

## Canonical quantization of (2+1)-dimensional gravity

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We consider the quantum dynamics of both open and closed two-dimensional universes with “wormholes” and particles. The wave function is given as a sum of freely propagating amplitudes, emitted from a network of mapping class images of the initial state. Interference between these amplitudes gives nontrivial scattering effects, formally analogous to the optical diffraction by a multidimensional grating; “bright lines” correspond to the most probable geometries.

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### I. INTRODUCTION

Roughly, 30 years after the first serious attempts at formulating a quantum theory of gravity [1], still very little is known about what such a theory might be, beyond the semiclassical approximation. It is nonrenormalizable as a quantum field theory, and there is a general feeling that one must seek a nonperturbative formulation of quantum gravity.

Unfortunately, the nonperturbative analysis of nonlinear dynamical systems is a notoriously difficult problem. Quantum gravity is far from the only case where nonperturbative effects are important. Other examples include some of the most pressing problems at hand, such as climatology, fusion physics, etc. Certainly, the challenge of developing new methods for this class of problems must be accepted. Theoretical work in the field has split along two currently nonintersecting trajectories: One deals with chaos; the other is related to knots (or, more generally, things that cannot be made to vanish by a succession of small deformations). Although both are studied in the context of gravity, knots appear to be most closely related to quantum gravity and the issue of the “small scale structure” of spacetime: Relations between quantum gravity and knots have appeared both in 3+1 dimensions [2] and in the model of (2+1)-dimensional gravity [3], which we will pursue in this paper.

Before we begin, it should be stressed that there is still no candidate for a consistent theory of “quantum gravity” or a proof that such a theory exists. On the other hand, numerous authors have suggested that (2+1)-dimensional gravity could be used as a simpler model, in which one could formulate a quantum theory and investigate issues of interpretation, the choice of time, etc. [4].

Witten’s work on the subject has led to a complete set of Heisenberg picture observables for quantum theory, an important step in the quantization program [5]. In spite of this, much remains to be done before one could say that quantum gravity in 2+1 dimensions has been

solved. The authors who have undertaken to follow up Witten’s work have encountered major obstacles in two different directions. Nelson and Regge, trying to make Witten’s reduced phase space explicit, found that they could do so up to genus 2 only by finding the ideal of a complicated system of relations among traces, some of which are cousins of the Cayley-Hamilton relations for  $SO(3)$  matrices [6]. On the other hand, the task of extracting the explicit time dependence of the observables (i.e., solving the Heisenberg equations for some choice of internal time) was solved in the case of genus-1 universes by Moncrief [7] and, in the context of Witten’s variables, by Carlip [8]. Interestingly, these authors differed as to what the Hamiltonian should be; in particular, Carlip’s proposed Hamiltonian was unbounded from below. Following Moncrief’s remarks to this effect, Carlip showed that his formulation led to a “Dirac square root” of the Wheeler-DeWitt equation, when one demands invariance of wave functions with respect to the mapping class group [9].

Our purpose in this paper is to pursue the quantization of (2+1)-dimensional gravity to the point of providing a complete set of Schrödinger observables, a Hamiltonian, a consistent choice of ordering of the operators, and, in computable form, wave functions for various types of scattering problems. We hope that this will help to formulate some of the fundamental problems of quantum gravity in terms of specific computable questions. We stress that some of these “problems” involve the very consistency of the formalism which we are proposing, and so we do not claim to have proven that the theory exists.

Many authors have worked on (2+1)-dimensional quantum gravity, and it sometimes seems that each one has a different approach. We will briefly list some of these approaches in terms of two main schools of thought. In 2+1 dimensions without continuous matter fields, there is only a finite number of independent observables. The issue is how to reduce the infinite dimensional phase space by exploiting the infinite number of gauge symmetries to obtain a finite-dimensional system which could be quan-

tized exactly. One approach is to fix the gauge by choosing a slicing of spacetime in constant curvature slices (York gauge [10]) or some other choice of slicing—we will call this the Arnowitt-Deser-Misner (ADM) school. The other relies on the fact that (2+1)-dimensional spacetimes can be regarded as a particular sheet of solutions (with “maximal Euler class”) of Chern-Simons gauge theory for an ISO(2,1) connection—the ISO(2,1) school.

## A. ADM school

### 1. Particles and cones

The geometry of the an  $\mathbb{R}^2$  universe with particles is a multicone, i.e., a flat surface with conical singularities, where the deficit angle at each singularity is related to the mass of the particle. The two-particle classical dynamics was solved explicitly by Deser, Jackiw, and 't Hooft [11], the quantum scattering of a light particle by a heavy one was worked out by Deser and Jackiw [12], and the general two-particle quantum scattering problem was solved by 't Hooft [13]. The three-particle classical problem was solved by Lancaster and Sasakura [14].

### 2. Riemann surfaces

The geometry is a genus- $g$  Riemann surface, from a constant extrinsic curvature foliation of spacetime. The surface coordinates are chosen so that the metric is conformally flat [26] or conformally hyperbolic [7]. To compute the Hamiltonian, one solves an equation for the conformal factor. In the first case, it is not a coordinate scalar, and so the equation must be solved on coordinate patches—it is not known whether it admits a solution for genus  $g > 1$ . In the conformally hyperbolic case, Moncrief finds a Lichnerowicz equation for the scalar conformal factor and proves the existence of a solution for all  $g$  [7]. The Schrödinger equation is easily written down for genus 1 [15], and the solutions can in principle be found from the modular invariant eigenfunctions of Laplacian on Teichmüller space, which are known as the Maass forms [16].

The virtue of the ADM approach is that it is most similar to (3+1)-dimensional gravity. Its drawback is that it encounters some technical problems that are most similar to those of (3+1)-dimensional gravity.

## B. ISO(2,1) school

### 1. Reduced phase space quantization

One finds a complete set of phase space observables which commute with the constraints. The reduced phase space is related to the moduli space of flat ISO(2,1) connections [5]. The problems begin when one attempts to parametrize this space and compute the Poisson brackets: One can form an infinite set of ISO(2,1) invariants

from the holonomy, but these are not independent variables. They are related by nonlinear constraints, some of which are cousins of the Cayley-Hamilton identities. Zertuche and co-workers [17] have succeeded in identifying the reduced phase space explicitly up to genus  $g = 2$ . The canonical quantization process leads, in some cases, to quantum groups [18]. The reduced phase space can be interpreted as a Hamilton-Jacobi formulation [19], and so the representations of the reduced algebra should give the wave functions in the Heisenberg picture. No attempt is made to compute the Hamiltonian or to formulate quantum dynamics.

