Periodic solutions of the Einstein equations for binary systems

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Solutions of the Einstein equations which are periodic and have standing gravitational waves, in the weak-field zone, are valuable approximations to more physically realistic solutions with outgoing waves. A variational principle for the periodic solutions is found which has the power to provide, for binary systems with weak gravitational radiation, an accurate estimate of the relationship between the mass and angular momentum of the system, the masses and angular momenta of the components, the rotational frequency of the frame of reference in which the system is periodic, the frequency of the periodicity of the system, and the amplitude and phase of each multipole component of gravitational radiation. Examination of the boundary terms of the variational principle leads to definitions of the effective mass and effective angular momentum of a periodic geometry which capture the concepts of mass and angular momentum of the source alone with no contribution from the gravitational radiation. These effective quantities are surface integrals in the weak-field zone which are independent of the surface over which they are evaluated, through second order in the deviations of the metric from flat space. The variational principle provides a powerful method to examine the evolution of, say, a binary black hole system from the time when the holes are far apart, through the stage of slow evolution caused by gravitational radiation reaction, up until the moment when the radiation reaction time scale, is comparable to the dynamical time scale.

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I. INTRODUCTION

The difficulties associated with the study of solutions of the Einstein equations are well known. However, a number of specialized techniques have been developed which allow analyses in rather narrow circumstances. For example, the post-Newtonian approximation yields relativistic corrections for systems which are restricted to slow speeds and weak gravitational fields. Or symmetric geometries with two or three Killing vectors allow for study of Kerr or Schwarzschild black holes and both rotating and nonrotating neutron star models. A particularly fruitful technique involves the perturbation analysis of analytically known solutions—perturbations of flat space comprise linearized gravity [1]; the perturbations of spherically symmetric geometries are used to study the emission of radiation from test particles orbiting a black hole and the quasinormal oscillations of black holes [2] and neutron stars [3].

This paper presents a new restriction of the Einstein equations which is sufficiently limiting that analysis can proceed and yet sufficiently general to encompass a wide variety of interesting applications. Interest focuses on those solutions of the Einstein equations which allow a coordinate system in which the geometry is periodic in time inside a bounded region of space-time. Such a geometry might have strong fields, gravitational waves, and high speeds; and it might involve black holes or neutron

stars.

In quantum mechanics the physically realistic solutions of the Schrödinger equation with, say, an incoming wave packet being scattered into an outgoing wave packet are constructed from a linear combination of the periodic solutions which contain standing waves. Thus, the scattering problem is reduced to the analysis of periodic solutions.

In general relativity the nonlinearity of the Einstein equations intrudes on using this same idea of constructing physically interesting solutions from the periodic ones. But with care and some limitations, the basic process still provides an avenue toward otherwise unapproachable physical systems.

Periodic solutions of the Einstein equations with radiation are not asymptotically flat [4]—this is not surprising: In the weak-field zone, where the linearized Einstein equations give an approximate description of the gravitational field, the mass density of standing waves falls off as r^{-2}. Thus, the contribution of radiation to the mass content inside a radius r grows linearly with r; and the linearized Einstein equations cannot give an accurate description of the gravitational field out to either spatial or null infinity. But this is *not* a critical limitation to periodic geometries.

In this paper much attention is focused on the "weak-field zone"—a region of space-time, surrounding a fully relativistic source, throughout which the linearized Einstein equations give an accurate description of the gravitational field. A further requirement which we impose upon our use of the weak-field zone is that its total energy content in gravitational waves be much less than

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the mass of the source. Thus the weak-field zone is restricted to a region where the nonlinear effects of the radiation are small. For modeling realistic binary systems the boundary of our region of interest is always within the weak-field zone. Thus, *for periodic geometries we consider only a bounded region of space, and describe and impose boundary conditions well away from spatial or null infinity,* which might not even exist for some continuations past the boundary of a periodic geometry.

For a periodic example, consider the binary pulsar—the neutron stars themselves are relativistic, collapsed objects, and the system has an orbital period of a bit less than 8 h and a total mass of about $3M_\odot$. A region bounded at a distance of 1 light month contains approximately 200 wavelengths of quadrupole radiation from the source; $m/r \sim 10^{-11}$; the approximate energy content in the gravitational waves out to this distance is $10^{-22}M_\odot$, and the approximate binding energy of the system is $Gm^2/R \sim 7 \times 10^{-7}M_\odot$. The true, physical system has outgoing radiation near the boundary, and the orbit decays as a consequence. But a similar bounded, periodic, exact solution to the Einstein equations could be constructed from this physically realistic one by sending gravitational radiation inward from the boundary with amplitude and phase chosen to keep the system from evolving. The nature of the periodic geometry outside the boundary need not be considered. However, if the periodic geometry were extended outward to a distance of about 10^{22} light years, then the energy of the radiation would dominate that of the source; and, even at linearized order and with ignorance of the time evolution, the periodic geometry would no longer resemble the physically realistic one. For this system, only at such a large distance does the lack of asymptotic flatness of the periodic geometry intrude upon the analysis.

A second example is provided by a test particle of small mass μ close to the innermost stable circular orbit about a black hole of mass M with $\mu \ll M$. A perturbation analysis shows that the test particle nearly moves along a geodesic; the secular corrections are from radiation reaction effects at order $(\mu/M)^2$. The physically reasonable solution to this system involves only outgoing radiation. But a corresponding exact periodic solution can be constructed by demanding standing wave boundary conditions at a large but finite distance from the black hole. The energy density of the gravitational waves at a distance r is proportional to $(\mu/Mr)^2$. As long as the radius r of the boundary is large enough that $r \gg M$ but small enough that $r(\mu/M)^2 \ll M$, then the energy content of the radiation will be dwarfed by the mass of the black hole, and the boundary will still be in the weak-field zone.

These periodic solutions of the Einstein equations are interesting, involve strong gravitational fields, high speeds, are not asymptotically flat, and cannot be described by the linearized Einstein equations except in the weak-field zone. These are typical of the solutions of interest in this paper.

The relationship between periodic geometries and physically realistic ones with outgoing waves was carefully considered in paper I [5]. There it was shown that a specific linear combination of similar periodic geome-

tries, of differing frequencies with each geometry containing only standing waves in the weak-field zone, was an approximation to an exact solution of the Einstein equations with outgoing waves. Similar methods were first used in the context of general relativity by Thorne [6] who analyzed perturbations of neutron star models.

The error in this approximation was also carefully analyzed in paper I. When the effect of radiation reaction is weak, the linear combination is sharply peaked at a resonant frequency. And some physical quantity, describing an aspect of the linear combination of metrics, differs from the corresponding quantity for the exact outgoing-wave metric by an amount comparable to its change in one cycle due to the effects of radiation reaction. Thus, a resonant periodic solution alone accurately models an outgoing-wave solution as long as the radiation reaction time scale is much longer than the dynamical time scale. And the error in the linear combination is inversely proportional to the ratio of these two time scales; this is also proportional to the ratio of the frequency width of the linear combination to the resonant frequency.

Relativistic binary systems are, perhaps, the most interesting of the systems which can be approximated in this way. In particular the gravitational wave luminosity and the location of the innermost stable orbit in a relativistic, binary system can be studied. Other phenomena of interest which are approachable via periodic geometries include axisymmetric, rapidly rotating neutron star models, and quasinormal oscillations of black holes or neutron stars. However, even in circumstances when radiation reaction forces are strong and evolution is normally rapid, the periodic solutions are still interesting in their own right—after all they are solutions of the Einstein equations with generally strong gravitational fields and gravitational radiation.

Thus far the two most useful methods of studying relativistic binaries are the post-Newtonian approximation and the test particle approximation. In both of these approaches the effects of radiation reaction are ignored at the first order of a small quantity, and the consequent orbits studied are conservative and periodic. The periodic assumption includes both of these approaches as special cases and is, therefore, of more general validity.

