Striking property of the gravitational Hamiltonian

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The Hamiltonian framework for (2+1)-dimensional gravity coupled with matter (satisfying positive energy conditions) is considered in the asymptotically flat context. It is shown that the total energy of the system is non-negative, vanishing if and only if space-time is (globally) Minkowskian. Furthermore, contrary to one's experience with usual field theories, the Hamiltonian is *bounded from above*. This is a genuinely nonperturbative result. In the presence of a spacelike Killing field, (3+1)-dimensional vacuum general relativity is equivalent to (2+1)-dimensional general relativity coupled to certain matter fields. Therefore, our expression provides, in particular, a formula for energy per-unit length (along the symmetry direction) of gravitational waves with a spacelike symmetry in 3+1 dimensions. A special case is that of cylindrical waves which have two hypersurface orthogonal, spacelike Killing fields. In this case, our expression is related to the "c energy" in a nonpolynomial fashion. While in the weak field limit the two agree, in the strong field regime they differ significantly. By construction, our expression yields the generator of the time translation in the full theory, and therefore represents the physical energy in the gravitational field.

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I. INTRODUCTION

Spaces of solutions to Einstein's equations admitting isometries have provided a useful and simplified arena to analyze a number of issues in (3+1)-dimensional classical and quantum gravity. An outstanding example is that of stationary space-times where the presence of the timelike Killing vector field can be used to introduce the notion of multipole moments, energy and angular momentum being only the first of a doubly infinite series. These moments can then be used to characterize the space-time geometry completely [1]. This class of space-time is of particular interest in astrophysics where sources can often be idealized as being stationary. Another class of models of interest is provided by Bianchi cosmologies. In this case, the isometries are spacelike and Einstein's equations are reduced to *ordinary* differential equations in time. The analysis therefore simplifies considerably. In some cases, the equations can be integrated completely by making appropriate canonical transformations in the phase space and the models can then be used to get insight into the conceptual issues of nonperturbative quantum gravity such as the problem of time (see, e.g., [2]). In other cases, particularly the Bianchi type-IX models, the equations continue to be sufficiently complicated so as to provide interesting examples of chaotic dynamical systems (see, e.g., [3]). These analyses have shed considerable light on the "generic behavior" of solutions of Einstein's equations as one approaches a singularity. Neither of these truncations of full general relativity is, however, well suited to tackle dynamical issues that are related to the fact that the gravitational field has an infinite number of degrees of freedom: In the stationary context, there is no time evolution, while in the Bianchi models the truncation is so severe that one is left with only a finite number of degrees of freedom.

There are two sets of dynamical issues that hinge on the presence of an infinite number of degrees of freedom. The first refers just to the classical theory: One would like to get insight into the nature of gravitational waves beyond the linear approximation. Of particular interest is the notion of energy in these waves and its properties. The second is quantum mechanical: One would like to learn more about the field theoretic difficulties associated with the existence of infinitely many modes that can be excited. To study these problems, a "midisuperspace" of solutions to Einstein's equations was studied in some detail in the 1970's. It consists of solutions to four-dimensional vacuum equations with cylindrical symmetry. In this case, the field equations again simplify. Because there still remains an infinite number of degrees of freedom, one continues to deal with *partial* differential equations. However, the presence of the two hypersurface orthogonal Killing fields reduces the problem to that of solving the *linear* wave equation for a scalar field in a three-dimensional (fictitious) Minkowski space. More precisely, given a solution to the wave equation on a three-dimensional Minkowski space, one can simply write down a four-metric with cylindrical symmetry which satisfies the full, nonlinear vacuum equations. In the classical theory, one is then led to the issue of the physical interpretation. What, in particular, is the energy carried by these waves? Is it always positive? Can one write down a simple "mass-loss" formula at null infinity?

These and related problems were examined by several authors, in particular, by Thorne [4] (using Cauchy surfaces), Stachel [5] (in terms of falloff at null infinity), and Kuchař [6] (in the context of canonical quantization). The analysis is far from being straightforward because of the following complication: Since the solutions have a "translational Killing field" ($\partial/\partial z$, parallel to the axis of rotation), the solution cannot be asymptotically flat either at spatial or null infinity in four-dimensions, whence the standard machinery of the Arnowitt-Deser-Misner (ADM) framework (see, e.g., [7]) at spatial infinity or of the Bondi-Penrose [8,9] framework at null infinity is simply not available. Indeed, the symmetry considerations tell us that the total energy in the wave must be infinite (or identically zero). The physically meaningful quantity would be the energy per unit length along $\partial/\partial z$. Thorne succeeded in making this notion, which he called the "c energy," precise. His final expression can be understood as follows. Since the four-metric is completely characterized by the solution to the wave equation in a three-dimensional reference Minkowski space, one can just compute the conserved energy of the scalar field (in Minkowski space) and declare that to be the total energy of the cylindrical wave per unit length. The quantity is then manifestly positive and, at least intuitively, satisfies the anticipated mass-loss formula. Furthermore, in the weak field limit, it reduces to the expected expression. Kuchař's strategy [6] to the problem of quantization can be understood in a similar fashion. The four-metric can be gauge fixed in such a way that the only degree of freedom in it is the solution to the wave equation. Since one knows how to quantize the free scalar field in three-dimensional Minkowski space, one can take over that operator-valued distribution $\hat{\Phi}$ and insert it in the gauge-fixed metric to provide the quantum operator corresponding to the four-geometry. (The same strategy has been used [10] in the "one polarization Gowdy models," which again have two commuting Killing fields, where, however, the spatial topology is that of a three-torus rather than R^3. The fact that the spacelike sections are compact does give rise to the usual problems in the definition of total energy. However, the essence of the quantization procedure is not affected.)

There is, however, another, and much more general, strategy. It is well known (see, e.g., [11]) that in the presence of a spacelike Killing field, Einstein's vacuum equations in 3+1 dimensions are equivalent to Einstein's equations in 2+1 dimensions with a source consisting of a triplet of scalar field constituting a [SO(2,1)] nonlinear σ model. If the 3+1 Killing field is "translational," it is natural to expect the (2+1)-dimensional fields to by asymptotically flat (both at spatial and null infinity). For such fields, one can imagine extending the (3+1)-dimensional ADM strategy [7] to define conserved quantities at spatial infinity and the Bondi-Penrose strategy [8,9] to define fluxes of energy momentum at null infinity. From the (2+1)-dimensional perspective, the Hamiltonian generating asymptotic time translations at spatial infinity would represent the total energy of the given isolated system. From the (3+1)-dimensional perspective, this would represent the energy per unit length (along the Killing trajectories). There is a similar dual interpretation of quantities at null infinity. Thus one can go back and forth between the two pictures. The strategy is attractive because it avoids the introduction and use of fictitious Minkowski spaces—the energy would arise as the generator of time translation directly in the *physical* picture. It is also natural from the perspective of quantum theory. Indeed, there exists a nonperturbative quantization of 2+1 gravity without sources. The next step from the (2+1)-dimensional perspective is to bring in matter. An exact Hamiltonian framework in the classical theory would be the first step toward such an analysis. Is the energy positive? If so, one can hope that there would be no problems with stability and/or unitarity of the quantum theory. Are the ultraviolet difficulties different, now that we have effectively a theory only in 2+1 dimensions?

Finally, this framework will encompass the cylindrical waves of [4,6] as a special case. These (3+1)-dimensional space-times have *two* spacelike Killing fields, which, moreover, are hypersurface orthogonal. In the 2+1 reduction with respect to the translational Killing field ($\partial/\partial z$), hypersurface orthogonality introduces a key simplification: The triplet of matter fields reduces simply to a single scalar field Φ (the logarithm of the norm of the translational Killing field) satisfying the wave equation in the (curved) (2+1)-dimensional geometry. The presence of the second Killing field then imposes a further rotational symmetry, which in turn implies that the scalar field Φ satisfies the wave equation with respect to the physical (2+1)-dimensional geometry if and only if it does so with respect to a fictitious Minkowskian metric on the (2+1)-dimensional manifold; we can thus recover the description used in the analysis of cylindrical waves. One can therefore ask: Does the notion of energy obtained here from the perspective of (2+1)-dimensional gravity reduced to the c energy? There is no *a priori* reason why the two should be the same: While the c energy is the generator of time translations for the scalar field propagating in a (2+1)-dimensional Minkowski space, the new energy would be the generator of time translations for the full 2+1 theory consisting of gravity plus matter. Similarly, in the quantum theory, new avenues open up. From the work of Allen [12], we can deduce that there *is* a consistent quantization of axisymmetric 2+1 gravity coupled to a scalar field satisfying the wave equation, which is equivalent to Kuchař's [6] quantization of cylindrical waves. However, from the general perspective of (2+1)-dimensional gravity, a number of new avenues also become available.

