frac{g^2}{2a}\sum_{y,j}\{[-i\theta_1 S_1, E_j^\alpha] + [-i\theta_2 S_2, E_j^\alpha], E_j^\alpha\} + \cdots + H_g + H_k^{\text{improved}}.
\end{aligned} \tag{4.1}
$$

The derivations of the commutators and anticommutators are straightforward, and the variational parameters

$$
\theta_1 = \frac{b_1}{g^2 K_{N_c}}, \quad \theta_2 = \frac{b_2}{2g^2 K_{N_c}}, \tag{4.2}
$$

are fixed by minimizing the vacuum energy, with $K_{N_c} = N_c(N_c-1)/(2N_c^2)$ being the Casmir invariant. In the strong coupling regime $1/g^2 \ll 1$, by neglecting the pure magnetic energy in H_g and using the condition Eq. (4.2), the transformed Hamiltonian (4.1) is simplified as

$$
\begin{aligned}
H_{\text{eff}} = {} & \frac{\alpha_1}{2a}\sum_x \bar{\psi}(x)\psi(x) + \frac{\alpha_2}{2a}\sum_{x,k}\bar{\psi}(x)U(x,k)\psi(x+k) \\
& + \frac{\alpha_3}{2a}\sum_{x,k}(2P - Q - Q^\dagger) + \frac{g^2}{2a}\sum_{y,j}E_j(y)E_j(y),
\end{aligned} \tag{4.3}
$$

where

$$\alpha_1 = \frac{d}{K_{N_c}(2b_1^2 + b_2^2)},$$

$$\alpha_2 = \frac{2b_1 b_2}{g^2 K_{N_c}},$$

$$\alpha_3 = \frac{b_1^2}{g^2 K_{N_c}}, \tag{4.4}$$

and

$$P = \frac{1}{N_c}\xi^\dagger_{C_1}(x)\sigma_k\eta^\dagger_{C_2}(x+k)\eta_{C_2}(x+k)\sigma_k\xi_{C_1}(x),$$

$$Q = \frac{1}{N_c}\eta_{C_1}(x)\sigma_k\xi_{C_2}(x+k)\eta_{C_2}(x+k)\sigma_k\xi_{C_1}(x), \tag{4.5}$$

with the color indices specified. In deriving Eq. (4.3), the two-link or more-link terms in the final step have been omitted (so the above approximation is referred as the one-link approximation), and as in [15,20] a linear approximation to the four fermion operators has been made. The first two terms in Eq. (4.3) are, respectively, the induced mass term and Wilson term. This is a very nice result of the improved theory: the original Hamiltonian is chiral invariant, while the Wilson term is induced by a unitary transformation and the addition of the next-to-nearest neighbor interactions. Based on this effective Hamiltonian, the decay of hadrons [21], mass splittings of ρ-ω and Δ-N would appear [20] even in the stronger coupling regime by means of one-link approximation. Here we pay special attention to the mass splitting of ρ-ω, and discuss in detail why the one-link approximation to the naive Hamiltonian fails in this case and how the one-link approximation to the improved Hamiltonian solves the problem.

We choose the creation operators for the mesons as

$$\begin{aligned} V^\dagger = {} & \xi^\dagger(x)\Gamma\eta^\dagger(x) + A\eta^\dagger(x)\Gamma\xi^\dagger(x) \\ & + B\xi^\dagger(x)\Gamma U(x,k)\eta^\dagger(x+k) \\ & + C\eta^\dagger(x)\Gamma U(x,k)\xi^\dagger(x+k), \end{aligned} \tag{4.6}$$

with

$$\Gamma_\pi = \tau_\alpha, \quad \Gamma_\rho = \tau_\alpha\sigma_i, \quad \Gamma_\omega = \sigma_i. \tag{4.7}$$

By evaluating the matrix elements as in Eq. (3.6), we obtain the eigenequations for the ρ meson in the one-link approximation,

$$M_\rho a = \alpha_1 + 6\alpha_2 B_\rho + \frac{\alpha_3}{3}A_\rho, \tag{4.8a}$$

$$M_\rho a A_\rho = -\alpha_1 A_\rho - 6\alpha_2 C_\rho - \frac{\alpha_3}{3}A_\rho, \tag{4.8b}$$

$$M_\rho a B_\rho = \alpha_1 B_\rho + \alpha_2 + \frac{2g^2}{3}B_\rho, \tag{4.8c}$$

$$M_\rho a C_\rho = -\alpha_1 C_\rho - \alpha_2 A_\rho + \frac{2g^2}{3}C_\rho, \tag{4.8d}$$

and those for the ω meson,

$$M_\omega a = \alpha_1 + 6\alpha_2 B_\omega + \frac{\alpha_3}{3}A_\omega, \tag{4.9a}$$

$$M_\omega a A_\omega = -\alpha_1 A_\omega - 6\alpha_2 C_\omega - \frac{\alpha_3}{3}, \tag{4.9b}$$

$$M_\omega a B_\omega = \left(\alpha_1 + \frac{2g^2}{3}\right)B_\omega + \alpha_2 + \frac{4\alpha_3}{9}(B_\omega - C_\omega), \tag{4.9c}$$

$$M_\omega a C_\omega = \left(-\alpha_1 + \frac{2g^2}{3}\right)C_\omega - \alpha_2 A_\omega - \frac{4\alpha_3}{9}(B_\omega - C_\omega). \tag{4.9d}$$