Paper I contains a variational principle restricted to time independent geometries; in this paper the variational principle is generalized to include periodic geometries. The generalization reveals the role of gravitational radiation more clearly and substantially simplifies the treatment of the radiative boundary terms in the weak-field zone. Also, in paper I the boundary terms had to be evaluated in the local wave zone; in this paper it is necessary to go out only to the weak-field zone where the geometry is accurately approximated by the linearized Einstein equations—for the Sun-Earth system this is the difference between going out to Alpha Centauri for the local wave zone and out to the orbit of Jupiter for the weak-field zone. And given an approximate periodic geometry, which differs from an exact solution to the Einstein equations by order δ, the variational principle has the power to yield accurate estimates, with an error of order δ^2, of *all* the interesting quantities which describe

the geometry, except the actual metric itself. These accurately estimated quantities include the effective mass and angular momentum of the total system as well as its individual components, the frequency of periodicity, the angular frequency of rotation, and the amplitudes and phases of the gravitational radiation in a multipole decomposition.

The quantity being extremized in the variational principle is called the effective mass of the system. It is defined as a surface integral in the weak-field zone, and has the interesting property of being independent of the actual surface chosen for the integral, through terms of second order in the deviation of the metric from flat space as long as the surface is in the weak-field zone. In particular, the effective mass does not include the mass content of the standing gravitational waves, which diverges and keeps the geometry from being asymptotically flat. And a second surface integral, involved in the variational principle, defines the effective angular momentum in a similar manner which also does not include a divergent contribution from the standing gravitational waves.

Section II first reviews the 3+1 formalism of general relativity and, then, reviews and modestly modifies Thorne's [1] unique notation for symmetric-trace-free tensors, outlines his general solution of the linearized Einstein equations, and transforms to a gauge more suitable for present purposes. Previous familiarity with Thorne's notation greatly facilitates the reading of this paper. Section III gives a variational principle for a generic, periodic solution to the Einstein equations. The actual quantity being extremized is closely related to the quasilocal energy as developed by Brown and York [7], and the appropriate boundary conditions for the variational principle include the specification of the three-metric on the sides of the bounded region of space-time under consideration. In Sec. IV the variational principle is modified for the special case that the boundary is in the weak-field zone, and the boundary integrals are rewritten in terms of the amplitude and phase of the gravitational waves of different multipoles. The variational principle leads to formal definitions of the effective mass and effective angular momentum of a periodic geometry. Some of the complicated analyses are described in the Appendix.

II. BACKGROUND AND NOTATION

A. Initial value formalism and dynamics of general relativity

A four-dimensional space-time with a metric g_{ab} may be foliated into spacelike hypersurfaces of constant t,

with a metric γ_{ab} [8]:

$$g_{ab}\,dx^a dx^b = -N^2 dt^2 + \gamma_{ab}(dx^a + N^a dt)(dx^b + N^b dt).$$

$$(1)$$

The quantity N is the lapse function, and N^a is the shift vector. The three-dimensional metric has a derivative operator D_a and Ricci tensor R_{ab}.

The extrinsic curvature K_{ab} of the hypersurface is defined from

$$\mathcal{G}_{ab} \equiv -2NK_{ab} + 2D_{(a}N_{b)} = \mathcal{L}_t\gamma_{ab},$$

$$(2)$$

where $\mathcal{L}_t$ is the Lie derivative with respect to the time translation vector, $t^a\partial/\partial x^a \equiv \partial/\partial t$, which points in the direction of increasing t with all spatial coordinates held fixed. In the Hamiltonian formulation of general relativity [9], the momentum conjugate to γ_{ab} is $\pi^{ab}/16\pi$ where

$$\pi^{ab} \equiv -\gamma^{1/2}\left(K^{ab} - \gamma^{ab}K^c{}_c\right),$$

$$(3)$$

and γ is the determinant of γ_{ab}.

The constraint equations on a given hypersurface are restrictions on γ_{ab} and π^{ab} from the Einstein equations. These are the Hamiltonian constraint

$$\mathcal{N} \equiv R + \gamma^{-1}(\tfrac{1}{2}\pi_a{}^a\pi_b{}^b - \pi^{ab}\pi_{ab}) = 16\pi\rho$$

$$(4)$$

and the momentum constraint

$$\mathcal{N}^a \equiv D_b(\pi^{ab}/\gamma^{1/2}) = -8\pi j^a,$$

$$(5)$$

where ρ is the energy density and j^a is the momentum density of the stress energy of matter,

$$\rho = N^2 T^{tt},$$

$$(6)$$

$$j^a = N\gamma^{ab}T_b{}^t,$$

$$(7)$$

and the spatial part of the stress-energy tensor,

$$S_{ab} \equiv \gamma_a{}^c\gamma_b{}^d T_{cd},$$

$$(8)$$

is used below.

The dynamical part of the Einstein equations gives

$$\mathcal{P}^{ab} \equiv -N\gamma^{1/2}\left(R^{ab} - \frac{1}{2}\gamma^{ab}R\right) + \frac{1}{2}N\gamma^{-1/2}\gamma^{ab}\left(\pi^{cd}\pi_{cd} - \frac{1}{2}\pi^c{}_c\pi^d{}_d\right)$$

$$-2N\gamma^{-1/2}\left(\pi^{ac}\pi_c{}^b - \frac{1}{2}\pi^c{}_c\pi^{ab}\right) + \gamma^{1/2}\left(D^a D^b N - \gamma^{ab}D^c D_c N\right)$$

$$+\gamma^{1/2}D_c\left(\gamma^{-1/2}\pi^{ab}N^c\right) - \pi^{ac}D_c N^b - \pi^{bc}D_c N^a + 8\pi N\gamma^{1/2}\left(S^{ab} - \rho\gamma^{ab}\right)$$

$$= \mathcal{L}_t\pi^{ab}.$$

$$(9)$$

B. Linearized analysis in the weak-field zone

The space-times of interest in this paper are periodic and may contain gravitational standing waves far from the source. And periodic, standing waves cannot extend to spatial infinity in an asymptotically flat geometry [4]. However, if the amplitude of the waves is small enough, then a weak-field zone around the source exists wherein the geometry is well approximated by the linearized Einstein equations. It is in this weak-field zone that we analyze the gravitational waves and define quantities similar to the mass and angular momentum of the system.

1. Notation

For the geometry in the weak-field zone, a slight modification is made of Thorne's [1] notation for coordinates and symmetric-trace-free tensors. The lapse function at zeroth order in the deviation from flat space is a constant N_0, not necessarily unity; thus the Minkowskii time coordinate is $N_0 t$, and the Cartesian coordinates (x, y, z) are the traditional flat space coordinates, with (r, θ, ϕ) being the corresponding spherical coordinates. Indices (i, j, k, p, q, r, s) run over (x, y, z) and denote the Cartesian components of a spatial tensor with a flat metric. Summation is implied when these indices are repeated, and such indices are sometimes both lowered without ambiguity. The flat space derivative operator is ∇_i. A comma denotes a partial derivative with respect to a Cartesian coordinate. And the quantity $r_i \equiv \nabla_i r$ is the unit outward radial vector.

Tensors with large numbers (say l) of indices are common. Thus in the convenient abbreviation

$$T_{J_l} \equiv T_{j_1 j_2 \ldots j_l} \tag{10}$$

the indices represent tensor components in a Cartesian coordinate system, just like (i, j, k, p, q, r, s) do. Furthermore, often such tensors are both symmetric and trace free (STF) on all pairs of indices and also have Cartesian components which are functions only of t and r and independent of θ and ϕ; the Cartesian components of such STF tensors are written as capital script letters. For example,

$$\mathcal{T}_{J_l} = \mathcal{T}_{j_1 j_2 \ldots j_l} = \mathcal{T}_{(j_1 j_2 \ldots j_l)}, \tag{11}$$

$$\mathcal{T}_{j_1 j_1 j_3 \ldots j_l} = 0, \tag{12}$$

and

$$\partial \mathcal{T}_{J_l} / \partial \theta = 0, \quad \partial \mathcal{T}_{J_l} / \partial \phi = 0. \tag{13}$$

Also the abbreviation

$$R_{K_l} \equiv r_{k_1} r_{k_2} \cdots r_{k_l} \tag{14}$$

is useful to represent the outer product of many unit radial vectors.