These considerations suggest that it is natural to investigate a more general "midisuperspace" consisting of 2+1 gravity coupled to physically reasonable matter fields. The purpose of this paper is to present a Hamiltonian framework for this system. In particular, we will show that the Hamiltonian has two interesting properties, one expected on physical grounds and the second somewhat unexpected and, at least at first, quite surprising. The first property is that, provided the matter fields satisfy a local energy condition, which they do if they are obtained by a symmetry reduction of the 3+1 theory, the total energy is non-negative and vanishes if and only if all matter fields vanish and space-time is globally Minkowskian. The second property is that the total energy is bounded *from above* as well. More precisely, the canonical framework breaks down beyond this limit in the sense that there is no function which can generate "the asymptotic time translation" in the part of the phase space where the bound would have been violated. Thus the Hamiltonian we obtain is quite different from the c energy. We will see that the two are nonpolynomially related. In the weak field limit, they agree. However, in the strong field limit, there is quite a difference. The existence of an upper bound for the Hamiltonian suggests that in the quantum theory the ultraviolet behavior may well be quite different from the one encountered in the 3+1 theory. This difference may well be the underlying reason behind the finding [13] that 2+1 gravity coupled to scalar fields is perturbatively finite.

Can we intuitively understand the existence of this upper bound? We will see that from a space-time point of view, when the bound on the Hamiltonian is violated, all resemblance to asymptotic flatness is lost in the sense that the points "at spatial infinity" can be reached by curves of finite length from any point in the interior. The qualitative picture of this "closing up" of space is perhaps not surprising in the light of the work by Deser, Jackiw, and 't Hooft [14] on 2+1 gravity in the presence of point particle sources. However, when one looks at the issue in detail, one sees that there are some important differences between point particles and smooth field sources. In geometrical terms, the locations of point particles do not, strictly speaking, belong to the space-time manifold since the geometry there has conical singularities. As a result, a complete Hamiltonian description is difficult to construct in that case: One must specify the boundary conditions not only at spatial infinity, but also at the location of the point particles, and then show that the resulting phase space has a well-defined symplectic structure and continuous Hamiltonian flows. Strictly speaking, therefore, in the case of point particle sources there is no clear-cut relation between the closing up of space and properties of the Hamiltonian. On the other hand, when the sources of smooth fields, the Cauchy surfaces can be taken to be topologically R^2 and geometry is smooth everywhere. There are no conical singularities in the physical space-time and no points need to be excised. Consequently, boundary conditions on all fields need to be specified only at spatial infinity and the task of constructing a complete Hamiltonian description is significantly simpler. Finally, at first, the phenomenon of the "closing up" of space may seem [15] to have an analogue also in the 3+1 theory in the vacuum initial data studied by Brill [16]. However, in that case, space closes up when the total energy is *infinite*, whence the result is not surprising—one does not expect a solution with infinite energy to be asymptotically flat. Here the "closing up" occurs for a *finite* value of the total Hamiltonian, and on general physical grounds, without the knowledge of the asymptotic behavior of the solutions to the constraint equations, it is hard to see why this should occur. Indeed, for cylindrical waves, for example, one can imagine just tuning up the value of the scalar field Φ by scaling it by an arbitrarily large constant factor, thereby obtaining a perfectly reasonable initial data for the scalar field (say, of compact support) with an arbitrarily large energy. It is therefore puzzling at first that the total Hamiltonian can be bounded from above. What actually happens is rather subtle: The total energy is a *nonpolynomial* function of the c energy, the Minkowskian energy of the scalar field Φ, and as one tunes the scalar field up, while the c energy diverges, the Hamiltonian tends to its upper bound.

In Sec. II, we will present the basic Hamiltonian framework. The boundary conditions, and hence also the details of the construction, are quite different from the 3+1 theory: There is no fixed, fiducial metric to which all metrics approach at spatial infinity, and extra care is needed in a number of steps. In Sec. III, we show that the Hamiltonian is bounded from above in the sense indicated above. In Sec. IV, we establish the positivity of the Hamiltonian. Again, while the general ideas are similar to those used in the proofs in the 3+1 theory [17], a number of subtle differences arise from the peculiar features of the (2+1)-dimensional boundary conditions. In Sec. V, we summarize the main results and discuss their implications to cylindrical waves.

This is a detailed account of the results presented by one of us (A.A.) at the Brill-Misner symposium.

II. HAMILTONIAN FRAMEWORK FOR 2+1 GRAVITY WITH MATTER FIELDS

Since we are interested in the Hamiltonian framework, we will assume that space-times M under consideration have a topology $\Sigma \times R$, where Σ is an arbitrary but fixed noncompact two-manifold, the complement of a compact set of which is diffeomorphic to the complement of a compact set in R^2. Thus the topological complications, if any, are confined to a world tube in the space-time M with compact spatial sections; outside this world tube M resembles R^3. The Cauchy surfaces in M are to be diffeomorphic to Σ. In the geometrodynamical framework, the basic phase space variables are the two-metrics q_{ab} on Σ and their canonically conjugate momenta P^{ab}. The momenta are related to the extrinsic curvature K^{ab} via $P^{ab} = \sqrt{q}(K^{ab} - Kq^{ab})$, where $\sqrt{q}$ is the square root of the determinant of the metric q_{ab} and $K = q^{ab}K_{ab}$ is the trace of the extrinsic curvature.

Our first task is to specify boundary conditions on q_{ab} and P^{ab}. As in (3+1)-dimensional general relativity, the choice of boundary conditions will be motivated by the asymptotic behavior of simple exact solutions. Let us therefore make a small detour to recall [14] the solution corresponding to a static point particle.

The spatial slice Σ in this case is topologically R^2. Fix a global "Cartesian" chart x, y on Σ. The particle resides at the origin, $x = y = 0$. The solution to the Einstein's equation is given by

$$dS^2 = -(dt)^2 + r^{-8GM}[(dr)^2 + r^2(d\theta)^2] , \qquad (2.1)$$

where G is Newton's constant (which in three dimensions has the dimensions of inverse mass) and (r, θ) are the polar coordinates obtained from the Cartesian ones in the usual way [i.e., $(x = 4 \cos\theta,\ y = r \sin\theta)$]. Thus $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$. A direct calculation shows that, as required, the stress-energy tensor T_{ab} is distributional and localized at the origin, $T_{ab} = M\delta^2(x, y)\nabla_a t \nabla_b t$.

Let us recast (2.1) in a more transparent form. Since the Ricci tensor vanishes for $r > 0$ and since we are in three space-time dimensions, the full Riemann tensor vanishes there as well. Thus, for $r > 0$, the metric (2.1) is flat. To exhibit it in the manifestly flat form, let us set

$$\alpha := 1 - 4GM \quad \text{and} \ \rho := \frac{r^\alpha}{\alpha} \ , \quad \bar{\theta} = \alpha\theta \ . \qquad (2.2)$$

The metric (2.1) then takes the manifestly flat form

$$dS^2 = -(dt)^2 + (d\rho)^2 + (\rho)^2(d\bar{\theta})^2 , \qquad (2.3)$$

where, however, $0 \leq \bar{\theta} \leq |\alpha|$. From the restricted range of $\bar{\theta}$, we immediately see that there is a conical singularity at $\rho = 0$. We also see that the deficit angle is directly related to the value of the mass M of the particle. Thus, although the space-time is flat, unless $M = 0$, it is not *globally* isometric to the three-dimensional Minkowski space. Indeed, the deficit angle persists even at infinity. Thus space-time metrics with different values of M differ from each other already in the *leading-order* terms at infinity. In four dimensions, all asymptotically flat metrics approach a fixed globally Minkowskian metric near infinity and the information about the mass resides in the leading-order $1/r$ deviations from this Minkowski metric. In three-dimensions, by contrast, the information about the mass resides already in the "zeroth-order" behavior of the metric at infinity; there is no universal Minkowski metric that they approach.