### 2. Covariant quantization

One works with the ISO(2,1) “homotopies” along loops which form an arbitrarily chosen basis of  $\pi_1(\Sigma_g)$  and later imposes invariance with respect to a change of basis or mapping class transformation [20]. The difficulties encountered by Zertuche and co-workers can be avoided by using these homotopies rather than the invariants derived from them, leaving out a global ISO(2,1) symmetry. The homotopies can be parametrized in the “polygon representation” of (2+1)-dimensional gravity, and the symplectic structure is known [21]. Mapping class invariant scattering amplitudes can be computed by the method of images, by summing over all mapping class images of the “in” state. Two problems are encountered: Each term in the sum can only be computed if one knows the Hamiltonian for a given choice of time. The maximal slicing choice only allows one to compute the Hamiltonian explicitly only for genus  $g = 1$  (see ADM school). The other problem is that the sum is over the non-Abelian, infinite, mapping class group, and it is not clear *a priori* which terms might be small or if the sum converges.

### 3. Planar multiple polygons

Recently, 't Hooft has proposed another approach based on a slightly different polygon representation, where one chooses a piecewise flat surface, which leads to a number of planar polygons, with SO(2,1) identifications of the boundary edges. There is a remarkably simple symplectic structure and Hamiltonian for this system; since these variables are directly related to a choice of generators of the first homotopy group, the mapping class group acts nontrivially on the wave functions [22].

This paper can now be summarized in one line: We combine our earlier calculation of the Hamiltonian in the polygon representation with the method of images and the “stationary phase theorem” to advance along the covariant quantization program [23].

The reader is referred to the previous paper on the polygon representation of (2+1)-dimensional gravity for the canonical variables, the Hamiltonian, and most importantly, the relation between the polygon representation and the ISO(2,1) homotopies [19]. We will begin this paper directly with the canonical quantization in Sec. II. The mapping class group invariance is discussed in Sec.

III, where we write down invariant amplitudes for various types of scattering problems. In Sec. IV we give the conclusions and discuss the problem of time.

### II. QUANTIZATION IN THE POLYGON REPRESENTATION

In this section, we will draw heavily on results from the classical theory; their derivations can be found in [19] and references therein, and will not be repeated here. The reduced phase space of (2+1)-dimensional gravity can be parametrized by  $2g + N$  three-vectors  $E(\mu)$  and as many SO(2,1) matrices  $M(\mu)$ , with the following Poisson brackets and constraints ( $\mu = 1, 2, \dots, 2g + N$ ):

$$\{E^a(\mu), E^b(\mu)\} = \epsilon^{abc} E_c(\mu) , \tag{2.1}$$

$$\{E^a(\mu), M^b{}_c(\mu)\} = \epsilon^{abd} M_{dc}(\mu) , \tag{2.2}$$

$$\{M^a{}_b(\mu), M^c{}_d(\mu)\} = 0 . \tag{2.3}$$

The vectors  $E^a(\mu)$  together with their identified partners  $-M^{-1}(\mu)E(\mu)$  form a closed polygon

$$J \equiv \sum [I - M^{-1}(\mu)]E(\mu) \approx 0 , \tag{2.4}$$

and the cycle condition for the SO(2,1) identification matrices leads to the constraints

$$P^a \equiv \frac{1}{2} \epsilon^{abc} W_{cb} \approx 0 , \tag{2.5}$$

where

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$$W = [M(1)M^{-1}(2)M^{-1}(1)M(2)] \cdots [M(2g-1)M^{-1}(2g)M^{-1}(2g-1)M(2g)]M(2g+1)M(2g+2) \cdots M(2g+N) . \tag{2.6}$$

The constraints  $J^a$  and  $P^a$  generate a Poincaré algebra with the brackets (2.1)–(2.3). The masses of the  $N$  particles are given by the constraints

$$H(\mu) \equiv P^2(\mu) + \sin^2[\Omega(\mu)] \approx 0 , \tag{2.7}$$

which generate translations of each particle along its world line.

It is convenient to define the canonical variables  $P^a(\mu) = \frac{1}{2} \epsilon^{abc} M_{cb}(\mu)$  and

$$X(\mu) = \frac{1}{P^2(\mu)} \left( P(\mu) \wedge J(\mu) - \frac{2[E(\mu) \cdot P(\mu)]P(\mu)}{\text{tr}M(\mu) - 1} \right) , \tag{2.8}$$

where

$$J(\mu) = [I - M^{-1}(\mu)]E(\mu) . \tag{2.9}$$

The variables  $X(\mu)$  and  $P(\mu)$  have canonical Poisson brackets:

$$\{P_a(\mu), X^b(\mu)\} = \delta_a{}^b . \tag{2.10}$$

The constraints can be written in terms of the canonical variables, using  $J(\mu) = X(\mu) \wedge P(\mu)$  and

$$M^a{}_b = \delta^a{}_b + P_c \epsilon^{ca}{}_b + (\sqrt{1 + P^2} - 1) \left( \delta^a{}_b - \frac{P^a P_b}{P^2} \right) \tag{2.11}$$

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for  $\mu = 1, \dots, 2g$  [for which  $M(\mu)$  is a hyperbolic (a boost) [24]] or, for  $\mu = 2g + 1, \dots, 2g + N$ ,

$$M^a{}_b = \delta^a{}_b + P_c \epsilon^{ca}{}_b + (\sqrt{1 - P^2} - 1) \left( \delta^a{}_b - \frac{P^a P_b}{P^2} \right) . \tag{2.12}$$

The SO(2,1) constraints become

$$J \equiv \sum_{\mu} X(\mu) \wedge P(\mu) \approx 0 , \tag{2.13}$$

while the translation constraints  $P \approx 0$  are defined implicitly in terms of  $P(\mu)$  by (2.11) and (2.12). The explicit form of the function  $P(P(\mu)) \approx 0$  can be derived from relations (2.11) and (2.12).