A convenient decomposition of the STF l tensors is in terms of a basis set of $2l + 1$ constant STF tensors $\mathcal{Y}_{J_l}^{lm}$, with $-l \leq m \leq l$, defined by Thorne [his Eq. (2.12)]. Two useful properties of the $\mathcal{Y}_{J_l}^{lm}$ are

$$Y^{lm} = \mathcal{Y}_{J_l}^{lm} R_{J_l}, \tag{15}$$

where Y^{lm} is the usual spherical harmonic function, and their orthogonality

$$\mathcal{Y}_{J_l}^{lm} \mathcal{Y}_{J_l}^{lm'} = \frac{(2l+1)!!}{4\pi l!} \delta^{mm'}. \tag{16}$$

With this basis set of STF tensors the decomposition of $\mathcal{T}_{J_l}(t, r)$ yields

$$\mathcal{T}_{J_l}(t, r) = \sum_{m=-l}^{l} T_{lm}(t, r) \mathcal{Y}_{J_l}^{lm}. \tag{17}$$

Also if $\mathcal{T}_{J_l}(t, r)/r$ is a solution to the flat space wave equation then it may be separated into outgoing and ingoing parts:

$$\mathcal{T}_{J_l}(t, r) = \mathcal{T}_{J_l}^{\text{out}}(N_0 t - r) + \mathcal{T}_{J_l}^{\text{in}}(N_0 t + r). \tag{18}$$

And the decompositions

$$\mathcal{T}_{J_l}^{\text{out}}(N_0 t - r) = \sum_{m=-l}^{l} T_{lm}^{\text{out}}(N_0 t - r) \mathcal{Y}_{J_l}^{lm}, \tag{19}$$

$$\mathcal{T}_{J_l}^{\text{in}}(N_0 t + r) = \sum_{m=-l}^{l} T_{lm}^{\text{in}}(N_0 t + r) \mathcal{Y}_{J_l}^{lm} \tag{20}$$

are natural. Furthermore, if T_{lm}^{out} and T_{lm}^{in} are each composed of a sum of periodic pieces of different frequencies ω_{mn}, then complex amplitudes T_{lm}^n and phases θ_{lm}^{Tn} are defined from

$$T_{lm}^{\text{out}}(N_0 t - r) = \sum_n T_{lm}^n \exp[-i\theta_{lm}^{Tn} + \tfrac{1}{2} i l \pi + i\omega_{mn}(N_0 t - r)] \tag{21}$$

and

$$T_{lm}^{\text{in}}(N_0 t + r) = \sum_n T_{lm}^n \exp[i\theta_{lm}^{Tn} - \tfrac{1}{2} i l \pi + i\omega_{mn}(N_0 t + r)]. \tag{22}$$

If the waves are standing, then the θ_{lm}^{Tn} are real so that the outgoing and ingoing magnitudes are equal. The above equations lead to

$$T_{lm}(t, r) = \sum_n 2 T_{lm}^n \cos(\theta_{lm}^{Tn} - \tfrac{1}{2} l \pi + \omega_{mn} r) e^{i\omega_{mn} N_0 t} \tag{23}$$

and

$$\mathcal{T}_{J_l}(t,r) = \sum_n \sum_{m=-l}^{l} 2T_{lm}^n \mathcal{Y}_{J_l}^{lm} \cos(\theta_{lm}^{Tn} - \tfrac{1}{2}l\pi + \omega_{mn}r)e^{i\omega_{mn}N_0 t}. \tag{24}$$

Thorne defines $\mathcal{Y}_{J_l}^{lm}$ as a special STF tensor which depends upon a particular orientation of the Cartesian axes. Thus one frame of reference rotated with respect to a second has a different corresponding $\mathcal{Y}_{J_l}^{lm}$. For a frame of reference rotating with a coordinate angular velocity Ω about the z axis with respect to an inertial frame,

$$\mathcal{Y}_{J_l \ \text{rot}}^{lm} = e^{-im\Omega t} \mathcal{Y}_{J_l \ \text{inertial}}^{lm}, \tag{25}$$

from Thorne's Eq. (2.12) if the axes are aligned at $t = 0$. And, as viewed from the rotating frame,

$$\mathcal{T}_{J_l}(t,r) = \sum_n \sum_{m=-l}^{l} 2T_{lm}^n \mathcal{Y}_{J_l \ \text{rot}}^{lm} \cos(\theta_{lm}^{Tn} - \tfrac{1}{2}l\pi + \omega_{mn}r)e^{i(\omega_{mn}+m\Omega/N_0)N_0 t}. \tag{26}$$

The final specialization of interest is that $\mathcal{T}_{J_l}$ be periodic with fundamental frequency ω_0 as viewed from the rotating frame of reference. Equation (26) then requires that

$$\omega_{mn} = n\omega_0 - m\Omega/N_0, \tag{27}$$

for n an integer.

If $\mathcal{T}_{J_l}$ is constant in time in the inertial frame of reference, then ω_{mn} must vanish for all m and n for which T_{lm}^n is not zero. If, in addition, Ω is nonzero and $\Omega/\omega_0 N_0$ is not rational, then the constant $T_{lm}^n = 0$ unless $m = n = 0$, i.e., $\mathcal{T}_{J_l}$ is constant in time and axisymmetric, or if Ω is zero, then $T_{lm}^n = 0$ for $n \neq 0$. Also, if $\Omega/\omega_0 N_0$ is an integer, then $\mathcal{T}_{J_l}$ is also periodic when viewed from an inertial frame of reference.

From Thorne's definition of $\mathcal{Y}_{J_l}^{lm}$ it follows that

$$\mathcal{Y}_{J_l}^{lm} = (-1)^m \mathcal{Y}_{J_l}^{l,-m*}, \tag{28}$$

and consequently if $\mathcal{T}_{J_l}$ is real then

$$\omega_{m,n} = -\omega_{-m,-n}, \tag{29}$$

$$\theta_{l,m}^{T,n} = -\theta_{l,-m}^{T,-n}, \tag{30}$$

and

$$T_{l,m}^n = (-1)^{l+m} T_{l,-m}^{-n*}. \tag{31}$$

2. General linearized solution

Thorne [1] gives a general solution to the linearized Einstein equations in terms of the mass and current moments evaluated at a retarded time, $\mathcal{I}_{K_l}(N_0 t - r)$ and $\mathcal{S}_{K_l}(N_0 t - r)$, in one specialization of the Lorentz gauge. It is convenient to separate the stationary, time independent parts of the metric from the (often radiative) time-dependent parts. The superscript zero refers to the constant moments so that $\mathcal{I}^0$ and $\mathcal{S}_j^0$ are the constant mass monopole and current dipole moment of the geometry— the mass and angular momentum of the geometry if there is no gravitational radiation. Useful definitions are

$$I \equiv \frac{2\mathcal{I}^0}{r} + \sum_{l=2}^{\infty} \frac{2(2l-1)!!}{r^{l+1}l!}\mathcal{I}_{K_l}^0 R_{K_l} \tag{32}$$

and

$$S^j \equiv \frac{-2f^{ji}\epsilon_{ipq}\mathcal{S}_p^0 r_q}{r^2} - f^{ji}\sum_{l=2}^{\infty} \frac{4l(2l-1)!!}{r^{l+1}(l+1)!}\epsilon_{ipq}\mathcal{S}_{pK_{l-1}}^0 r_q R_{K_{l-1}}. \tag{33}$$

The boldface ϵ_{jpq} is the Levi-Civita tensor.