With these motivating remarks, we are ready to specify the boundary conditions. Since, outside the compact set, Σ is diffeomorphic to the complement of a compact set in R^2, in the asymptotic region of Σ we can fix coordinates r, θ, with $r_0 < r < \infty$ and $0 \leq \theta < 2\pi$. Let x, y be the Cartesian coordinates corresponding to r, θ. Denote by e_{ab} the Euclidean metric defined by these coordinates, $e_{ab} = \nabla_a x \nabla_b x + \nabla_a y \nabla_b y$. Note that the coordinates $(r, \theta), (x, y)$ and the metric e_{ab} are defined only in the asymptotic region of Σ where there are no topological nontrivialities. We will require that the metric q_{ab} have the asymptotic form

$$q_{ab} = r^{-\beta}[e_{ab} + O(1/r)] \qquad (2.4)$$

for some real constant β, which we leave arbitrary for the time being. [As in the 3+1 theory, the falloff conditions will refer to the components of the tensor field being considered in the Cartesian chart x, y, which is fixed in the asymptotic region. Also, if $f = O(r^n)$, we assume that the derivatives in the Cartesian chart fall off as $\partial_a f = O(r^{n-1})$, $\partial_a\partial_b f = O(r^{n-2})$, etc.] Comparison with (2.1) leads one to expect that, in the final picture, the ADM mass would be coded in β through $\beta = 8GM$. This will turn out to be the case. To begin with, we will allow β to assume negative values; it is the positive energy theorem of Sec. IV that will force β to be positive.

Thus the gravitational or geometric part $\mathcal{C}^{\text{geo}}$ of our configuration space will consist of smooth metrics on Σ which have the asymptotic behavior given by (2.4). Note that q_{ab} is assumed to be smooth *everywhere* on Σ; in particular, it cannot have conical singularities such as the one at the origin in the point particle geometry. Put differently, Σ is assumed not to have "interior boundaries." This in particular means that we will only consider *smooth matter sources.* Had we allowed singular sources such as point particles, the Hamiltonian analysis would have been significantly more difficult: To obtain a consistent framework, *all* sources must be included in the construction of the phase space and it is generally difficult to do symplectic geometry with singular fields. Thus the metric (2.1) was used only to motivate the boundary conditions at *infinity*; in the interior, the geometries included in the phase space will be quite different.

Since the geometric part Γ^{geo} of the phase space is the cotangent bundle over $\mathcal{C}^{\text{geo}}$, it is completely determined by $\mathcal{C}^{\text{geo}}$. To exhibit the induced boundary conditions on the momenta, let us first examine the asymptotic behavior of tangent vectors δq_{ab} at a generic point q_{ab} of $\mathcal{C}^{\text{geo}}$. By varying (2.4), we obtain

$$\delta q_{ab} \sim r^{-\beta}[-\delta\beta \ln(r) e_{ab} + O(1/r)] \ . \qquad (2.5)$$

(Note that, unlike in the 3+1 theory, the fall off of the tangent vectors δq_{ab} varies from point to point on $\mathcal{C}^{\text{geo}}$.) The momenta P^{ab} are to be such that, regarded as cotangent vectors, their action $P \circ [\delta q]$ on any tangent vector δq should be well defined; i.e., the following integral should exist:

$$P \circ [\delta q] := \int_\Sigma d^2x\, P^{ab}\delta q_{ab} \ .$$

(Here and in what follows, we integrate only scalar densities over Σ; a fiducial volume element is therefore unnecessary.) This requirement fixes the boundary conditions on P^{ab}:

$$P^{ab}e_{ab} \sim r^{\beta-3} \ , \quad P := P^{ab}q_{ab} \sim r^{-3} \ ,$$
$$[P^{ab} - \tfrac{1}{2}Pq^{ab}] \sim r^{\beta-2} \ . \qquad (2.6)$$

Thus the phase space Γ^{geo} is to consist of smooth fields (q_{ab}, P^{ab}), where q_{ab} is a positive definite metric on Σ and P^{ab} a tensor density of weight 1, satisfying the boundary conditions (2.4) and (2.6). Since the momenta P^{ab} have a well-defined action on tangent vectors, it follows that the gravitational part of the symplectic structure is well defined.

Let us next consider matter fields. We do not wish to commit ourselves to specific types of sources; our main restriction will only be that the matter fields satisfy the energy condition $T_{ab}t^a n^b \geq 0$, where T_{ab} is the stress-energy tensor of matter and t^a and n^a are any future-directed timelike vector fields. Consequently, the form of the boundary conditions will now be rather general. First, we require that the falloff on the fields and their momenta should be such that the matter part of the symplectic structure is well defined. Second, we demand that the components of the matter stress-energy tensor T_{ab} in the Cartesian chart should be $O(r^{\beta-3})$. Note that it is easy to satisfy this last condition *and* have interesting solutions to the constraints. For example, the matter sources could have compact spatial support. In this case, the space-time metric would be flat in a neighborhood of spatial infinity. Nonetheless, in contrast with the situation in the four-dimensional case where the constraints would have forced the initial data to correspond to Minkowski space-time globally, there is an infinite dimensional family of nontrivial solutions to constraints (e.g., the ones corresponding to cylindrical waves.)

To conclude this section, let us list some consequences of these conditions which will be needed in the subsequent analysis. Equation (2.4) implies that the asymptotic behavior of $\sqrt{q}$ is given by

$$\sqrt{q} \sim r^{-\beta} \tag{2.7}$$

and that of the Ricci scalar is given by

$$R \sim r^{\beta-3} \quad \text{and} \quad \sqrt{q}R \sim r^{-3} \ . \tag{2.8}$$

Finally, (2.6) implies that the asymptotic behavior of the extrinsic curvature is given by

$$K_{ab} \sim r^{-2} \quad \text{and} \quad K \equiv K_{ab}q^{ab} \sim r^{\beta-3} \ . \tag{2.9}$$

III. CONSTRAINTS AND THE HAMILTONIAN

Let us begin by recalling the situation in the (3+1)-dimensional case in the asymptotically flat context. The phase space has two sets of constraints, a vector $C_a(x)$ and a scalar $C(x)$. To analyze the canonical transformations they generate, one smears them by shift $N^a(x)$ and lapse $N(x)$ fields to obtain functions $C_{\mathbf{N}}(q,p)$ and $C_N(q,p)$ on the phase space and computes the corresponding Hamiltonian vector fields, i.e., the infinitesimal canonical transformations they generate. Now, because of the falloff conditions [of the (3+1)-dimensional theory] on the canonical variables, it follows that the constraint functions fail to be differentiable unless the smearing fields N and N^a go to zero at infinity. Thus what constraints generate are spatial diffeomorphisms and time evolutions which are asymptotically identity. Assuming these falloffs on the lapse-shift pairs, one can compute the Poisson brackets between $C_{\mathbf{N}}$ and C_N. They constitute a first-class system. Hence the canonical transformations they generate should be thought of as a gauge. The space-time translations, on the other hand, correspond to lapse-shift pairs which are asymptotically constants. These do induce canonical transformations on the phase space, but to obtain their generating functions, one must add suitable boundary terms to the smeared versions of the constraint functions. Consequently, even when the constraints are satisfied, the generating functions do not vanish; they are simply reduced to surface terms, the ADM three-momentum and energy [7]. To summarize, on the mathematical side, there is a delicate interplay between the boundary conditions and the differentiability of the constraint functions. This in turn gives rise to a physical distinction between gauge and dynamics. The former corresponds to spatial diffeomorphisms which are asymptotically identity and the bubble-time evolutions which fail to move the Cauchy surfaces at infinity. The latter correspond to asymptotic space and time translations. On the constraint surface, the numerical values of the generators of gauge transformations vanish, while those of dynamics are given by boundary terms. Thus there is a clean separation between gauge and dynamics. (For further details, see, e.g., [19].)

The overall structure is similar in 2+1 dimensions. In Sec. III A, we will analyze the vector constraint and, in Sec. III B, the scalar constraint. Section III C discusses these results from various angles.

A. Vector constraint

Given a shift N^a on Σ, the smeared vector constraint can be written as

$$C_{\mathbf{N}} = 2\int_\Sigma d^2x\, N^a D_c(P^{cd}q_{da}) + \text{matter terms} \ . \tag{3.1}$$

With our assumptions on the matter fields, the integral involving matter fields is well defined and will play no role in the discussion of this section. We will therefore focus just on the gravitational part, i.e., the first term on the right-hand side of (3.1), which we will refer to as $C^{\text{geo}}_{\mathbf{N}}$. Using the boundary conditions (2.4) and (2.6) on q_{ab} and P^{ab}, we conclude that $D_b(P^{bc}q_{ac}) = O(1/r^3)$. Since the volume element is $d^2x = r\,dr\,d\theta$, it follows that $C^{\text{geo}}_{\mathbf{N}}$ is well defined (i.e., finite) provided the shift N^a behaves asymptotically as $N^a \sim N^a_0(\theta) + O(1/r)$. Thus, as far as the issue of existence of $C^{\text{geo}}_{\mathbf{N}}$ is concerned, we can let N^a be a vector field which remains asymptotically bounded with respect to e_{ab}; it does not have to vanish in the limit.