We will assume the universe is a closed surface with genus  $g > 1$ ; for the other cases, the arguments below carry over if one substitutes for the “time” and “Hamiltonian,” the appropriate expressions drawn from the previous paper. We propose the following choice of “internal time” in a frame where  $M(1)$  is a boost in the  $(yt)$  plane:

$$T = - \frac{E^x(1)}{P^x(1)} . \tag{2.14}$$

The Hamiltonian is given in terms of the variables  $M(\mu)$ ,  $\mu > 1$ , by solving the constraint

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$$W \equiv [M(1)M^{-1}(2)M^{-1}(1)M(2)] \cdots [M(2g-1)M^{-1}(2g)M^{-1}(2g-1)M(2g)] \approx I \tag{2.15}$$

for  $\text{tr}[M(1)]$ , where  $M(1)$  is a boost of magnitude  $b$  with axis  $x$ . One finds

$$H = \frac{P_t(2)Q_t - P_y(2)Q_y}{P_y^2(2) - P_t^2(2)}, \quad (2.16)$$

where

$$Q^a = \frac{1}{2}\epsilon^{abc}\{M(2)[M(3)M^{-1}(4)M^{-1}(3)M(4)] \cdots \\ \times [M(2g-1)M^{-1}(2g)M^{-1}(2g-1)M(2g)]\}_{cb}. \quad (2.17)$$

There are two residual "translation" constraints and one "boost" constraint. The former come from

$$M(1)M^{-1}(2)M^{-1}(1) \approx [M(2)M(3)M^{-1}(4)M^{-1}(3) \\ \times M(4) \cdots M(2g)]^{-1} :$$

$$T = \frac{X^x(2g+1)}{P^x(2g+1)}, \quad (2.21)$$

$$P^x \equiv \epsilon^{abc}[M(2g+3) \cdots M(2g+N)M(1) \cdots M(2g)M(2g+1)]_{cb}^{-1} \approx 0, \quad (2.22)$$

$$P^y \equiv \epsilon^{ybc}[M(2g+3) \cdots M(2g+N)M(1) \cdots M(2g)M(2g+1)]_{cb}^{-1} \approx 0, \quad (2.23)$$

$$H = \text{Tr}[M(2g+2) \cdots M(2g+N)M(1)M^{-1}(2) \cdots M^{-1}(2g-1)M(2g)]. \quad (2.24)$$

Note that the internal times (2.14) and (2.21) have a linear time evolution, as well as any choice which is linear in the  $E$ 's, since  $d^2 E(\mu)/dt=0$ . Such times are also "inertial" in the sense that the position of a particle as seen from the observer,  $E(\mu)$ , follows a straight trajectory at constant "velocity."

We begin by constructing the quantum theory in the most naive way and then pick up the problems as they appear (see the mapping class group, next section). We quantize the canonical phase space  $X(\mu)$ ,  $P(\mu)$  in the usual way ( $\mu = 2, 3, \dots, 2g+N$ ):

$$P_a(\mu) = -i \frac{\partial}{\partial X^a(\mu)}, \quad (2.25)$$

$$[P_a(\mu), X^b(\mu)] = -i\delta_a^b. \quad (2.26)$$

To prove the equivalence of this quantization to one which could be carried out in the variables  $E(\mu)$ ,  $M(\mu)$ , one must express these variables in terms of  $X(\mu)$  and  $P(\mu)$  and show that there exists an ordering of these expressions such that the commutators form a representation of the algebra (2.1)–(2.3). An acceptable ordering is obtained by placing all the  $X$ 's on the right:

$$P^x(2) - Q^x \approx 0, \quad (2.18)$$

$$P^2(2) - Q^2 \approx 0. \quad (2.19)$$

The boost generator is

$$J^x \equiv \sum_{\mu>1} [I - M^{-1}(\mu)]_a^x \times E^a(\mu) \approx 0. \quad (2.20)$$

The remaining phase space variables are  $E(\mu)$ ,  $M(\mu)$  for  $\mu = 2, \dots, 2g$ . Their time evolution is generated by the Hamiltonian (2.16) and any linear combination of the first-class constraints (2.18)–(2.20), which commute with the Hamiltonian.

If  $N > 1$ , one can choose a gauge where the vector  $X(2g+1)$  is aligned with the  $x$  axis and  $X(2g+2)$  vanishes at all times;  $M(2g+2)$  is then a pure rotation with angle  $\Omega(2g+2)$ , and

$$E(\mu) = \frac{M^{1/2}}{P\sqrt{2\sqrt{P^2+1}-2}} P \wedge (P \wedge X) \\ - \frac{\sqrt{P^2+1}}{P^2} P(P \cdot X), \quad (2.27)$$

where the index  $\mu = 2, \dots, 2g+N$  has been omitted in the right-hand side expressions, and  $M^{1/2}$ , the "square root" of  $M(\mu)$ , was given explicitly in the previous paper [Eqs. (6.7) and (6.9)].

As mentioned, the ordering of the operators in (2.27) leads to the correct commutator algebra for  $E(\mu)$ ,  $M(\mu)$ . For example, using  $P^a(\mu) = -i\eta^{ab}\partial/\partial X^b(\mu)$ , one finds  $[E^a(\mu), E^b(\mu)] = -i\epsilon^{abc}E_c(\mu)$ . This solution to the ordering problem is not unique.

The Wheeler-DeWitt quantization would consist in taking the set of square-integrable functions on  $\mathbb{R}^{3 \times (2g+N)}$ , which are annihilated by the constraint operators

$$\psi = \psi(\{X(\mu), \mu = 1, \dots, 2g+N\}), \quad (2.28)$$

$$J^a \psi = 0, \quad (2.29)$$

$$P^a \psi = 0. \quad (2.30)$$

Equation (2.29) states that  $\psi$  must be a Lorentz scalar

in the variables  $X(\mu)$ . The other constraints are the Wheeler-DeWitt equations. These can in principle be obtained explicitly from (2.11) and (2.12) by replacing everywhere  $P_a(\mu)$  by the operator  $-i\hbar\partial/\partial X^a(\mu)$ . The result is a nonrational function of the derivatives, not likely to give a physically acceptable quantum theory. For this reason, we will prefer to partly fix the gauge, as indicated above, by choosing the internal time (2.21)

and the corresponding Hamiltonian (2.24) and quantize in the Schrödinger picture:

$$i\hbar\frac{\partial\psi}{\partial T} = H(\{P(\mu), \mu = 2, \dots, 2g + N\})\psi. \quad (2.31)$$

One must still impose the two “momentum constraints” (2.22) and (2.23),

$$\epsilon^{abc}[M(2g+3)\cdots M(2g+N)M(1)\cdots M(2g)M(2g+1)]_{cb}^{-1}\psi = 0, \quad (2.32)$$

$$\epsilon^{ybc}[M(2g+3)\cdots M(2g+N)M(1)\cdots M(2g)M(2g+1)]_{cb}^{-1}\psi = 0, \quad (2.33)$$

and require that the wave function be a Lorentz scalar.