Thorne's metric is just a perturbation away from flat space-time, and consequences of his Eq. (8.13), through first order in the perturbation, are

$$N = N_0 \left\{ 1 - \frac{1}{2}I - \sum_{l=2}^{\infty} \frac{(-\epsilon)^l}{l!}\left[r^{-1}\mathcal{I}_{K_l}(N_0 t - \epsilon r)\right]_{,K_l} \right\} \tag{34}$$

and

$$N^j = N_0 S^j + N_0 f^{ji}\sum_{l=2}^{\infty} \frac{4l(-\epsilon)^{l+1}}{(l+1)!}\left[r^{-1}\epsilon_{ipq}\mathcal{S}_{pK_{l-1}}(N_0 t - \epsilon r)\right]_{,qK_{l-1}}$$

$$+ N_0 f^{ji}\sum_{l=2}^{\infty} \frac{4(-\epsilon)^l}{l!}[r^{-1}\dot{\mathcal{I}}_{iK_{l-1}}(N_0 t - \epsilon r)]_{,K_{l-1}} \tag{35}$$

for the lapse and shift; a dot represents a derivative with respect to the function argument $(N_0 t - \epsilon r)$ and ϵ is $+1$ for outgoing radiation and -1 for ingoing. The three-metric, through first order, has the stationary part separated from the time-dependent part by

$$\gamma_{ij} = f_{ij}(1 + I) + h_{ij}, \tag{36}$$

where f_{ij} is the flat three-metric in Cartesian components, and the time-dependent terms are all contained in

$$h_{ij} = f_{ij}\sum_{l=2}^{\infty}\frac{2(-\epsilon)^l}{l!}\left[r^{-1}\mathcal{I}_{K_l}(N_0 t - \epsilon r)\right]_{,K_l} + \sum_{l=2}^{\infty}\frac{4(-\epsilon)^l}{l!}\left[r^{-1}\ddot{\mathcal{I}}_{ijK_{l-2}}(N_0 t - \epsilon r)\right]_{,K_{l-2}}$$
$$+\sum_{l=2}^{\infty}\frac{8l(-\epsilon)^{l+1}}{(l+1)!}\left[r^{-1}\epsilon_{pq(i}\dot{\mathcal{S}}_{j)pK_{l-2}}(N_0 t - \epsilon r)\right]_{,qK_{l-2}}. \tag{37}$$

A different version of the Lorentz gauge is preferred here wherein all of the time dependence is removed from the lapse and shift vector. The gauge change generated by (ξ_t, ξ_i), where

$$\partial\xi_t/\partial t = -N_0^2\sum_{l=2}^{\infty}\frac{(-\epsilon)^l}{l!}\left[r^{-1}\mathcal{I}_{K_l}(N_0 t - \epsilon r)\right]_{,K_l},$$
$$\partial\xi_j/\partial t = -\nabla_j\xi_t - (N_j - N_0 S_j), \tag{38}$$

accomplishes this. The ultimate result is that the general solution to the linearized Einstein equations, in the preferred gauge, has a lapse and shift vector

$$N = N_0(1 - I/2) \tag{39}$$

and

$$N^j = N_0 S^j. \tag{40}$$

And γ_{ij} is still given by Eq. (36) but with

$$h_{ij} = \sum_{l=2}^{\infty}(-\epsilon)^l\frac{2}{l!}\left\{f_{ij}[r^{-1}\mathcal{I}_{K_l}]_{,K_l} + 2[r^{-1}\mathcal{I}_{ijK_{l-2}}^{(+2)}]_{,K_{l-2}} + [r^{-1}\mathcal{I}_{K_l}^{(-2)}]_{,ijK_l} - 4[r^{-1}\mathcal{I}_{K_{l-1}(i}]_{,j)K_{l-1}}\right\}$$
$$+\sum_{l=2}^{\infty}\frac{(-\epsilon)^{l+1}8l}{(l+1)!}\left\{[r^{-1}\epsilon_{pq(i}\mathcal{S}_{j)pK_{l-2}}^{(+1)}]_{,qK_{l-2}} - [r^{-1}\mathcal{S}_{pK_{l-1}}^{(-1)}\epsilon_{pq(i}]_{,j)qK_{l-1}}\right\}, \tag{41}$$

where the parenthesized superscript, e.g., $\mathcal{S}_{pK_{l-1}}^{(-1)}$, denotes differentiation with respect to the implied argument $(N_0 t - \epsilon r)$ the appropriate number of times, positive or negative. Also

$$K_{ij} = -\frac{1}{2N_0}\frac{\partial h_{ij}}{\partial t} + \nabla_{(i}S_{j)}. \tag{42}$$

In this gauge a number of useful identities hold:

$$f^{ij}h_{ij} = 0, \tag{43}$$
$$f^{ij}K_{ij} = 0, \tag{44}$$
$$\nabla_i h^i{}_j = 0, \tag{45}$$

$$r_i S^i = 0, \tag{46}$$
$$\nabla_i S^i = 0, \tag{47}$$
$$\nabla_k\nabla^k I = 0, \tag{48}$$
$$\nabla_k\nabla^k S^i = 0, \tag{49}$$

and

$$\nabla_k\nabla^k h_{ij} - \frac{1}{N_0^2}\frac{\partial^2 h_{ij}}{\partial t^2} = 0. \tag{50}$$

In the local wave zone the leading $1/r$ contributions to h_{ij} from the time-dependent moments give

$$h_{ij} = \sum_{l=2}^{\infty}\frac{4}{rl!}\mathcal{I}_{pqK_{l-2}}^{(l)}R_{K_{l-2}}(\sigma_{0(i}^p\sigma_{0j)}^q - \tfrac{1}{2}\sigma_0^{pq}\sigma_{0ij})$$
$$+\sum_{l=2}^{\infty}\frac{8l}{r(l+1)!}\epsilon_{rsp}\mathcal{S}_{qrK_{l-2}}^{(l)}r_s R_{K_{l-2}}(\sigma_{0(i}^p\sigma_{0j)}^q - \tfrac{1}{2}\sigma_0^{pq}\sigma_{0ij}) + O(r^{-2}), \tag{51}$$

where σ_{0ij} is the two-dimensional metric of a constant-r two-sphere. Also in the wave zone $r^i h_{ij} = O(r^{-2})$.

3. Standing wave solutions in the rotating frame of reference

The amplitudes I_{lm}^n and phases θ_{lm}^{In} of the gravitational radiation are defined in an inertial frame of reference in a manner similar to Eqs. (21) and (22):

$$(-\epsilon)^l \mathcal{I}_{K_l}(N_0 t - \epsilon r) = \sum_{m,n} I_{lm}^n \mathcal{Y}_{K_l}^{lm} \exp[-i\epsilon(\theta_{lm}^{In} - \tfrac{1}{2}l\pi) + i w_{mn}(N_0 t - \epsilon r)] \qquad (52)$$

and similarly for $\mathcal{S}_{K_l}$.

For the special case that the geometry is periodic in a rotating frame of reference, and contains no traveling waves (only standing waves) the notation of Sec. II B 1 allows the linearized geometry in the rotating frame of reference to be written in terms of the stationary moments I_{lm}^0 and S_{lm}^0 and the periodic, radiative moments I_{lm}^n and S_{lm}^n as

$$N = N_0(1 - I/2) \qquad (53)$$

and

$$N^j = N_0 S^j + \Omega \Phi^j, \qquad (54)$$

where

$$\Phi^i \partial/\partial x^i \equiv \partial/\partial\phi, \qquad (55)$$

$$I = \frac{2\mathcal{I}^0}{r} + \sum_{l,m} \frac{2(2l-1)!!}{r^{l+1} l!} I_{lm}^0 \mathcal{Y}_{K_l}^{lm} R_{K_l}, \qquad (56)$$

and

$$S^j = -\frac{2f^{ji}\epsilon_{ipq} r_q S_p^0}{r^2} - f^{ji}\sum_{l,m} \frac{4l(2l-1)!!}{r^{l+1}(l+1)!} \epsilon_{ipq} S_{lm}^0 \mathcal{Y}_{pK_{l-1}}^{lm} r_q R_{K_{l-1}}. \qquad (57)$$