The question then is that of differentiability of $C^{\text{geo}}_{\mathbf{N}}$. Let us begin with differentiability with respect to P^{ab}. Let us first write (3.1) as an integral over the interior of the disk $r \leq R_0$, integrate by parts, use Stokes' theorem, and then take the limit as $R_0 \to \infty$. We then have

$$C^{\text{geo}}_{\mathbf{N}} = \lim_{R_0\to\infty}\left[-\int_{r\leq R_0} d^2x(\mathcal{L}_{\mathbf{N}}q_{ab})P^{ab} + 2\oint_{r=R_0} d\theta\, r N^a P^{bc} q_{ab}\partial_c r\right] \ . \tag{3.2}$$

Now the integrand in the surface integral behaves asymptotically as R_0^{-1}; the surface integral therefore vanishes in the limit. Hence the expression of the smeared constraint reduces to

$$C_{\mathbf{N}}^{\text{geo}} = -\int_\Sigma d^2x(\mathcal{L}_{\mathbf{N}}q_{ab})P^{ab} . \tag{3.3}$$

The volume integral is now manifestly differentiable with respect to P^{ab}. We have

$$\frac{\delta C_{\mathbf{N}}^{\text{geo}}}{\delta P^{ab}} = -\mathcal{L}_{\mathbf{N}}q_{ab} , \tag{3.4}$$

which confirms our expectation that the canonical transformations generated by $C_{\mathbf{N}}^{\text{geo}}$ should correspond to the diffeomorphisms generated by N^a on Σ. Let us now consider differentiability with respect to q_{ab}. For this, let us again write $C_{\mathbf{N}}^{\text{geo}}$ of (3.3) as a limit of the integral over a disk $r \leq R_0$, integrate by parts, and use Stokes' theorem to obtain

$$C_{\mathbf{N}}^{\text{geo}} = \lim_{R_0\to\infty}\left[\int_{r\leq R_0} d^2x\, q_{ab}(\mathcal{L}_{\mathbf{N}}P^{ab}) - \oint_{r=R_0} R_0 d\theta\, N^c q_{ab}P^{ab}\partial_c r\right] . \tag{3.5}$$

Because of our boundary conditions, the integrand in the surface term now falls off as R_0^{-2}, whence in the limit we have

$$C_{\mathbf{N}}^{\text{geo}} = \int_\Sigma d^2x\, q_{ab}(\mathcal{L}_{\mathbf{N}}P^{ab}) . \tag{3.6}$$

Thus $C_{\mathbf{N}}^{\text{geo}}$ is now manifestly differentiable with respect to q_{ab},

$$\frac{\delta C_{\mathbf{N}}^{\text{geo}}}{\delta q_{ab}} = \mathcal{L}_{\mathbf{N}}P^{ab} , \tag{3.7}$$

and the value of the derivative again confirms our expectation that $C_{\mathbf{N}}^{\text{geo}}$ is the generator of spatial diffeomorphisms. For simplicity, in the above discussion, we have left out matter terms. When they are added, the total constraint function $C_{\mathbf{N}} = C_{\mathbf{N}}^{\text{geo}} + C_{\mathbf{N}}^{\text{matter}}$ generates diffeomorphisms on the entire phase space consisting of geometrical and matter variables.

We conclude this subsection with two remarks.

(1) While the general structure of the argument is the same as the one normally used in 3+1 theories, there is nonetheless a key difference in the final result. In the 3+1 analysis, the vector constraint generates only those diffeomorphisms which are asymptotically *identity*. These, in turn, are interpreted as a gauge. In the present case, we have found that the vector constraints generate diffeomorphisms which can remain *bounded* asymptotically; the shifts do not have to vanish asymptotically. This conclusion may seem counterintuitive at first since in the 3+1 theory, space translations on Σ remain asymptotically bounded and their generator on the phase space is the ADM three-momentum [7]. In the present case, on the other hand, because of the presence of deficit angles at infinity, asymptotic space translations are *not* symmetries of the theory. That is, unless $\beta = 0$, the translation Killing fields of the fiducial e_{ab} are not asymptotic Killing fields of the q_{ab} being considered because of the $r^{-\beta}$ term in the boundary condition (2.4). The only asymptotic symmetries of the class of space-times under consideration are the ones associated with time translation and spatial rotation. (This observation was made by several authors; see, in particular, [14,18]. Note incidentally that, had we introduced a θ dependence in the conformal factor relating q_{ab} and e_{ab}, we would not have had the rotational symmetry.) Thus there is complete consistency: There are neither space translations nor nonvanishing Hamiltonians associated with asymptotically bounded diffeomorphisms which could have, potentially, played the role of a (generalized) ADM two-momentum.

(2) We could have carried out the above analysis for a shift field N^a which is an asymptotic rotational Killing field, i.e., behaves asymptotically as $N^a \sim (\partial/\partial\theta)^a + N_0^a(\theta)$. We would then have found that the surface term is nonzero, whence $C_{\mathbf{N}}^{\text{geo}}$ would not have been a differentiable function in the phase space Γ. Thus the asymptotic rotation is *not* generated by the constraint it does *not* correspond to a gauge transformation. Indeed, it is easy to find the Hamiltonian $J_{\mathbf{N}}$ on the phase space Γ which generates the corresponding canonical transformation. As in the 3+1 theory, one just has to add to the constraint functional the appropriate boundary term to restore differentiability and rescale the result by $1/16\pi G$ to conform to the standard normalization (which comes from the overall constant in the expression of the action):

$$J_{\mathbf{N}} = \frac{1}{16\pi G}\left[\int_\Sigma d^2x\, q_{ab}(\mathcal{L}_{\mathbf{N}}P^{ab}) + \text{matter terms}\right] \approx \frac{1}{8\pi G}\oint dS_c N^a P^{cd} q_{da} , \tag{3.8}$$

where $dS_c = r\,\partial_c r\, d\theta$ is the line element on the boundary that arises in the Stokes' theorem and it is understood that the expression is first evaluated on a circle $r = R_0$ in the asymptotic region and then the limit $R_0 \to \infty$ is taken. The last step provides the numerical value of the angular momentum $J_{\mathbf{N}}$ *on the constraint surface.* Our boundary conditions ensure that the integral is well defined over the entire phase space Γ. As in the 3+1 theory, the surface integral involves only the gravitational variables; the matter terms enter only through the constraints. The expression (3.7) agrees with the formulas for angular momentum given by Deser, Jackiw, and 't Hooft [14] and by Henneaux [18].

B. Scalar constraint

The steps in this analysis are the same as in the previous subsection. However, since the final result is somewhat unexpected, we will provide the relevant details.

Given a lapse function N on Σ, we can write the smeared constraint function as

$$C_N = \int_\Sigma d^2x\, N\left[\sqrt{q}R - \frac{1}{\sqrt{q}}(P^{ab}P_{ab} - P^2)\right] + \text{matter terms} = C_N^{\text{geo}} + C_N^{\text{matter}} . \tag{3.9}$$

Again, matter terms will play no role in our discussion.

Now, from (2.8), we see that the "potential term" $\sqrt{q}R$ falls off as r^{-3}, independently of the value of β, while (2.4), (2.6), and (2.7) imply that the falloff of the "kinetic term" does depend on β: $(1/\sqrt{q})(P^{ab}P_{ab} - P^2) \sim r^{\beta-4}$. Hence the integral containing the potential term will exist only if N/r goes to zero, while that containing the kinetic term will exist if $Nr^{\beta-2}$ goes to zero, in the limit $r \to \infty$. Now an asymptotic time translation corresponds to $N \sim 1 + O(1/r)$. We therefore see that the *kinetic integral will not exist for the N corresponding to time translations unless β is less than 2.* Furthermore, the addition of surface terms cannot improve the situation since the kinetic terms are purely algebraic in the canonical variables. From the phase space viewpoint, this, in essence, will turn out to be the reason why the Hamiltonian is forced to the bounded from above.