It is known what contributions actually cause such mass splittings in the real world: the valence-quark–antiquark pair can annihilate in the ω meson but not in the ρ meson because of the isospin matrix τ_α in Eq. (4.7). If we compare the above eigenequations for the ρ and ω mesons, we find that the differences between Eqs. (4.8c) and (4.9c) and those between Eqs. (4.8d) and (4.9d) come from the ω meson's nonvanishing matrix elements of such terms as P, Q, and $Q^\dagger$ in the Hamiltonian (4.3), which can produce the ρ-ω mass splitting since for the ρ meson, $\mathrm{Tr}(\tau_i) = 0$.

However, for the unimproved theory ($b_1 = 1$ and $b_2 = 0$), α_2 vanishes identically. In this case, according to Eqs. (4.8) and (4.9), the consistent solutions are

$$B_\rho = B_\omega = 0, \quad C_\rho = C_\omega = 0, \quad \frac{M_\rho}{M_\omega} = 1. \tag{4.10}$$

It is obviously seen that the consistency condition of the eigenequations forces the ρ and ω meson masses to be degenerate. The mass ratio M_ρ/M_ω is plotted in Fig. 4, where the experimental value (0.98) is represented by the dashed line and the result from the unimproved Hamiltonian with the one-link approximation by the circles.

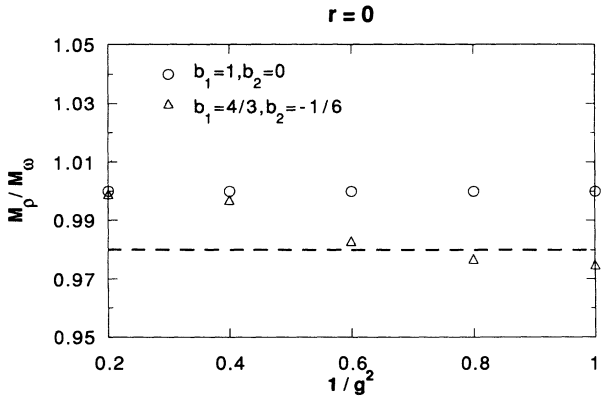

FIG. 4. M_ρ/M_ω as a function of $1/g^2$ for $r = 0$. The dashed line is the experimental value.

With the improved Hamiltonian ($b_1 = 4/3$ and $b_2 = -1/6$), α_2 is nonzero, and therefore the mass splitting indeed appears. Our numerical data are shown by the triangles in Fig. 4. One sees that even in the one-link approximation the results from the improved one are in good agreement with the experimental one in the strong and intermediate coupling regions.

Of course, these results should be regarded as preliminary ones, because it is still not clear about the effect of the magnetic fluctuations and scaling behavior of $M_\rho a$ and $M_\omega a$ beyond the strong coupling regime.

V. DISCUSSIONS

In this paper, we have proposed some improved Hamiltonians for lattice gauge theory with fermions to reduce the finite spacing effects in the strong or intermediate coupling region and then employed them to compute the mass spectrum of the Schwinger model and QCD. Some major results may be recapitulated here.

(1) Even for the Schwinger model with naive fermions ($r = 0$) in which only the kinetic Hamiltonian is modified, $M_v a/g$ shown in Fig. 1 enters the scaling region faster than that with the unimproved one, and the addition of the next-to-nearest neighbor interaction to the conventional one seems enough for naive fermions in the intermediate coupling region.

(2) In the Schwinger model with Wilson fermions, for $r = 1$, Fig. 3 tells us that the improved Hamiltonian gives much better result.

(3) Most apparently in Fig. 4, one can see the importance of using the improved Hamiltonian in (3+1)-dimensional QCD. With the unimproved Hamiltonian, the result (the circles) would completely disagree with experiment and there would be no mass splitting between the ρ and ω mesons in the one-link approximation. Therefore, the validity of the one-link approximation in this case is limited to the strong coupling regime and it requires more-gauge links to extend the validity of the approach to the weaker coupling. With the use of the improved one, the result (the triangles) is in agreement with experiment at least from $1/g^2 = 0.6$ through $1/g^2 = 1.0$ even by means of the one-link approximation.

To summarize, from the test of the Schwinger model and preliminary applications to QCD, we see that the improved Hamiltonians indeed lead to better results. Therefore, the use of the improved theory is a more economical and efficient procedure for significantly reducing the major source of systematic errors due to the finite a. Even from the naive fermion model, the successful improvement of the mass spectrum implies that the improved theory with Kogut-Susskind fermions would also work better than the original one.

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