Also

$$h_{ij} = \sum_{l,m,n} \frac{4}{l!} I_{lm}^n e^{inw_0 N_0 t}\{f_{ij}\mathcal{Y}_{K_l}^{lm}[r^{-1}\cos(\theta_{lm}^{nI} - \tfrac{1}{2}l\pi + w_{mn}r)]_{,K_l} - 2w_{mn}^2 \mathcal{Y}_{ijK_{l-2}}^{lm}[r^{-1}\cos(\theta_{lm}^{nI} - \tfrac{1}{2}l\pi + w_{mn}r)]_{,K_{l-2}}$$

$$-w_{mn}^{-2}\mathcal{Y}_{K_l}^{lm}[r^{-1}\cos(\theta_{lm}^{nI} - \tfrac{1}{2}l\pi + w_{mn}r)]_{,ijK_l} - 4\mathcal{Y}_{K_{l-1}(i}^{lm}[r^{-1}\cos(\theta_{lm}^{nI} - \tfrac{1}{2}l\pi + w_{mn}r)]_{,j)K_{l-1}}\}$$

$$-\sum_{l,m,n} \frac{16l}{(l+1)!} S_{lm}^n e^{inw_0 N_0 t}\left\{w_{mn}\epsilon_{pq(i}\mathcal{Y}_{j)pK_{l-2}}^{lm}[r^{-1}\sin(\theta_{lm}^{nS} - \tfrac{1}{2}l\pi + w_{mn}r)]_{,qK_{l-2}}\right.$$

$$\left. +w_{mn}^{-1}\mathcal{Y}_{pK_{l-1}}^{lm}\epsilon_{pq(i}[r^{-1}\sin(\theta_{lm}^{nS} - \tfrac{1}{2}l\pi + w_{mn}r)]_{,j)qK_{l-1}}\right\}. \qquad (58)$$

In the wave zone this becomes

$$h_{ij} = \sum_{l,m,n} \frac{8(w_{mn})^l}{rl!} I_{lm}^n \cos(\theta_{lm}^{nI} + w_{mn}r)\mathcal{Y}_{K_{l-2}pq}^{lm} R_{K_{l-2}}(\sigma_{0i}^p \sigma_{0j}^q - \tfrac{1}{2}\sigma_0^{pq}\sigma_{0ij})e^{inw_0 N_0 t}$$

$$+\sum_{l,m,n} \frac{16l(w_{mn})^l}{r(l+1)!} S_{lm}^n \cos(\theta_{lm}^{nS} + w_{mn}r)\epsilon_{rsp}\mathcal{Y}_{qrK_{l-2}}^{lm} r^s R_{K_{l-2}}(\sigma_{0i}^p \sigma_{0j}^q - \tfrac{1}{2}\sigma_0^{pq}\sigma_{0ij})e^{inw_0 N_0 t}. \qquad (59)$$

These equations give the general periodic solution to the linearized Einstein equations, in the gauge described above, with stationary moments I_{lm}^0 and S_{lm}^0 and constant wave amplitudes and phases $(I_{lm}^n, \theta_{lm}^{In})$ and $(S_{lm}^n, \theta_{lm}^{Sn})$. And for h_{ij} to represent real, standing waves it is necessary that

$$I_{lm}^n = (-1)^{l+m} I_{l,-m}^{-n*}, \qquad (60)$$

$$\theta_{lm}^{In} = -\theta_{l,-m}^{I,-n}, \qquad (61)$$

and similarly for S_{lm}^n and θ_{lm}^{Sn}.

Finally, it is useful in Sec. IV and in the Appendix to separate out the part of h_{ij} in Eq. (58) which includes contributions from both I_{lm}^n and S_{lm}^n and is summed over l, with both m and n held fixed. Thus

$$h_{ij} = \sum_{m,n} h_{ij}^{mn} \qquad (62)$$

and

$$\nabla_k \nabla^k h_{ij}^{mn} - \omega_{mn}^2 h_{ij}^{mn} = 0. \qquad (63)$$

III. VARIATIONAL PRINCIPLE FOR PERIODIC SOLUTIONS OF THE EINSTEIN EQUATIONS

A variational principle comes from the traditional Hamiltonian formalism of general relativity and is very closely related to the Hamiltonian and the concept of quasilocal energy as developed by Brown and York [7], their Eq. (4.13). The starting point is the definition

$$
16\pi H_1 \equiv -\int \left\{ N\left[R + \frac{1}{\gamma}\left(\frac{1}{2}\pi_a{}^a\pi_b{}^b - \pi^{ab}\pi_{ab}\right)\right] + 2N^a D_b(\pi^b{}_a/\gamma^{1/2}) \right\} \sqrt{\gamma}\, d^3x
$$

$$
+ \oint_{r_B} 2N^b\gamma^{-1/2}\pi_b{}^a r_a \sqrt{\sigma}\, d^2x - \oint_{r_B} 2N\sigma^{ab}D_a r_b \sqrt{\sigma}\, d^2x. \tag{64}
$$

This volume integral is over a spacelike hypersurface of finite extent which is bounded by a two-surface defined by a scalar field, r, which is constant on the boundary, $r = r_B$. Here in Sec. III the vector r^a is the outward-pointing unit normal to the bounding two-surface, σ_{ab} is both the metric of the two-surface and the projection operator onto the two-surface,

$$
\sigma_{ab} \equiv \gamma_{ab} - r_a r_b, \tag{65}
$$

σ is its determinant, and there is no restriction on the location of the boundary of the finite region in which we are interested—in particular it is necessary neither that the boundary extend out to the weak-field zone nor that the scalar field r which defines the boundary be related to a flat space radial coordinate.

An arbitrary, infinitesimal variation of N, N^a, γ_{ab}, and π^{ab}, which holds the location of the boundary fixed, results in an infinitesimal change in H_1:

$$
16\pi\delta H_1 = -\int \left(\delta N\mathcal{N}\sqrt{\gamma} + 2\delta N^a\mathcal{N}_a\sqrt{\gamma} + \delta\gamma_{ab}\mathcal{P}^{ab} - \delta\pi^{ab}\mathcal{G}_{ab}\right) d^3x
$$

$$
+ \oint_{r_B} \left(r_a N^a\gamma^{-1/2}\pi^{bc}\delta\gamma_{bc} + 2\delta N^b\gamma^{-1/2}\pi_b{}^a r_a\right) \sqrt{\sigma}\, d^2x
$$

$$
- \oint_{r_B} \left[N\delta\sigma^{ab}D_a r_b + (2\delta N + N\sigma^{-1}\delta\sigma)D_a r^a + \sigma^{-1}\delta\sigma r^a D_a N\right] \sqrt{\sigma}\, d^2x. \tag{66}
$$

The indices of the perturbed quantities are neither raised nor lowered by a metric. We now assume that a gauge may be chosen with $r_a N^a = 0$ on the boundary. Under some circumstances this might not be possible as recently emphasized by Hayward [10]; but for the applications which we currently envision, namely, a static boundary in the weak-field zone, this is possible. Thus the first term in the first surface integral of Eq. (66) vanishes, and a comparison of the coefficients of the variations of N, N^a, γ_{ab}, and π^{ab} with the Einstein equations (4), (5), (9), and (2), respectively, shows that H_1 is a Hamiltonian for the vacuum Einstein equations when the boundary conditions are that N, N^a, and σ^{ab} are given functions of time on the surface at r_B so that the remaining variations within the surface terms all vanish.

A tradition in introductory electricity and magnetism is to consider solutions to Maxwell's equations inside a "box," with the boundary conditions, say, of the potential being given on the surface of the box. The Hamiltonian H_1 and the boundary conditions of the previous paragraph are appropriate for the gravitational analog of this tradition.

We have particular interest in periodic geometries, when all of N, N^a, γ_{ab}, and π^{ab} are periodic in t with period $T = 2\pi/N_0\omega_0$. It is most convenient to consider the geometrical quantities to be functions of the spatial coordinates and a dimensionless time coordinate $\tau \equiv 2\pi t/T$, and T is left as unknown and yet to be determined.