The solution to the Schrödinger equation, for an initial state  $\psi(X(\mu), 0)$ ,  $\mu = 2, 3, \dots, 2g + N$ , is given by

$$\psi(\{X(\mu)\}; T) = e^{iTH\{-i\partial/\partial X(\mu)\}}\psi(\{X(\mu)\}; 0). \quad (2.34)$$

The Hamiltonian operator is defined by the Fourier decomposition

$$e^{iTH\{-i\partial/\partial X(\mu)\}}e^{iK_0\cdot X} = e^{iTH(K_0)}. \quad (2.35)$$

Finding initial states which satisfy the constraints is not difficult in principle. Only the constraints (2.32) and (2.33), which depend on the momenta in a nonpolynomial way, appear to present a difficulty, but in  $P$  space these become algebraic relations, which summarize the cycle conditions, for which explicit solutions can be constructed (this is not always easy).

### III. MAPPING CLASS GROUP AND QUANTUM SCATTERING

The preliminary theory of (2+1)-dimensional quantum gravity, which we constructed in Sec. II, is formally similar to the nonrelativistic quantum mechanics of a set of free particles propagating in a flat three-dimensional background, but with an unusual “kinetic energy” function. This noninteracting picture can be understood as follows: We have defined the polygon variables by cutting up and unfolding the manifold to get a polygon embedded in Minkowski space. In this classical picture, the corners of the polygon follow straight timelike lines—so indeed they behave very much like free particles.

#### A. Mapping class group

The variables which we are using are associated with a particular choice of generators of the fundamental group for a genus- $g$  surface with  $N$  punctures. A different choice of generators can lead to a different set of variables for the same spacetime. The group of transformations from one set of generators to another, called “mapping class group,” is a discrete symmetry of the polygon representation. One must demand that the wave function be mapping class invariant and this leads to interesting “scattering” effects [23].

Before we begin to construct invariant wave functions, we will review the mapping class group and its action on the polygon variables, including “time” and the Hamiltonian. We must also make sure that our choice of operator ordering is consistent with this symmetry group.

The mapping class group is presented by the following generators [25] (the loops  $a_i b_i$  are a standard basis, where each  $a_i$  intersects only the corresponding  $b_i$  at only one point;  $\tau_{u_i}, \tau_{y_i}, \tau_{z_i}$  are Dehn twists [20]):

$$\tau_{u_i} : b_i \rightarrow b_i a_i, \quad (3.1)$$

$$\tau_{y_i} : a_i \rightarrow a_i b_i^{-1}, \quad (3.2)$$

$$\tau_{z_i} : \begin{cases} a_i \rightarrow a_i b_i^{-1} a_{i+1} b_{i+1} a_{i+1}^{-1}, & (3.3) \\ b_i \rightarrow a_{i+1} b_{i+1}^{-1} a_{i+1}^{-1} b_i a_{i+1} b_{i+1} a_{i+1}^{-1}, & (3.4) \\ a_{i+1} \rightarrow a_{i+1} b_{i+1}^{-1} a_{i+1}^{-1} b_i a_{i+1}. & (3.5) \end{cases}$$

For genus  $g=2$ , the representation of the elements  $\{a_i, b_i\}$  in ISO(2,1) was given in [19] (with the change in notation  $u_i \rightarrow a_i, v_i \rightarrow b_i$ ). For example,

$$\rho(a_1) = \begin{pmatrix} M(1) & [I - M(1)]E(1) - M(1)E(2) \\ & + [I - M(1)]OA \\ 0 & 1 \end{pmatrix}, \quad (3.6)$$

$$\rho(b_1) = \begin{pmatrix} M^{-1}(2) & [I - M^{-1}(2) - M^{-1}(1)]E(1) \\ & + [I - M^{-1}(2)]E(2) + [I - M^{-1}(2)]OA \\ 0 & 1 \end{pmatrix}, \quad (3.7a)$$

so that we find, applying Eq. (3.1), that  $\rho(a_1)$  is unchanged under  $\tau_{u_i}$  and  $\rho(b_1)$  becomes

$$\rho(b_1) \rightarrow \begin{pmatrix} M^{-1}(2)M(1) & [I - M^{-1}(1) - M^{-1}(2)M(1)]E(1) \\ & + [I - M^{-1}(2) - M^{-1}(2)M(1)]E(2) \\ & + [I - M^{-1}(2)M(1)]OA \\ 0 & 1 \end{pmatrix}. \quad (3.7b)$$

By comparing this to expression (3.7a), one can deduce the action of the mapping class transformation on the variables  $E(\mu)$  and  $M(\mu)$  (this is not easy):

$$\tau_{u_i} : \begin{cases} E(1) \rightarrow E(1) + E(2), & (3.8) \\ E(2) \rightarrow M^{-1}(1)E(2), & (3.9) \\ M(1) \rightarrow M(1), & (3.10) \\ M(2) \rightarrow M^{-1}(1)M(2), & (3.11) \end{cases}$$

and similarly for  $\mu=3$  and 4 and for any other element of the mapping class group. These relations can also be derived from the polygon representation of the variables  $E(\mu)$  and  $M(\mu)$  (Fig. 1). The loop  $a_1$  is a loop starting at  $O$ , going through the segment  $E(1)$  at the point  $X$ , through  $M^{-1}(1)E(1)$  at  $X'$ , and back to  $O$ . Likewise,  $b_1$  starts at  $O$ , goes through the segment  $M^{-1}(2)E(2)$  at  $Y'$ , then continues through  $Y$ , and returns to  $O$ . Now one can deform the combined loop  $b_1 a_1$  smoothly into the loop which is represented in Fig. 1, where a small semicircle surrounds the corner  $B$  of the polygon. One can then choose a new set of cuts which avoids the corner  $B$ : The first cut is now  $E(1) + E(2)$  (dashed line), while the second cut is as before, but in a frame which has been transported around the loop  $E(2)$  once:  $E(2) \rightarrow M^{-1}(1)E(2)$ . Any mapping class group transformation can be obtained in a similar fashion from the polygon representation by sliding one corner of the polygon around a loop (a Dehn twist).