Next, we turn to the analysis of differentiability. It is straightforward to verify that if we demand that the lapse go to zero asymptotically as (i) $O(1/r)$ if $\beta \leq 2$ and (ii) $O(r^{-\beta+1})$ if $\beta \geq 2$, not only does $C_N^{\rm geo}$ exist, but it is also differentiable on the phase space. The canonical transformations it generates correspond to "bubble-time evolutions"; the time translation vanishes identically at infinity. As in the (3+1)-dimensional theory, these correspond to "gauge motions" in the sense that they can be taken care of by appropriate gauge fixing. (For details on this interpretation, see, e.g., [19].)

Let us now consider lapse functions which correspond to time translations, i.e., have the asymptotic behavior $N \sim 1 + O(1/r)$. Using evolution equations of the initial value formulation, we can formally write an infinitesimal transformation on the phase space corresponding to this time translation. Since the evolution equation (for zero shift) on q_{ab} is simply

$$\dot{q}_{ab} = 2NK_{ab} \equiv \frac{1}{\sqrt{q}}(P_{ab} - Pq_{ab}) ,$$

it follows that, if a Hamiltonian is to exist, its kinetic piece must be the same as that in (3.9). Since this does not converge for $\beta \geq 2$, it follows that on the $\beta \geq 2$ part of the phase space, there is simply no Hamiltonian which can generate a canonical transformation corresponding to this evolution. Thus, while one can formally write down the "evolution equations," they do not induce canonical transformations on this part of the phase space. We will discuss this point in some detail at the end of this section.

From now on, therefore, let us focus on the "physical" part of the phase space defined by $\beta < 2$.

Now, if $N \sim N_\infty + O(1/r)$, where N_∞ is a constant, the functional $C_N^{\rm geo}$ does exist on the physical part of the phase space. However, as in the (3+1)-dimensional case, it is not differentiable. Thus, again, the "evolution equations" are not generated by the scalar constraints. However, as in the (3+1)-dimensional theory, this evolution *does* correspond to a well-defined canonical transformation and its generator is obtained by adding a suitable surface term to the constraint functional. Let us now see how this arises. It is clear by inspection that the kinetic integral, being algebraic in the canonical variables, is differentiable with respect to both q_{ab} and P^{ab}. Thus we can focus just on the potential term. Furthermore, since this term is independent of momenta, we need only be concerned with its derivative with respect to the configuration variable q_{ab}. Taking the variation of the potential term, we obtain

$$\begin{aligned}\delta \int_\Sigma d^2x\, N\sqrt{q}R &= \int_\Sigma d^2x \sqrt{q}(-D_aD_bN + D_cD^cNq_{ab})\delta q^{ab} \\ &\quad + \oint_{r=\infty} d\theta \sqrt{h}[Nv_a + (D_aN)q^{bd}\delta q_{bd} - D^cN\delta q_{ac}]r^a ,\end{aligned} \tag{3.10}$$

where

$$v_a = D^b\delta q_{ab} - D_a(q^{bd}\delta q_{bd}) ,$$

r^a is the unit normal to the circle at spatial infinity, and $\sqrt{h}$ is the determinant of the induced metric h_{ab} on this circle. [As before, it is understood that the integrals are first evaluated at a fixed radius where integrations by parts are carried out and then the radius is made to approach infinity. Also, in (3.10), we have used the fact that the Einstein tensor $R_{ab} - \frac{1}{2}Rg_{ab}$ vanishes identically in two dimensions.] The second and third terms in the surface integral, involving derivatives of the lapse function, vanish identically because of the choice of the boundary conditions, while the first term can be simplified. The final result is

$$\begin{aligned}\delta \int_\Sigma d^2x\, N\sqrt{q}R = \int_\Sigma d^2x \sqrt{q}(-D_aD_bN + D_cD^cNq_{ab})\delta q^{ab} \\ +\delta\beta \oint d\theta\, N_\infty .\end{aligned} \tag{3.11}$$

It is the presence of the surface term that spoils the differentiability of $C_N^{\rm geo}$. In the case when N goes to zero, the surface integral vanishes and $-C_N^{\rm geo}$ generates canonical transformations corresponding to the "bubble-time evolution" by an amount dictated by N. Hence we have an obvious strategy to obtain the generator of the canonical transformation: Subtract the surface integral from $-C_N^{\rm geo}$. This strategy does work, and the Hamiltonian generating the time translation which is unit at infinity is given by

$$H = -\frac{1}{16\pi G}\left[C_N^{\rm geo} + C_N^{\rm matter} - \beta \oint_0^{2\pi} d\theta\right] , \tag{3.12}$$

where we again divided by the factor $1/16\pi G$ to conform to the standard normalization.

Let us summarize. In the part of the phase space corresponding to $\beta \geq 2$, "time translations" do not induce canonical transformations; there is no Hamiltonian generating them. In the part with $\beta < 2$, the Hamiltonian does exist and is given by (3.12). On physical states, constraints are satisfied and its numerical value is given simply by $\beta/8G$. Together, these results lead us to the conclusion that the Hamiltonian is bounded from above, $H < 1/4G$.

C. Discussion

(1) The result that the Hamiltonian is bounded from above is quite unsettling at first. Let us therefore probe it from various angles.

In certain Bianchi type-II models, although the space-time picture and the initial value formulation are perfectly well defined, the standard ADM-type Hamiltonian formulation fails to exist (see, e.g., [20]), and as of now, one does not have viable replacements. Is the situation similar here? That is, does the main result of this section have to do only with the Hamiltonian framework or does something strange happen at $\beta = 2$ also from the viewpoint of space-time geometry? Let us begin with the point particle example [14] discussed in the beginning of Sec. II. For $\beta < 2$, there is a conical singularity at the origin. At $\beta = 2$, on the other hand, the cone simply opens up to become a cylinder and the distinction between the origin and infinity is blurred. For $\beta > 2$, the old origin becomes the point at infinity and the particle can be thought of as residing at the old point at infinity. Thus something strange does happen to the geometry. However, in our case, there are only smooth matter sources, and in particular, there are no conical singularities or even preferred points on Σ. Therefore the point particle picture can only be taken as an indication.

This indication is correct. Something remarkable does happen to the spatial geometry at $\beta = 2$ even in the smooth case. For $\beta < 2$, the points $r = \infty$ are, as one would expect, at an infinite proper distance from any point in the interior. Indeed, assuming that the matter sources have compact support, one can calculate the geodesic distance from any point in the asymptotic region to a point at infinity and find that it diverges as a power of r. For $\beta = 2$, the divergence is logarithmic. For $\beta > 2$, there is no divergence; the points $r = \infty$ *are at a finite distance with respect to any point on* Σ. Thus, for $\beta > 2$, space simply "curls up" and there is no resemblance to asymptotic flatness. Note, however, that Σ is *not* compactified; the two-metric q_{ab} does not extend to "the point at infinity" in a smooth manner. Σ is still noncompact, but it is geodesically incomplete; there is, effectively, a singularity at "the point at infinity."

Indeed, it is not clear if there are any physically admissible solutions to the constraints on the Cauchy surface Σ when $\beta \geq 2$. The simplest case would be to consider matter fields in the (2+1)-dimensional theory which arise from the symmetry reduction of (3+1)-dimensional cylindrically symmetric space-times. In this case, global analysis has recently been carried out, without requiring that the two Killing fields be hypersurface orthogonal [21]. It was found that asymptotically flat solutions to constraints exist only if $0 \leq \beta < 2$. This may seem surprising at first since one might expect the energy to grow unboundedly as one keeps scaling the matter fields by a constant. However, as one "tunes up" the matter fields, as a result of the curling up of Σ, the effective gravitational "potential energy" also goes up such that β remains bounded below 2. (This point is discussed further in Sec. V.)

Even if one assumes that physically reasonable solutions with $\beta > 2$ exist to the constraint equations, at least on heuristic grounds it would appear that, as a result of the effective singularity at the point at infinity, difficulties should arise with finite evolution. Given any $\epsilon > 0$, one would expect under the evolution by proper time ϵ that singularities would appear in the neighborhood of the point at infinity of radius ϵ since "the past light cones of points in this neighborhood would contain the singular point at infinity."

Thus it is reasonable to expect that the difficulties we encountered at $\beta = 2$ are not artifacts of the Hamiltonian formulation. These points of the phase space are pathological also from the viewpoint of space-time geometry.

(2) Let us restrict ourselves to the part of the phase space where $\beta < 2$. It is easy to check that the Hamiltonian H and the angular momentum J have vanishing Poisson brackets with all the constraints. Hence they are gauge invariant; they are observables of the theory in the sense of Dirac [22]. Since H and J are differentiable functions on the (restricted) phase space, one can take their Poisson brackets. It vanishes, reflecting the fact that the time translation and the rotation symmetries commute. Thus the overall picture is internally consistent and the situation is completely analogous to that in the 3+1 theory.