Along with the periodicity, the definition of the time average of a dynamical quantity, say H_1, is

$$
\langle H_1 \rangle \equiv T^{-1}\int_0^T H_1\, dt = \frac{1}{2\pi}\int_0^{2\pi} H_1\, d\tau. \tag{67}
$$

The action used in Maupertuis's principle of least action plays an important role here. This is referred to as the M action A defined by

$$
16\pi A \equiv \int_0^T \int \pi^{ab}\mathcal{L}_t\gamma_{ab}\, d^3x\, dt
$$

$$
= \int_0^{2\pi} \int \pi^{ab}\partial\gamma_{ab}/\partial\tau\, d^3x\, d\tau. \tag{68}
$$

It follows from Eqs. (66) and (68) that

$$
16\pi\delta\langle H_1\rangle = -\frac{1}{2\pi}\int_0^{2\pi}\int\left[\delta N\mathcal{N}\sqrt{\gamma} + 2\delta N^a\mathcal{N}_a\sqrt{\gamma} - \delta\gamma_{ab}(\mathcal{L}_t\pi^{ab} - \mathcal{P}^{ab}) + \delta\pi^{ab}(\mathcal{L}_t\gamma_{ab} - \mathcal{G}_{ab})\right] d^3x\, d\tau
$$

$$
+ 16\pi\delta A/T + \frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B} 2\delta N^b\gamma^{-1/2}\pi^a{}_b r_a\sqrt{\sigma}\, d^2x\, d\tau
$$

$$
- \frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B}\left[N\delta\sigma^{ab}D_a r_b + (2\delta N + N\sigma^{-1}\delta\sigma)D_a r^a + \sigma^{-1}\delta\sigma r^a D_a N\right]\sqrt{\sigma}\, d^2x\, d\tau. \tag{69}
$$

Equation (69) reveals the use of $\langle H_1 \rangle$ in a variational principle. Consider the class of $(N, N^a, \gamma_{ab}, \pi^{ab})$ restricted to periodic functions of τ, with period 2π, with a specific, predetermined value for A and which satisfy boundary conditions at r_B specified as particular periodic functions of τ for N, N^a, and σ^{ab}. Then $\langle H_1 \rangle$, considered a functional of the $(N, N^a, \gamma_{ab}, \pi^{ab})$ in this class, is an extremum under arbitrary infinitesimal variations of the $(N, N^a, \gamma_{ab}, \pi^{ab})$, remaining in the class, if and only if the $(N, N^a, \gamma_{ab}, \pi^{ab})$ satisfy the Einstein equations.

Furthermore, the unknown T can be found from the variational principle as well. If only A is changed by a small amount, ΔA, and the variational principle reapplied to find a corresponding change $\Delta \langle H_1 \rangle$, then Eq. (69) shows that

$$T = \Delta A / \Delta \langle H_1 \rangle. \qquad (70)$$

IV. VARIATIONAL PRINCIPLE EXTENDED TO THE WEAK-FIELD ZONE

The importance of boundary terms in the Hamiltonian of general relativity was emphasized by Regge and Teitelboim [11]—different boundary conditions necessitate the inclusion of different boundary integrals in the Hamiltonian. For data consisting of N, N^a, and σ^{ab} on the boundary, H_1 is the appropriate Hamiltonian. Instead the data could consist of, essentially, the derivatives of some of these quantities normal to the boundary, in which case the appropriate Hamiltonian would be that of Eq. (64) but without the boundary integrals at r_B. The variational principle of Sec. III is closely related to an action principle, which is usually the source of definitions for quantities such as mass and angular momentum as particular boundary integrals. Thus, for each different choice of a description of the data at the boundary, similar analysis results in a variational principle involving different appropriate boundary integrals, different definitions of quantities similar to the mass and angular momentum, and a different $\langle H \rangle$. For the periodic, radiative geometries of this paper the amplitudes and phases of the multipole components (rather than the fields and their normal derivatives) are the convenient independent boundary data. And this choice determines the boundary integrals of the variational principle and the quantities which correspond to mass and angular momentum.

In this section we discuss quantities similar to the mass, angular momentum, and amplitude and phase of gravitational radiation, all in the context of periodic solutions of the Einstein equations. Now, these quantities are generally not well defined except at spatial or null infinity; and even there technical difficulties persist except under the best of circumstances. In addition our analysis is limited to a bounded region of space-time which specifically excludes any infinity. It is not surprising then that our approach does not directly involve an elegant, gauge invariant definition of any of these quantities.

Thus we forsake explicit gauge invariance for a precise variational principle involving the multipole moments of the geometry. While the physical interpretation of these quantities may be suspect, they have the virtues of being precisely defined in the context of the chosen gauge and straightforward to calculate. Furthermore, after the fact, a periodic geometry can be analyzed and if the amplitude of gravitational radiation (as defined in this gauge dependent manner) is sufficiently small, then the geometry near the boundary is of the form of the general linearized solution of Sec. II B. And the mass monopole $\mathcal{I}^0$ and current dipole $\mathcal{S}_j^0$ are reminiscent of mass and angular momentum.

A. Specification of boundary data

The first task of this section is to describe the data on the boundary in terms of the amplitudes (I_{lm}^n, S_{lm}^n) and phases $(\theta_{lm}^{In}, \theta_{lm}^{Sn})$ of the multipole moments. The physical metric near the boundary is not necessarily flat and may, in fact, be far from flat. None the less, in the vicinity of the boundary we sometimes use a spatial, Cartesian coordinate system, (x, y, z) along with the Euclidean tensors h_{ij} and S^i, defined in Eqs. (58) and (57), in terms of the multipole moments; also, the two-sphere at constant r has a unit normal r_0^i when embedded in flat space and r^a when embedded in the geometry described by the metric γ_{ab}. The notation of Sec. II B is employed here except that the symbol r used as a tensor index on one of the Cartesian tensors denotes the implied contraction of the index with r_0^i.

The geometry is still assumed to be periodic, and so all tensors are periodic functions of the dimensionless time coordinate τ with period 2π. Also the frame in which the geometry is periodic is assumed to be uniformly rotating with respect to the frame of reference tied to (x, y, z) with angular velocity Ω about the z axis.

A prescription for uniquely specifying data on the boundary follows: Choose N_0 and r_B, once and for all—these are never changed. Now choose values of T, Ω, I_{lm}^n, θ_{lm}^{In}, S_{lm}^n, and θ_{lm}^{Sn}. In terms of these chosen values let each of the following geometrical quantities have a value *exactly* equal to that given by just the linearized theory as described in Eqs. (53)–(58): $\sqrt{\sigma}$, $N^2\sqrt{\sigma}$, $\sqrt{\sigma}\sigma^{ab}$, N^a, $N(\sigma_a^c \sigma_b^d D_c r_d - \frac{1}{2}\sigma_{ab} D_c r^c)$, $N^{-1}D_a r^a$, $r^a D_a N$, and $\gamma^{-1/2}\pi_a{}^b r_b \sqrt{\sigma}$. Also, let S^a and Φ^a represent the contravariant tensors in generic coordinates whose components in the Cartesian coordinates are given in Eqs. (57) and (55), respectively. Note that each of these quantities is independent of the others. For example, the determinant of $\sqrt{\sigma}\sigma^{ab}$ is unity and does not depend upon $\sqrt{\sigma}$. And $N(\sigma_a^c \sigma_b^d D_c r_d - \frac{1}{2}\sigma_{ab} D_c r^c)$ is trace free and independent of $D_a r^a$. These specific choices for data on the boundary are made so that the boundary integrals in Eq. (73) below are *precisely* equal to what would be expected through second order in an expansion in powers of the deviation from flat space.