Mapping class transformations leave the cycle conditions invariant, since they are automorphisms of  $\pi_1(\Sigma)$ . This implies that the Hamiltonian, which was computed in the previous section by solving the cycle conditions

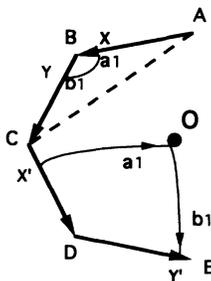


FIG. 1. The loop  $a_1 b_1$  crosses twice through the edges of the original polygon at  $X = X'$  and  $Y = Y'$ . By recutting the surface in a different way prior to unwrapping it as a polygon, one obtains a first cut which goes directly from  $A$  to  $C$  (dotted line), so that the loop  $a_1 b_1$  crosses only once an edge of the new polygon, exactly in the same way as  $a_1$  crossed one edge of the original polygon. The switch from one set of cuts to another, and from  $a_1$  to  $a_1 b_1$ , is an example of a mapping class transformation.

in a frame where  $M(1)$  is a pure  $x$  boost, is invariant as long as the frame is adjusted so that this condition is preserved. We will thus choose the frame independently for each term in the sum over the mapping class group (next section). This requires adjusting the initial and final conditions appropriately by an overall  $SO(2,1)$  transformation. Time, defined as the variable canonically conjugate to this Hamiltonian, is mapping class invariant up to a constant.

One must check that the mapping class group action is consistent with the choice of operator ordering. If one chooses to place the  $E$ 's and  $X$ 's on the right, the bracket algebra, as well as the relations that give  $X(\mu)$  as a function of  $E(\mu)$ , are preserved under the mapping class group transformations (this can be checked explicitly and derives from the fact that all the relations are linear and homogeneous in  $E$  and  $X$ ). This differs significantly from the situation in York's extrinsic gauge, where the task of finding an operator ordering consistent with the mapping class group implies that one consider multivalued wave functions, with a modular weight which depends on the ordering chosen [9]. We feel that this implication is due to the choice of slicing and is not a fundamental property of (2+1)-dimensional gravity. Indeed, the polygon variables do not exclude the possibility of choosing the extrinsic curvature slicing; rather, they do not specify any foliation: We have decoupled the slicing (gauge) from the global variables (observable) and simply dropped the former. The implication discovered by Carlip comes from the difficulty to find a consistent operator ordering for the relations which give the ADM operators in terms of the quantized  $ISO(2,1)$  homotopies.

## B. Invariant wave functions: The method of images

One must find mapping class invariant wave functions which solve the Schrödinger equation. We first review what has already been achieved in both the ADM and  $ISO(2,1)$  schools. In the ADM approach [7,26], the Hilbert space is the space of modular-invariant square-integrable functions over Teichmüller space. To find the solutions to the Schrödinger equation, one considers the eigenfunctions of the Hamiltonian operator. The Hamiltonian is not yet known for genus greater than 1 (one must solve a Lichnerowicz equation), but it is expected that if a solution could be found, then the ordering problems would be intractable. This means that one is probably limited to the case of the torus, where the Hamiltonian is known and the wave functions are the Maass forms [16,32].

In the  $ISO(2,1)$  approach, one computes the amplitude for the scattering of  $N$  particles on a plane as a sum of

amplitudes over all possible braidings of the world lines of the particles; if one is dealing with a compact universe with handles and punctures, then one sums over the mapping class group, rather than the braid group. Each term in this sum is a path integral with the Chern-Simons action, over all Poincaré connections on three-manifolds with fixed “initial” and “final” conditions, and a given topology and mapping (or “braiding,” for particles). Besides the difficulty of computing the path integral with fixed boundary conditions (which is related to the difficulty of finding the Hamiltonian), one must deal with an infinite sum over a non-Abelian group; it is not known whether the sum can be ordered as a converging series or if the result is computable.

In both approaches, one can separate the task at hand into two subtasks: (1) the dynamical problem (finding the Hamiltonian or computing a sum over histories with initial and final conditions); (2) the mapping class problem (finding modular-invariant eigenstates of the Hamiltonian operator or performing the sum over the mapping class group).

Both the dynamical and the mapping class problems have been solved for the scattering of two particles on  $\mathbb{R}^2$  [13], and the Hamiltonian is known for the quantum torus [7], with work under way on the mapping class problem [32].

So far in this paper, we have solved the dynamical problem in the general case of a genus- $g$  surface with  $N$  punctures. In what follows, we will attempt to provide a solution to the mapping class problem by constructing the invariant wave functions as a series of computable terms. We will give a procedure to decide on which terms [or elements of the mapping class group (MCG)] give a significant contribution to the amplitude and argue that the number of such terms is finite.

We will consider the amplitude to go from an initial state  $|X_1\rangle$ , which we choose to be an eigenstate of the “position” operators at  $T_1=0$   $\{X^\alpha(\mu), \mu = 2, \dots, 2g + N\}$ , to a final state  $|X_2\rangle$  at  $T_2 > 0$ . Neither of these states is mapping class invariant, but one can construct an invariant states  $|X\rangle_{\text{inv}}$  by summing over all mapping class images of  $X$ :

$$|X\rangle_{\text{inv}} \sim \sum_{i \in \{\text{MCG}\}} |\rho(g_i)X\rangle. \tag{3.12}$$

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$$\langle 1|2\rangle \sim \sum_{i,j \in \{\text{MCG}\}} \langle \rho(g_j)(X_2, T_2) | e^{iH(T_2 - T_1)} | \rho(g_i)(X_1, T_1) \rangle \tag{3.15}$$

$$\sim \sum_{i,j \in \{\text{MCG}\}} \langle X_2, T_2 | e^{-iH(T_2 - T_1^{-j})} | \rho(g_j^{-1})\rho(g_i)(X_1, T_1) \rangle, \tag{3.16}$$

where  $\{X_2^j, T_2^j\} = \rho(g_j)\{X_2, T_2\}$ , etc. Finally, since no loop of  $\pi_1(\Sigma)$  is privileged with respect to any other, the products  $(g_j^{-1}g_i)$  will cover the mapping class group uniformly, and absorbing the infinite factor equal to the cardinality of the group,

This eigenstate has infinite norm, equal to the cardinality of the mapping class group. We will use the formal expression (3.12) as a starting point and derive physically acceptable wave functions below.