(3) The surface term which provides the numerical value of the Hamiltonian on the constraint surface agrees with that of Deser, Jackiw, and 't Hooft [14] and of Henneaux [18]. As was pointed out by Henneaux, in two-dimensions, the Ricci scalar is a pure divergence and therefore can be expressed as a surface term which, apart from an overall constant, coincides with the surface term in the Hamiltonian. Note, however, that the Hamiltonian is *not* given by the integral of the Ricci scalar; indeed, as we saw above, this term is not even differentiable with respect to q_{ab}. Thus, to obtain the correct evolution even at points of the constraint surface, we must use the full Hamiltonian given in (3.12). Finally, after this work was completed, it was pointed out to us that a number of authors had noted that in special contexts, such as cylindrical symmetry [21], time symmetric initial data or cosmic strings [23], etc., the deficit angle at infinity is bounded both from above and below. However, the generality of the result and especially its relation to the boundedness of the Hamiltonian generating time translations in a proper phase space formulation were not analyzed in these references.

IV. POSITIVITY OF ENERGY

In the previous section we saw that the Hamiltonian is bounded from above. We now wish to show that it is also bounded below; when the constraints are satisfied with matter fields satisfying our energy condition, the Hamiltonian is non-negative and vanishes if and only if the matter fields vanish and the initial data is that of Minkowski space. Under certain restrictive assumptions, positivity was established by a number of authors. For example, if one restricts oneself to matter fields that arise from a symmetry reduction of (3+1)-dimensional cylindrically symmetric space-times, a theorem due to Berger *et al.* [21] ensures that $\beta \geq 0$ and vanishes if and only if we are in Minkowski space. Similarly, Henneaux [18] has observed that since the surface integral in the expression of the Hamiltonian is proportional to the integral $\int d^2x\sqrt{q}R$, it is straightforward to establish positivity of the Hamiltonian on a $K = 0$ surface. Here we will use techniques similar to those introduced by Witten [17] in the 3+1 theory to establish positivity without such restrictions.

In the first part of this section, we recall basic facts about SU(1,1) spinors and, in the second part, establish the main result.

A. SU(1,1) spinors

Since the reader is likely to be more familiar with SU(2) spinors than SU(1,1), we will adopt conventions that are geared to the SU(2) case.

Let us begin by recalling the elements of spinor algebra. Let S denote a two-dimensional complex vector space and let $\alpha^A, \beta^D, \ldots$ denote its elements. These will be called (one index) spinors. Let us fix a second-rank nonzero tensor ϵ^{AB} over S and denote its inverse by ϵ_{AB}; thus $\epsilon_{AC}\epsilon^{BC} = \delta_A^B$. Following the Penrose-Rindler [24] convention, we will raise and lower the spinor indices using these tensors: $\alpha^A\epsilon_{AB} = \alpha_B$ and $\epsilon^{AB}\alpha_B = \alpha^A$. Next, we introduce a Hermitian conjugation operation (†) on S satisfying

$$(\alpha^A + k\beta^A)^\dagger = (\alpha^\dagger)^A + \bar{k}(\beta^\dagger)^A \, ,$$
$$(\alpha^\dagger)^\dagger = -\alpha^A \, , \quad \alpha^{\dagger A}\alpha_A \geq 0 \, , \tag{4.1}$$

where k is any complex number and the equality in the last condition holds if and only if $\alpha^A = 0$. We then extend this operation to spinors of arbitrary rank by demanding that

$$(\epsilon^\dagger)_{AB} = \epsilon_{AB} \quad \text{and} \quad (\alpha_{A\cdots B}^{C\cdots D}\beta_{M\cdots N}^{P\cdots Q})^\dagger = \alpha_{A\cdots B}^{\dagger C\cdots D}\,\beta_{M\cdots N}^{\dagger P\cdots Q} \, . \tag{4.2}$$

(For details, see, e.g., [25], Chap. 5.)

We can now consider the space V of trace-free Hermitian, second-rank spinors $\alpha_A{}^B$. V is a three-dimensional real vector space, equipped with a natural positive-definite inner product: $(\alpha,\beta) := -\alpha_A{}^B\beta_B{}^A \equiv -\mathrm{tr}\alpha\beta$. To define SU(2) spinor fields on a three-dimensional Riemannian manifold, one sets up a metric preserving isomorphism between V and the tangent space at each point of the manifold. In our case, the space-time manifold M is equipped with a metric of signature $-++$ and hence a "Wick rotation" is required. To accomplish this, choose a spinor n_{AB} satisfying $n_{AB}^\dagger = -n_{AB}$, $n_A{}^A = 0$, and $\mathrm{tr}n\cdot n = 1$. Denote the one-dimensional real subspace spanned by the real multiples of n^{AB} by $\mathcal{N}$ and the two-dimensional real subspace of *Hermitian* spinors $\alpha_A{}^B$, which is orthogonal to n_{AB} (so $\mathrm{tr}\,\alpha n = 0$), by $\mathcal{V}$, and let $V' = \mathcal{N}\oplus\mathcal{V}$. Then V' is a real three-dimensional vector space, equipped with a natural metric of signature $-++$. To define SU(1,1) spinor fields, one then has to fix a metric preserving isomorphism, or, soldering form, e_a^{AB} between V' and the tangent space at each point M.

Since we are interested in the canonical framework, however, we will need a slightly weaker structure. Let us fix a foliation of M by spacelike two-manifolds Σ and denote by n^a the vector field which is unity, future pointing, and everywhere orthogonal to Σ. Let us assume that $e_a{}^{AB}$ is so chosen that it maps n^a to n^{AB}. Set $E_a{}^{AB} = e_a{}^{AB} + n_an^{AB}$. Then $E_{aA}{}^B$ solders the two-dimensional real tangent space at any point of Σ to the vector space $\mathcal{V}$ of trace-free Hermitian spinors which are orthogonal to n_{AB}. In particular, therefore, the two-metric q_{ab} on Σ is given by $q_{ab} = -\mathrm{tr}E_aE_b$. For future use, we note the following algebraic properties of the Σ-soldering forms $E_{aA}{}^B$, which will be useful in the next subsection:

$$E_{aA}{}^C E_{bC}{}^B = -\tfrac{1}{2}q_{ab}\delta_A^B + \frac{i}{\sqrt{2}}\epsilon_{ab}n_A{}^B \, , \tag{4.3}$$

$$E_{mA}{}^C n_C{}^B = \frac{i}{\sqrt{2}}\epsilon_m{}^a E_{aA}{}^B \, , \tag{4.4}$$

where ϵ_{ab} is the alternating tensor on Σ compatible with q_{ab}. This completes the discussion of spinor algebra.

We can now introduce the basic notions of spinor calculus. First, it is straightforward to establish that Σ admits a unique derivative operator D (which acts on both the spinor and the tensor indices) which is compatible with the given $E_{aA}{}^B$: The equation

$$0 = D_aE_{bA}{}^B = \partial_aE_{bA}{}^B + [\Gamma_a, E_b]_A{}^B - \Gamma_{ab}^cE_{cA}{}^B \tag{4.5}$$

determines the Christoffel symbols $\Gamma_{ab}{}^c$ and the spin connection $\Gamma_{aA}{}^B$ uniquely. A particularly convenient connection $A_{aA}{}^B$ on spinors is obtained by adding to the spin connection a suitable multiple of the extrinsic curvature $K_{aA}{}^B := K_{ab}E_A^{bB}$:

$$GA_{aA}{}^B := \Gamma_{aA}{}^B - \frac{i}{\sqrt{2}}K_{aA}{}^B \, . \tag{4.6}$$

In the (2+1)-dimensional theory, $A_{aA}{}^B$ will play essentially the same role as that played by the Sen connection in (3+1)-dimensional general relativity. We will therefore refer to it as the Sen connection also in the present case. Denote by $\mathcal{D}$ the derivative operator on spinors defined by the Sen connection: $\mathcal{D}_a\alpha_A := \partial_a\alpha_A + GA_{aA}{}^B\alpha_B$. Then we have the two useful properties

$$(\mathcal{D}_a\alpha_A)^\dagger =: \mathcal{D}_a^\dagger\alpha^\dagger{}_A = \mathcal{D}_a\alpha^\dagger{}_A + i\sqrt{2}K_{aA}{}^B\alpha^\dagger{}_B \, ,$$
$$\mathcal{D}_aE_A^{a}{}^B = 0 \, , \tag{4.7}$$

where, in the second equation, we have used the fact that K_{ab} is symmetric and $K_{ab}n^b = 0$. Finally, as in the (3+1)-dimensional theory, the constraints can be expressed succinctly in terms of the curvature of the Sen connection. We have

$$\mathrm{tr}E^aF_{ab} = 4\sqrt{2}\pi i G T_{ab}n^a \text{ and } \mathrm{tr}E^aE^bF_{ab} = 8\pi G T_{ab}n^an^b. \quad (4.8)$$

B. Positivity of the Hamiltonian

With the machinery of SU(1,1) spinors at hand, we are now ready to prove the positive energy theorem. The beginning of the argument is the same as in the 3+1 theory. However, because of the difference in the boundary conditions, there is a departure from the (3+1)-dimensional procedure at an intermediate stage, and unlike in the (3+1)-dimensional case, the proof is now by contradiction.