B. Variational principle

The next step toward a variational principle is the definition

$$16\pi H_2 \equiv -\int \left\{ N\left[R + \frac{1}{\gamma}\left(\frac{1}{2}\pi_a{}^a\pi_b{}^b - \pi^{ab}\pi_{ab}\right)\right] + 2N^a D_b(\pi^b{}_a/\gamma^{1/2})\right\}\sqrt{\gamma}\,d^3x$$

$$+ \oint_{r_B} 2N_0 S^a \gamma^{-1/2}\pi_a{}^b r_b \sqrt{\sigma}\,d^2x - \oint_{r_B} 2N\sqrt{\sigma}\sigma^{ab}D_a r_b\,d^2x. \tag{71}$$

An arbitrary, infinitesimal variation of N, N^a, γ_{ab}, and π^{ab}, which holds fixed the location of the boundary, results in an infinitesimal change of $\langle H_2\rangle$ which is similar to the change in $\langle H_1\rangle$ in Eq. (69) and is simplified by the substitutions of A, defined in Eq. (68), and of a quantity, J, which reduces to the current dipole moment for the specific boundary data given:

$$8\pi J \equiv -\frac{1}{2\pi}\int_0^{2\pi} \oint_{r_B} \Phi^a \gamma^{-1/2}\pi_a{}^b r_b\sqrt{\sigma}\,d^2x\,d\tau$$

$$= 8\pi S_z^0. \tag{72}$$

Then

$$16\pi\delta\langle H_2\rangle = -\frac{1}{2\pi}\int_0^{2\pi}\int E_{\mathrm{Eqs.}}\,d^3x\,d\tau + 16\pi\delta A/T + 16\pi\Omega\delta J + \frac{1}{\pi}\int_0^{2\pi}\oint_{r_B}\delta(N_0 S^a)\gamma^{-1/2}\pi_a{}^b r_b\sqrt{\sigma}\,d^2x\,d\tau$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B}[\delta(\sqrt{\sigma}\sigma^{ab})N(D_a r_b - \tfrac{1}{2}\sigma_{ab}D_c r^c) + \delta(N^2\sqrt{\sigma})N^{-1}D_a r^a + 2\delta(\sqrt{\sigma})r^a D_a N]\,d^2x\,d\tau. \tag{73}$$

The symbol $E_{\mathrm{Eqs.}}$ is an abbreviation for the first integrand in Eq. (69) and vanishes when the Einstein equations are satisfied.

The two surface integrals in Eq. (73) are examined in the Appendix where it is shown that

$$\frac{1}{\pi}\int_0^{2\pi}\oint_{r_B}\delta(N_0 S^a)\gamma^{-1/2}\pi_a{}^b r_b\sqrt{\sigma}\,d^2x\,d\tau$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B}[\delta(\sqrt{\sigma}\sigma^{ab})N(D_a r_b - \tfrac{1}{2}\sigma_{ab}D_c r^c) + \delta(N^2\sqrt{\sigma})N^{-1}D_a r^a + 2\delta(\sqrt{\sigma})r^a D_a N]\,d^2x\,d\tau.$$

$$= \frac{\delta N_0}{2\pi}\int_0^{2\pi}\oint_{r_B}\left[2S^a\gamma^{-1/2}\pi_{ab}r^b - \frac{2N}{N_0}D_a r^a\right]\sqrt{\sigma}\,d^2x\,d\tau$$

$$+\frac{N_0}{4\pi}\delta\left\{\int_0^{2\pi}\oint_{r_B}[\tfrac{1}{2}h^{ij}r_0^k\nabla_k h_{ij} - r_{0p}\nabla_i(h^{ij}\sigma_{0jq}h^{qp}) + \tfrac{5}{2}h_{rr}h_{rr}/r - h^{ik}h_{ki}/r + Ir_0^i\nabla_i I\right.\right.$$

$$\left.+ S^i S_i/r - S^i r_0^j\nabla_j S_i + \tfrac{1}{2}rr_0^p\nabla_p(h_{ij})r_0^q\nabla_q(h^{ij}) - \tfrac{1}{2}h_{ij}r_0^p\nabla_p(rr_0^q\nabla_q h^{ij})]\sqrt{\sigma_0}\,d^2x\,d\tau\right\}$$

$$-\frac{N_0}{8\pi}\sum_{m,n}\left(n\omega_0 - \frac{m\Omega}{N_0}\right)\delta\left\{\int_0^{2\pi}\oint_{r_B}\omega_{mn}^{-1}[rr_0^p\nabla_p(h_{ij}^{mn})r_0^q\nabla_q(h_{ij}^{mn*}) - h_{ij}^{mn*}r_0^p\nabla_p(rr_0^q\nabla_q h_{ij}^{mn})]\sqrt{\sigma_0}\,d^2x\,d\tau\right\}$$

$$+N_0\sum_{lmn}\left[\frac{8(\omega_{mn})^{2l+1}(l+1)(l+2)}{(l!)^2 l(l-1)}I_{lm}^n I_{lm}^{n*}\delta\theta_{lm}^{In} + \frac{32(\omega_{mn})^{2l+1}l(l+1)(l+2)}{[(l+1)!]^2(l-1)}S_{lm}^n S_{lm}^{n*}\delta\theta_{lm}^{Sn}\right]. \tag{74}$$

The different terms on the right hand side of this equation come from expressions in the Appendix. Inside the total δ, the first term is built into expression (A11); the next three are the second integral of expression (A14); the three terms involving the stationary moments are expressions (A13) and (A10); and the last two terms come from expression (A20). The part of Eq. (74) proportional to $(n\omega_0 - m\Omega/N_0)$ is the second integral of expression (A20). And the last summation in Eq. (74), which is proportional to the variation of the phases, is expression (A16).

Consideration of Eqs. (73) and (74) leads to natural re-definitions of J and A which move the parts of Eq. (74) proportional to $16\pi\Omega$ into δJ and the parts proportional to $8N_0\omega_0(= 16\pi/T)$ into δA. These changes dramatically simplify Eq. (73) to the result in Eq. (79) below. Thus the effective angular momentum J_* is defined by

$$8\pi J_* \equiv -\frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B}\Phi^b\gamma^{-1/2}\pi_b{}^a r_a\sqrt{\sigma}\,d^2x\,d\tau$$

$$+\sum_{m,n}\frac{m}{8\omega_{mn}}\oint_{r_B}[rr_0^p\nabla_p(h_{ij}^{mn})r_0^q\nabla_q(h_{ij}^{mn*}) - h_{ij}^{mn*}r_0^p\nabla_p(rr_0^q\nabla_q h_{ij}^{mn})]\sqrt{\sigma_0}\,d^2x. \tag{75}$$

And the effective M action A_* is defined by

$$16\pi A_* \equiv \int_0^{2\pi}\int \pi^{ab}\partial\gamma_{ab}/\partial\tau\,d^3x\,d\tau - \sum_{m,n}\frac{\pi n}{2\omega_{mn}}\oint_{r_B}[rr_0^p\nabla_p(h_{ij}^{mn})r_0^q\nabla_q(h_{ij}^{mn*}) - h_{ij}^{mn*}r_0^p\nabla_p(rr_0^q\nabla_q h_{ij}^{mn})]\sqrt{\sigma_0}\,d^2x. \tag{76}$$

Similarly it seems natural to redefine $\langle H_2 \rangle$ by having it absorb the terms inside the total δ in Eq. (74). But, instead, we introduce the effective mass which is a boundary integral independent of N_0:

$$
\begin{aligned}
16\pi m_* \equiv & -\frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B} \frac{2N}{N_0} D_a r^a \sqrt{\sigma}\, d^2x\, d\tau + \frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B} 2\nabla_i r_0^i \sqrt{\sigma_0}\, d^2x\, d\tau \\
& +\frac{1}{2\pi}\int_0^{2\pi}\oint_{r_B} 2S^a \gamma^{-1/2}\pi_a{}^b r_b \sqrt{\sigma}\, d^2x\, d\tau \\
& -\frac{1}{4\pi}\int_0^{2\pi}\oint_{r_B}[\tfrac{1}{2}h^{ij}r_0^k \nabla_k h_{ij} - r_0^p \nabla_i(h^{ij}\sigma_{0jq}h^q{}_p) + \tfrac{5}{2}h_{rr}h_{rr}/r - h_{ik}h^{ki}/r \\
& +I r_0^i \nabla_i I + S^i S_i/r - S^i r_0^j \nabla_j S_i \\
& +\tfrac{1}{2}rr_0^p \nabla_p(h_{ij})r_0^q \nabla_q(h^{ij}) - \tfrac{1}{2}h_{ij}r_0^p \nabla_p(rr_0^q \nabla_q h^{ij})]\sqrt{\sigma_0}\, d^2x\, d\tau.
\end{aligned}
\tag{77}
$$

The variation of the second integral is identically zero; it is included so that m_* has the expected value for, say, the Schwarzschild geometry.