### C. Scattering amplitudes

Even though the spacetimes considered here are not asymptotically flat, one can define noninteracting “in” and “out” states as follows. Since there is no potential energy term in the Schrödinger equation, the system is noninteracting as long as the mapping class group symmetry is not taken into account. It is easy to see from the construction of invariant states in (3.12) how the mapping class group affects this noninteracting picture: A realistic state might be a superposition of position eigenstates (3.12), with some variance  $\sigma^2$ . Thus the support of each term in the sum is a fuzzy patch centered at  $\rho(g_i)X$ . If the patches for various mapping class images overlap, then one expects to pick up interference terms—this is the essence of “interactions” in (2+1)-dimensional gravity. We will define “in” and “out” states by demanding that such overlaps do not occur—practically, this requires that the  $X(\mu)$  be “sufficiently large” so that for every element  $g_i$  of the mapping class group the image of  $X(\mu)$  under  $g_i$  lies in the tail of the wave packet, which is exponentially suppressed:

$$\forall \{\mu, i\}, \| X(\mu) - \rho(g_i)X(\mu) \|^2 \gg \sigma^2. \tag{3.13}$$

This criterion depends on the variance of the wave packet; one can get a more universal condition if one assumes that the variance is of order  $\hbar^2$ . In units  $\hbar = 1$  this gives

$$\forall \{\mu, i\}, \| X(\mu) - \rho(g_i)X(\mu) \|^2 \gg 1. \tag{3.14}$$

We are now ready to calculate the scattering amplitude: Taking two asymptotic invariant states (3.12) and the Hamiltonian (2.24) and recalling that the mapping class group leaves  $H$  invariant and translates  $T$  by a constant, the amplitude is just

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$$\langle 1|2\rangle \sim \sum_{i \in \{\text{MCG}\}} \langle X_2, T_2 | e^{-iH(T_2 - T_1)} | \rho(g_i)(X_1, T_1) \rangle. \tag{3.17}$$

Thus the amplitude at  $\{X_2, T_2\}$  is a sum over the

contribution of each mapping class image of the source,  $\{X_1^i, T_1^i\}$ , propagated to  $\{X_2, T_2\}$  by the “free-particle” Hamiltonian  $H(P(\mu), \mu = 2, \dots, 2g + N)$ . Note that the sum is over an infinite group so the expression (3.17) is not necessarily well defined. We will argue below that (3.17) is computable as a distribution on a certain limited set of wave functions  $\psi(\{X_1, T_1\})$ ; it would be very interesting to identify an appropriate space of test functions for which the distribution (3.17) is well defined, but this lies beyond the scope of this paper—the difficulties arise because the mapping class group does not act properly discontinuously on the space of polygons.

#### D. Stationary phase theorem and the sum over mapping class images

We will consider an initial wave packet centered at a suitable  $X_1^0$ , where by “suitable” we mean that the images  $\{X_1^i, T_1^i\}$  have no accumulation point ( $\exists \epsilon : \forall i, \sum_{\mu} [X(\mu) - \rho(g_i)X(\mu)]^2 > \epsilon$ ). The orientation of the “beam” is defined in a frame-independent way, to get an  $SO(2,1)$ -invariant wave function. We further require that the wave packet  $a(K)$  have support only over  $P$ 's which correspond to solutions of the cycle conditions which form a faithful representation of the fundamental group [5,21] (other sheets of solutions include totally collapsed handles or curvature singularities with a surplus angle equal to a multiple of  $4\pi$ ):

$$\psi(X_1, T_1) = \int dK a(K) e^{iK \cdot (X_1 - X_1^0)}, \quad (3.18)$$

where

$$K = \{P^\alpha(\mu), \mu = 2, 3, \dots, 2g + N\}, \quad (3.19)$$

$$dK = \prod_{\mu} d^3 P(\mu). \quad (3.20)$$

The wave function at  $(X_2, T_2)$  is given by propagating the initial state (3.18) with the invariant propagator (3.17). With the notation  $\omega_k = H(K)$  and the definition

$$\{X_{(i)}, T_{(i)}\} = \rho(g_i)(X_1, T_1) - (X_2, T_2), \quad (3.21)$$

the wave function becomes

$$\psi(X_2, T_2) \sim \sum_{i \in \{\text{MCG}\}} \int dK a(K) e^{i(K \cdot X_{(i)} - \omega_k T_{(i)})}. \quad (3.22)$$

The stationary phase argument implies that the only contributions to the integral are from values of  $K$  where the phase varies slowly as compared to  $a(K)$ ; this picks out the trajectories near the classical solutions:

$$X_{(i)} = \frac{\partial \omega_k}{\partial K} T_{(i)}, \quad (3.23)$$

where the first factor on the right-hand-side (RHS) is

the group velocity of the packet, which can be calculated explicitly given the Hamiltonian (2.24). In the scattering problem, the initial state, final state, and  $i$  are fixed, so that (3.23) can be solved for  $K$ ; let us denote this solution by  $K(i)$ . The only elements of the mapping class group which give a large contribution to the sum (3.22) are such that  $K(i)$  is not in the exponentially suppressed tail of the wave packet:  $a(K(i)) > \epsilon$  for some appropriately chosen  $\epsilon$ . Since we have assumed that the images of the source have no accumulation point, only a finite number of terms will contribute and the wave function (3.22) has a finite norm. We have examined a number of specific examples to various degrees of detail and found that the expression (3.22) is computable for initial data peaked about sufficiently nonsingular polygons. One encounters the following situations.

(1) If the images are widely spaced and their emissions do not interfere, only one of the classical trajectories  $K(i)$  lies within the wave packet  $a(K)$ ; one recovers the free propagator. This occurs when the images  $X(i)$  are sufficiently distant for each other as compared to the distance between  $X_2$  and  $X_1$ , so that the images  $K(j)$  different from  $K(i)$  do not lie within the support of the wave packet  $a(K)$ . There is no scattering in this case.