Let us begin with the analogue of the (3+1)-dimensional Witten identity [17]. For any spinor field λ_A on Σ, we have

$$\begin{aligned}(E_D^{b\ A}\mathcal{D}^\dagger{}_b)(E_A^{a\ B}\mathcal{D}_a)\lambda_B &= -\tfrac{1}{2}\mathcal{D}^{\dagger a}\mathcal{D}_a\lambda_D + E_D^{a\ A}E_A^{b\ B}\mathcal{D}_{[a}\mathcal{D}_{b]}\lambda_B\\ &= -\tfrac{1}{2}\mathcal{D}^{\dagger a}\mathcal{D}_a\lambda_D - \frac{i}{4\sqrt{2}}\epsilon^{ab}F_{ab}{}^{EF}n_{EF}\lambda_D + \tfrac{1}{4}\epsilon^{ab}F_{ab}{}^{EF}E^m{}_{EF}\epsilon_{mn}E_D^{n\ A}\lambda_A\\ &= -\tfrac{1}{2}\mathcal{D}^{\dagger a}\mathcal{D}_a\lambda_D + 4\pi G(T_{ab}n^an^b\lambda_D - i\sqrt{2}E_D^{a\ A}\lambda_A T_{ab}n^b)\ ,\end{aligned} \quad (4.9)$$

where, in the last step, we have used (4.3) and (4.8). Now let us choose for the spinor field λ_D a solution to the analogue of the (3+1)-dimensional Witten equation:

$$E_A^{a\ B}\mathcal{D}_a\lambda_B = 0\ , \quad (4.10)$$

so that the left-hand side of the Witten identity (4.9) vanishes. Then, if we multiply both sides by $\lambda^{\dagger D}$, integrate over the disk $r \le R_0$, and use Stokes' theorem to simplify the first term on the right-hand side, we obtain

$$\begin{aligned}\oint_{r=R_0} d\theta\sqrt{h}r^b(\lambda^{\dagger D}\mathcal{D}_b\lambda_D)\\ = \int_{r\le R_0} d^2x\sqrt{q}(\mathcal{D}_b\lambda^D)^\dagger(\mathcal{D}^b\lambda_D)\\ +4\pi G\int_{r\le R_0} d^2x\sqrt{q}T_{ab}n^a(Nn^b+N^b)\ ,\end{aligned} \quad (4.11)$$

where

$$N = (\lambda^\dagger)^D\lambda_D \text{ and } N^a = i\sqrt{2}E^{n\ AB}\lambda_A\lambda_B^\dagger\ . \quad (4.12)$$

It is straightforward to verify that (N^a is real and) $N^a + Nn^a$ is timelike.

The idea now is to take the limit of this equation as $R_0 \to \infty$. For this, we need a control over the asymptotic behavior of the solution λ_A to (4.10). In the proof of the (3+1)-dimensional positive energy theorem, one simply requires λ_A to asymptotically approach a constant spinor. In the present case, however, a more subtle choice is necessary because of the difference in the asymptotic conditions on the metric and extrinsic curvature. Fortunately, the analysis is simplified because Eq. (4.10) is again conformally invariant [with, however, a conformal weight for λ_A which is different from the one of the (3+1)-dimensional theory]: *λ_A solves (4.10) with respect to $(q_{ab}, K_{aA}{}^B)$ if and only if $\phi^{-1/2}\lambda_A$ solves (4.10) with respect to $(\phi^2 q_{ab}, K_{aA}{}^B)$ for any smooth, positive function ϕ on Σ.*

In view of the boundary condition (2.4) on the metric q_{ab}, let us consider $\hat{q}_{ab} = r^\beta q_{ab}$ (and set $\hat{K}_{aA}{}^B = K_{aA}{}^B$). Then $\hat{q}_{ab} \sim e_{ab} + O(1/r)$ [and $\hat{K}_{aA}{}^B \sim O(r^{-2+\beta/2})$]. Therefore, for the pair $(\hat{q}_{ab}, \hat{K}_{aA}{}^B)$, we can apply arguments which are completely parallel to the ones used in the (3+1)-dimensional theory (see, e.g., [26], Sec. 3). The conclusion is the following: Given a spinor field λ_A^0 in the asymptotic region of Σ, which is constant with respect to the connection D^0 defined by (an ${}^0E_{aA}{}^B$ compatible with) e_{ab}, there is a unique solution $\hat{\lambda}_A$ to (4.10) with respect to $(\hat{q}_{ab}, \hat{K}_{aA}{}^B)$ with the asymptotic behavior $\hat{\lambda}_A \sim \lambda_A^0 + O(1/r)$. Using the conformal invariance of (4.10), we therefore conclude that there is a unique solution λ_A to (4.10) with respect to $(q_{ab}, K_{aA}{}^B)$, with the asymptotic falloff $\lambda_A \sim r^{\beta/4}\lambda_A^0 + O(1/r)$.

We can now return to (4.11) and the main argument. Let us suppose that $\beta \le 0$. Then, *not only do the limits of all integrals in (4.11) exist, but the surface term goes to zero.* Therefore, for $\beta \le 0$, the sum of the volume terms is zero. However, the first of these terms is manifestly non-negative and our energy condition is precisely that the integrand of the second term is also non-negative. Hence each must vanish in the limit $R_0 \to \infty$. The vanishing of the first term implies $\mathcal{D}_a\lambda_A = 0$ everywhere on Σ. As in the 3+1 theory, (4.10) admits two solutions which are linearly independent almost everywhere on Σ. The availability of two independent spinors which are constant with respect to $\mathcal{D}$ implies that the curvature $F_{abA}{}^B$ of $\mathcal{D}$ must vanish, which in turn implies that the matter terms must vanish and that the initial data are that for Minkowski space. Thus, if $\beta \le 0$, we must have $\beta = 0$: for $\beta < 0$, there are no (globally well-defined) solutions to the constraints satisfying the asymptotic conditions if the matter fields are to obey our energy condition. Thus we have established the desired result.

As we indicated above, the final argument is somewhat different from that in the (3+1)-dimensional theory. In particular, we do not have a manifestly positive expression for energy in the case $\beta > 0$; both the surface and volume integrals in (4.11) diverge in that case.

V. CONCLUSION

In the last three sections, we analyzed the Hamiltonian formulation of general relativity coupled to matter in 2+1 dimensions in the asymptotically flat context. The analysis we presented can also be carried out in the connection-dynamics framework, which is in fact simpler in 2+1 dimensions since all canonical variables can be taken to be real [25]. Furthermore, the proof of the positive energy theorem would have been more direct in that framework. However, since some of the results are rather unexpected, we chose to present the material in the more familiar geometrodynamical language to emphasize the fact that they are not artifacts of connection dynamics.

We saw that the Hamiltonian framework differs from that in 3+1 dimensions in a number of important respects. First, the boundary conditions on the geometrical fields are such that, while we can fix an Euclidean metric e_{ab} near infinity, if the mass in the space-time is nonzero, the physical two-metrics q_{ab} do not approach it even asymptotically. The two are related by a conformal factor $r^{-\beta}$, which goes to zero or diverges at infinity depending on the sign of β. Therefore the construction of the Hamiltonian framework is somewhat more involved. In particular, the asymptotic symmetry group is just the two-dimensional Abelian group of time translation and spatial rotation.