(2) Two classical trajectories connect the initial and final states, such as with geodesics on a cone (two-particle scattering). If the initial wave packet is wide enough, e.g., includes both  $P(1)$  and  $M(2)P(1)$ , then the scattering amplitude involves the interference between these two terms, as in a two-slit experiment. Other terms in the sum, such as  $[M(2)]^n P(1)$  for  $n > 1$ , correspond to the events where one particle winds around the other  $n$  times—the corresponding  $K$  may lie within the packet  $a(K)$  for small enough  $n$ , but these trajectories are not “nearly classical” and will contribute little, by the stationary phase argument. According to 't Hooft's more complete analysis [13], such terms are suppressed by a factor of order  $(\epsilon/2\pi)^{4n}$ , where  $\epsilon$  is the deficit angle at the conical singularity.

(3) One of the braid generators of the mapping class group (winding around a small loop) acts as a small translation, so that one picks up interference from a large number of images spaced on a line: This amplitude is related to scattering of light by a grating and describes the quantum dynamics of a thin wormhole. If one of the mapping class generators  $g_\alpha$  modifies  $K(i)$  very little, where  $K(i)$  is a classical solution which lies within the packet  $a(k)$ , then by applying this generator again and again one obtains a sequence of contributions from  $K_{(j_n)}$ , with  $g_{(j_n)} = (g_\alpha)^n g(i)$ . The series (3.22) may be exactly summable in this case; in the example of the torus, the result should be related to the Maass forms [16].

(4) Various mapping class generators contribute significantly to the sum: In the general case where more than one generator has “almost nonproper” action on  $K$ , one must consider the interference of the amplitudes propagating freely from a large number of images, which are distributed in a complicated fashion. This corresponds to a multiple scattering amplitude, and the expression (3.22) is related to the scattering by a complicated multidimensional “grating.”

### E. Quantum gravity near the “big bang” and through the “big bounce”

Besides the scattering from asymptotically free states, another problem of interest is the (2+1)-dimensional equivalent of the “big bang.” Many solutions of the classical theory expand from an initial singularity (although smoothly, unlike in 3+1 dimensions). For  $N=0$  and genus  $g \geq 2$ , all solutions have either an initial or a final singularity [27]. This brings up a very interesting situation: As one approaches the singularity, the topological features are increasingly tiny and a wave packet of given variance will contain an increasing amount of mapping class images. This means that the number of terms in the invariant wave function (3.22) increases, and it becomes less and less reasonable to interpret the result in the semiclassical picture by referring to a background geometry with interference terms acting as effective interactions. The “geometry” becomes completely fuzzy, but the wave function can still be calculated—this gives one a window on the nonperturbative regime, and small scale structure, of a theory of quantum geometry.

The wave function (3.22) is probably computable up to any desired level of accuracy at a given point away from the singularity, although the number of terms which must be considered in the sum can be large if one is close to the initial singularity. Short of actually performing this calculation, one can draw a few tentative conclusions. The increasing density of mapping class images as one approaches the big bang indicates that the inevitable anisotropy of classical universes with nontrivial topology is likely to be smoothed out: Any event which occurs before the quantum gravity effects lose significance is likely to appear isotropic to a future observer, even in a topologically nontrivial cosmology. By the same argument, it appears plausible that the wave function at  $\{X, T\}$  away from the big bang would depend very little on the precise initial condition on  $\psi(X_1, T_1)$ . This would be very convenient, since it eliminates the problem of having to specify an initial condition for the very early universe; it places all the burden of selecting among the possible histories on the measurements which are presumably performed later on and collapse the wave function to something resembling a classical background geometry.

The last compact universe problem which we wish to discuss is the “big bounce.” The initial state is a wave packet centered about a contracting universe, which would classically contract to a singularity and then expand on the other side. The semiclassical picture breaks down as one approaches the singularity, since the wave function becomes a superposition of an increasingly large number of mapping class images and the wave function extends to cover all possible universes with the given topology. It is impossible to think in terms of a “geometry” in this region, yet interference patterns from the mapping class images are likely to create “dark regions” and “bright regions” in the space of geometries. The large number of mapping class images is likely to lead to randomization of the wave packet and a very nonlocalized wave function on the other side, as in the previous case. One arrives at a similar conclusion if one considers

the chaotic behavior of particles close to the singularity [28], together with the finite spread of the initial wave packet.

### F. Scattering of particles and wormholes on $\mathbb{R}^2$

If the universe has the topology  $\mathbb{R}^2$  with  $g$  handles and  $N$  punctures, the amplitudes are calculated in much the same way, but the Hamiltonian function is simpler. The geometry at infinity is that of a cone with a helical shift, and the axis of the cone defines the direction of the “time” vector. The deficit angle of the conical geometry gives the total energy  $H$ , and the helical shift corresponds to the total angular momentum [29,11]. The cycle condition for closed topologies is replaced by a single equation, which gives the overall holonomy for a loop which goes around the circle at infinity; we will consider only wave functions with support limited by the condition that this overall holonomy be a rotation, since otherwise closed timelike curves occur in the semiclassical region (this implies that there is at least one particle, since wormholes are always tachyonic [21]). The angle of this rotation is the Hamiltonian

$$H = \arcsin(P) , \quad (3.24)$$

where

$$P^a = \frac{1}{2} \epsilon^{abc} [M(1)M^{-1}(2)M^{-1}(1)M(2) \cdots M(2g+N)]_{cb} . \quad (3.25)$$

This function has been written down explicitly as a function of the momenta for two particles [11] and three particles [14] and can be calculated, with an increasing amount of work, for any number of particles. Note that there is an ambiguity in the definition of the Hamiltonian, which is only given modulo  $2\pi$ . This ambiguity is related to the fact that the holonomies are representations of the fundamental group in the covering group of  $SO(2,1)$ . It is usual to restrict one’s attention to the case where the holonomy  $M(\mu)$  associated with each individual particle ( $\mu = 2g + 1, \dots, 2g + N$ ) lies on the component simply connected to the identity and similarly for the overall holonomy at infinity—this corresponds to particles of reasonably small mass [21] and so-called “physically reasonable” conditions at infinity [30]; the consequence of these assumptions is that for the  $g=0$  case one can show that there exists a global spacelike slicing (no time machines) [31]; this fact was first stated without proof in [11]. We will enforce these conditions by choosing wave packets with support limited to these “reasonable geometries” for  $g=0$ . In the general case, there is no avoiding this ambiguity, and one must decide if the wave function is multivalued or if it transforms nontrivially under a  $2\pi$  rotation [23]. As before, we will make the choice of a scalar wave function and postulate

$$\Psi(X, T) \sim \sum_{i \in \{\text{MCG}\}} \int dK a(K) e^{i(K \cdot X_{(i)} - \omega_k T_{(i)})} . \quad (3.26)$$

This expression is similar to Carlip’s proposal for ex-

act scattering amplitudes [23], the only difference being that we are considering wave packets rather than the pure propagator. This allows us to appeal to the stationary phase theorem and argue that the sum is limited to a finite number of mapping class images, which are such that the classical trajectory which connects an image to  $\{X, T\}$  corresponds to a momentum  $K(i)$  which lies within the support of the packet  $a(K)$ . It would be very interesting to see an explicit solution of the three-particle scattering problem in a specific situation by using (3.28) and the expanded expression for the Hamiltonian [14].

#### IV. CONCLUSION: THE “PROBLEM OF TIME”

In this paper, we have developed the covariant quantization program which was initiated by Witten and Carlip using the “polygon representation” of the reduced phase space. In this representation, classical (2+1)-dimensional gravity appears formally similar to a set of free particles propagating in  $\mathbb{R}^{2+1}$ . We gave the explicit expression for the Hamiltonian function and quantized the theory canonically. The mapping-class-invariant wave function can be written as a sum of freely propagating amplitudes, which each term represents the propagation from one mapping class image of the initial state. Interactions occur when these amplitudes interfere. This can produce interference patterns analogous to either the two-slit experiment or the diffraction by a regular grating, in some special cases.

It is not obvious that the sum over mapping class images is computable for wave functions which have support over singular geometries. Since singular geometries occur rather generically in (2+1)-dimensional gravity, it would be of great interest to prove that the sum is well defined for a specific Hilbert space of wave functions.

Also of interest is the opportunity to compare this quantum theory for the torus ( $g = 1$ ) to other author’s results in the extrinsic time. Joining our results and those of Carlip [9], we have indirectly shown that the theories are closely related if one considers only *static* issues, since it was shown that the ADM, Witten-Carlip, and polygon variables are different representations of the same reduced phase space and that the relations between these representations are consistent with the operator orderings chosen [9]. However, it is not clear whether the quantum dynamical theories, given by Schrödinger’s equation in York’s time or in our internal time  $T$ , are equivalent. We will argue below that they are not, but first we must define what we mean by “equivalent.” We will use the term “observable” in the traditional sense of quantum mechanics, without requiring that they should be constants of the motion. An example of an observable might be, e.g.,  $L_\mu^2 = E^2(\mu)$ .

*Definition.* Two quantum systems, described by the wave functions  $\psi$  and  $\psi'$  in times  $t$  and  $t'$ , are said to be “equivalent” if for any set of commuting observables  $A_i$  ( $i = 1, \dots, N$ ) the expectation values  $\langle A_i \rangle_t$  and  $\langle A_i \rangle_{t'}$  follow the same trajectories in  $\mathbb{R}^N$ .

Let  $\{C_i, H\}$  be a complete set of commuting observ-

ables (CSCO) where  $H$  is the Hamiltonian for time  $t$ , and let  $K$  be the Hamiltonian for time  $t'$ . The wave functions  $\psi$  and  $\psi'$  can be expanded in a basis of eigenstates of the commuting observables. We now explain why the theories of (2+1)-dimensional gravity in different times are not equivalent.

(1)  $K$  must be diagonal in this basis (if it were not, then the expectation values of the observables in the states  $\psi'$  would not be constant and the equivalence criterion would be violated).

(2) The initial state  $\psi'(t' = 0)$  must have the same probability for each CSCO eigenstate as  $\psi(t = 0)$  (a trivial consequence of the definition above; set  $C_i = A_i$ ).

(3) The wave functions  $\psi(t)$  and  $\psi'(t')$  differ only by the phase factor of each coefficient in the expansion in the basis of eigenstates. The time dependences of these phases in the two formulations are given by  $\phi'_n(t') = \phi'_n(0) + K_n t'$  and  $\phi_n(t) = \phi_n(0) + H_n t$ . One easily checks, by comparing the form of the Hamiltonian for our choice of interval time  $T$  to that which corresponds to the extrinsic time, that these phases do not follow the same trajectory in  $\mathbb{R}^d$ , where  $d$  is the number of commuting observables. This further implies that one can easily find a set of  $N = 2$  observables (which do not commute with  $H$ ) such that their expectation values do not follow the same trajectory in  $\mathbb{R}^2$ . Thus either the quantum theories in different times give different dynamics or they cannot be compared in the sense that it is impossible to set up the same experiment in both pictures. The attempts to avoid this fundamental problem by appealing to the path integral formulation of quantum mechanics, as a unified formalism to which both Schrödinger quantizations are presumably equivalent, usually end up hiding the problem in either an inadequate definition of the boundary conditions or of the measure: As Guven and Ryan have emphasized [33], two inequivalent theories cannot be simultaneously equivalent to a third one.

Until one can gain a better grasp on this problem and find an appropriate Hilbert space for which the wave function (3.22) is computable, it is impossible to claim that this canonical formalism for (2+1)-dimensional gravity is acceptable as such.

One might consider extending this work to 3+1 dimensions along the lines of the exactly solvable theories proposed by Horowitz [34]. A reduced phase space for Horowitz’s topological  $B \wedge F$  theory can be found [35,36], and it is probable that the canonical quantization program can be carried out along the same lines as those laid out in this paper.

The canonical quantization program of (2+1)-dimensional gravity was faced with two problems: that of computing the Hamiltonian to address quantum dynamical issues and that of finding wave functions that are invariant under the action of the mapping class group. The first problem has essentially been solved, although this required abandoning the extrinsic gauge which is favored by the ADM school for its relevance in 3+1 dimensions; as we argued above, the change of internal time is not a trivial step in a quantum theory. As for the second problem, an explicit construction of the invariant wave functions was given. Even if the computability can-

not be established in the general case, we hope that this paper has provided a practical scheme with which one can tackle specific nonperturbative problems of quantum gravity in 2+1 dimensions.

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