We carried out a detailed analysis of constraints of the theory and found that their role is somewhat different from that in the 3+1 theory. The smeared vector constraint is differentiable on the phase space even when the smearing shift field N^a remains asymptotically bounded, i.e., even when $N^a \sim N_0^a(\theta) + O(1/r)$. Thus the diffeomorphisms generated by all such shifts N^a are to be regarded as a gauge in the 2+1 theory. In the 3+1 case, by contrast, only the diffeomorphisms generated by shifts which *vanish* asymptotically that are regarded as a gauge; the generators of asymptotically constant vector fields are the ADM three-momenta. In the present case, there is no conserved quantity analogous to the three-momentum, in agreement with the fact that the asymptotic symmetry group does not admit space translations. Spatial rotations, on the other hand, are not generated by constraints since their falloff is given by $N^a \sim r$. They do induce canonical transformations on the phase space whose generating function can be obtained by adding a surface term to the constraint functional. The generator is the angular momentum. Thus, when we are "on shell," angular momentum is given by a surface integral at infinity.

In the case of the scalar constraint, we found that the differentiability requirement forces the lapse to go to zero at infinity at a rate that depends on β. Thus, to obtain a constraint function which is differentiable on the entire phase space, the lapse has to vanish faster than any inverse power of r. A more significant surprise is that, if we ask that the lapse be asymptotically constant, say, $N = 1 + O(1/r)$ so that it corresponds to an unit time translation at infinity, the resulting infinitesimal motion on the phase space, although formally defined, fails to be a canonical transformation unless $\beta < 2$. Thus the Hamiltonian framework simply fails to exist if $\beta > 2$. [In retrospect, therefore, without loss of generality, we could have added the requirement $\beta < 2$ in the boundary condition (2.4), i.e., in the very construction of the configuration space $\mathcal{C}^{\rm geo}$.] We then focused on the "physical part" of the phase space where $\beta < 2$ and computed the Hamiltonian generating the unit time translation. We found that it can be obtained by adding a surface terms to the smeared constraint. The value of the surface term is simply $\beta/8G$. Thus, on the physically relevant phase space, when the constraints are satisfied, the numerical value of the Hamiltonian is bounded from above by $1/4G$. Finally, using SU(1,1) spinors, we analyzed the issue of the lower bound. Using an argument along the lines given by Witten [17] in the (3+1)-dimensional theory, we showed, if the matter fields satisfy a local energy condition, β, and hence the value of the Hamiltonian on physical states, is necessarily non-negative, vanishing if and only if space-time is globally Minkowskian. Thus, on physical states, the Hamiltonian is bounded by $0 < H < 1/4G$.

We will now discuss the implications of these results to general relativity in 3+1 dimensions.

As we recalled in Sec. I, (3+1)-dimensional vacuum general relativity in the presence of a spacelike Killing field is equivalent to (2+1)-dimensional general relativity coupled to certain scalar fields [11] (which satisfy our energy condition). If the spatial Killing field in the (3+1)-dimensional theory is translational, the induced geometry in 2+1 dimensions can be expected to be asymptotically flat. An exhaustive analysis of such space-times was carried out recently under the assumption that there is an additional axial Killing field, and it was shown, in particular, that there exists a large class of examples in which the scalar fields have spatially compact support [21]. In all these cases, we can use our expression of the Hamiltonian to represent the energy per unit length (along translational isometry) in (3+1)-dimensional gravity waves. In particular, contrary to what one might have initially expected, this energy is bounded from above.

For concreteness and simplicity, let us restrict our detailed discussion to cylindrical waves [4–6] where both $\partial/\partial z$ and $\partial/\partial\theta$ Killing fields are hypersurface orthogonal. In the (2+1)-dimensional picture, these space-times correspond to gravity coupled to a single scalar field, where the scalar field and, well, the geometry have an additional rotational symmetry. In this case, one can go to coordinates t, r, θ with $-\infty < t < \infty$, $0 \le r < \infty$, $0 \le \theta < 2\pi$, in which the space-time metric takes the form

$$ds^2 = e^{\Gamma(r,t)}(-dt^2 + dr^2) + r^2 d\theta^2 \ , \tag{5.1}$$

where the coefficient Γ is completely determined by the scalar field Φ via

$$\Gamma(r,t) = \tfrac{1}{2}\int_0^r dr'\, r'[(\partial_t\Phi)^2 + (\partial_{r'}\Phi)^2] \ , \tag{5.2}$$

the integration being performed on a t =const slice. The scalar field Φ satisfies the wave equation with respect to the space-time metric (5.1). However, as remarked in Sec. I, because of axisymmetry, this is equivalent to the condition that it satisfy the wave equation with respect to the globally Minkowskian metric, obtained by setting $\Gamma(r,t) = 0$, i.e., with respect to

$$dS_0^2 = -dt^2 + dr^2 + r^2 d\theta^2 \ . \tag{5.3}$$

Because of this, the problem decouples: One can first solve for the scalar field Φ on the Minkowski space (5.3), construct Γ from Φ using (5.2), and just write down the resulting metric (5.1) to obtain a solution to the combined Einstein–scalar-field equations. (A similar decoupling occurs also when the Killing fields are not hypersurface orthogonal.)

Thorne's c energy [7] is easy to express in this framework: Apart from an overall constant, it is just the conserved energy associated with Φ propagating on the flat metric (5.2),

$$c := \tfrac{1}{16} \int_0^\infty r\, dr[(\partial_t\Phi)^2 + (\partial_r\Phi)^2] \;, \tag{5.4}$$

which, however, is to be interpreted as the energy associated with the coupled system, consisting of gravity and the scalar field. It is obvious that the c energy is nonnegative, vanishes if and only if $\Phi = 0$, and dS^2 is the flat Minkowskian metric dS_0^2, and that, even if one restricts oneself to scalar fields with compact spatial support, it is *unbounded* from above.

Let us compare it with our Hamiltonian. For the metric (5.1), the value of the Hamiltonian reduces to

$$H = \frac{1}{4G}(1 - e^{-4Gc}) \;. \tag{5.5}$$

Thus the relation is nonpolynomial. However, H *is* a monotonic function of c; both attain the value zero, their minimum, simultaneously, and as c tends to infinity, H tends to its upper bound $1/4G$. In the weak field limit, where the field Φ and hence the c energy can be taken to be small compared to $1/G$, the two agree. However, as one scales up Φ, space "curls up" and the "gravitational contribution" to the energy becomes significant. The total energy then is quite different from the c energy. Note also that the *boundedness of the Hamiltonian is a genuinely nonperturbative result.* Indeed, if we expand out the exponent, we obtain a power series in G:

$$H = c - 2Gc^2 + \tfrac{8}{3}G^2c^3 + \cdots \;, \tag{5.6}$$

where the individual terms, being proportional to the powers of c, are all unbounded. One can take the c energy as a function on the (gauge-fixed) phase space and ask for the canonical transformation it generates. Since it is a function only of H, one would expect it also to correspond to a time evolution for *some* lapse. This expectation is correct. The lapse is simply $N = \exp(\Gamma/2)$, which tends asymptotically to $\exp(4c)$. Thus, while the lapse corresponding to the Hamiltonian H is asymptotically identity (by the very definition of the Hamiltonian), that corresponding to c is not; even its asymptotic value is a "q number"—it depends on the phase space variables. Furthermore, as we approach the bound $\beta = 2$, the lapse corresponding to the c energy diverges. Within symplectic geometry, this is the origin of the unboundedness of the c energy. Note also that H is defined more generally, e.g., in the case when the four-geometry has only one (space-translational) Killing field which is not necessarily hypersurface orthogonal. Finally, the example of cylindrical waves brings out the fact that although the Hamiltonian is bounded, there is nothing unusual about time; it is not cyclic. This is because the points at which $H = 0$ and $1/4G$ are not identified; they correspond to entirely different geometries.

Let us briefly compare our results with those obtained in the twistorial approach to "quasilocal" quantities [27]. Cylindrical waves have been analyzed by Tod [28] in this framework. He found that in the limit appropriate to obtaining the total ADM-like energy per unit length, the prescription of [27] yields twice the c energy and is thus nonpolynomially related to our Hamiltonian. This is perhaps not surprising because it is known that the results of [27] are not always in agreement with those obtained by Hamiltonian methods. Quasilocal expressions which are geared to Hamiltonian methods were proposed in [29]. It would be interesting to evaluate them for cylindrical waves and compare the result with the one obtained here.

Finally, in this paper we have restricted ourselves to the behavior of the (2+1)-dimensional gravitational field at spatial infinity. A similar analysis can be carried out also at null infinity and again leads to some result which are surprising from a (3+1)-dimensional perspective [30].

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