Now a variational principle for m_* is based upon

$$
16\pi N_0 m_* = -\frac{1}{2\pi}\int_0^{2\pi}\int\left\{N\left[R + \frac{1}{\gamma}\left(\frac{1}{2}\pi_a{}^a\pi_b{}^b - \pi^{ab}\pi_{ab}\right)\right] + 2N^a D_b(\pi^b{}_a/\gamma^{1/2})\right\}\sqrt{\gamma}\, d^3x\, d\tau
$$
$$
+N_0 \times [\text{right hand side of Eq. (77)}].
\tag{78}
$$

And the variation of this equation yields

$$
16\pi N_0 \delta m_* = \frac{1}{2\pi}\int_0^{2\pi}\int E_{\text{Eqs.}}\, d^3x\, d\tau + 16\pi \delta A_*/T + 16\pi \Omega \delta J_*
$$
$$
+N_0 \sum_{lmn}\left[\frac{8(\omega_{mn})^{2l+1}(l+1)(l+2)}{(l!)^2 l(l-1)}I_{lm}^n I_{lm}^{n*}\delta\theta_{lm}^{In} + \frac{32(\omega_{mn})^{2l+1}l(l+1)(l+2)}{[(l+1)!]^2(l-1)}S_{lm}^n S_{lm}^{n*}\delta\theta_{lm}^{Sn}\right].
\tag{79}
$$

To apply the variational principle, choose specific values for J_*, A_*, and each of the θ_{lm}^{In} and θ_{lm}^{Sn}, and consider the class of periodic geometries described by $(N, N^a, \gamma_{ab}, \pi^{ab})$ which have these specific values. Equation (79) shows that m_*, evaluated by Eq. (78), is an extremum for a member of this class if and only if the geometry is a solution of the Einstein equations.

As might be anticipated, the coefficients of $\delta\theta_{lm}^{In}/2\pi$ and $\delta\theta_{lm}^{Sn}/2\pi$ are just 16π times the energy in one wavelength of the gravitational waves in the wave zone as derived from Thorne's Eq. (4.16') [1].

C. Interpretations of A_*, J_*, and m_*

In this section we show that when the boundary is in the weak-field zone, each of the effective quantities is independent of the exact location of the boundary through terms of second order in the deviation of the geometry from flat space. Thus, for each of the equations of this section, equality is understood to hold only through terms of second order.

The quantity A_* is defined as a sum of a volume integral and a surface integral in Eq. (76). And the difference in A_* corresponding to two different boundary surfaces, at r_1 and r_2, is

$$
16\pi \Delta A_* \equiv 16\pi(A_{2*} - A_{1*}) = \int_0^{2\pi}\int_{r_1}^{r_2}\pi^{ab}\partial\gamma_{ab}/\partial\tau\, d^3x\, d\tau
$$
$$
-\sum_{m,n}\frac{\pi n}{2\omega_{mn}}\left(\oint_{r_2} - \oint_{r_1}\right)[rr_0^p \nabla_p(h_{ij}^{mn})r_0^q \nabla_q(h_{ij}^{mn*}) - h_{ij}^{mn*}r_0^p \nabla_p(rr_0^q \nabla_q h_{ij}^{mn})]\sqrt{\sigma_0}\, d^2x.
\tag{80}
$$

Through terms of second order in the weak-field zone this is

$$
16\pi \Delta A_* = \sum_{m,n}\int_{r_1}^{r_2}\pi n\omega_{mn}h_{ij}^{mn}h_{ij}^{mn*}\sqrt{\sigma_0}\, d^3x
$$
$$
-\sum_{m,n}\frac{\pi n}{2\omega_{mn}}\int_{r_1}^{r_2}\nabla_p[r\nabla_p(h_{ij}^{mn})r_0^q \nabla_q(h_{ij}^{mn*}) - h_{ij}^{mn*}\nabla_p(rr_0^q \nabla_q h_{ij}^{mn})]\sqrt{\sigma_0}\, d^3x.
\tag{81}
$$

And by use of the wave equation, Eq. (63), for h_{ij}^{mn} this simplifies to

$$16\pi\Delta A_* = \sum_{m,n}\int_{r_1}^{r_2}\pi n\omega_{mn}h_{ij}^{mn}h_{ij}^{mn*}\sqrt{\sigma_0}\,d^3x - \sum_{m,n}\frac{\pi n}{2\omega_{mn}}\int_{r_1}^{r_2}2\omega_{mn}^2h_{ij}^{mn}h_{ij}^{mn*}\sqrt{\sigma_0}\,d^2x$$
$$= 0. \tag{82}$$

Thus the effective M action is independent of the surface over which it is evaluated, through second order in the deviation from flat space.

Similarly, the difference in effective angular momentum evaluated at two different surfaces in the weak-field zone is

$$8\pi\Delta J_* \equiv 8\pi(J_{2*} - J_{1*}) = -\frac{1}{2\pi}\int_0^{2\pi}\left(\oint_{r_2} - \oint_{r_1}\right)\Phi^b\gamma^{-1/2}\pi_b{}^ar_a\sqrt{\sigma}\,d^2x\,d\tau$$

$$+ \sum_{m,n}\frac{m}{8\omega_{mn}}\left(\oint_{r_2} - \oint_{r_1}\right)[rr_0^p\nabla_p(h_{ij}^{mn})r_0^q\nabla_q(h_{ij}^{mn*}) - h_{ij}^{mn*}r_0^p\nabla_p(rr_0^q\nabla_q h_{ij}^{mn})]\sqrt{\sigma_0}\,d^2x. \tag{83}$$

The first integral may be rewritten as

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_{r_1}^{r_2}D_a(\gamma^{-1/2}\pi_b{}^a\Phi^b)\sqrt{\gamma}\,d^3x\,d\tau = -\frac{1}{2\pi}\int_0^{2\pi}\int_{r_1}^{r_2}\left[D_a(\gamma^{-1/2}\pi_b{}^a)\Phi^b + \gamma^{-1/2}\pi^{ab}D_a\Phi_b\right]\sqrt{\gamma}\,d^3x\,d\tau. \tag{84}$$

The first term vanishes from the momentum constraint, Eq. (5), and the second is easily written in terms of the deviation from flat space. Also use of the wave equation, Eq. (63), for h_{ij}^{mn} in Eq. (83) results in

$$8\pi\Delta J_* = -\sum_{mn}\int_{r_1}^{r_2}\frac{1}{4}\omega_{mn}mh_{ij}^{mn}h_{ij}^{mn*}\sqrt{\sigma_0}\,d^3x + \sum_{m,n}\frac{m}{8\omega_{mn}}\int_{r_1}^{r_2}2\omega_{mn}^2h_{ij}^{mn}h_{ij}^{mn*}\sqrt{\sigma_0}\,d^3x. \tag{85}$$

Thus the effective angular momentum is independent of the surface over which it is evaluated in the weak-field zone, through second order in the deviation from flat space. At first order J_* is just the first term in Eq. (75), which evaluates to S_z^0, the current dipole moment as defined by the multipole structure of the geometry in the weak-field zone. Thus we have the following interpretation of the effective angular momentum: In the weak-field zone, at linear order, J_* is the constant current dipole moment S_z^0. But when the second order corrections are added to the linearized theory, then S_z^0, has a small contribution from the standing gravitational waves which is linear in r. However, J_* does not change at second order. Thus, through second order in the deviation from flat space J_* is the current dipole moment of the source alone without a contribution from the standing gravitational waves.

In a similar fashion the difference in the effective mass evaluated at two different surfaces, Δm_*, is defined from Eq. (77).

Now, Δm_* is quadratic in the deviation of the geometry from flat space—the linear contribution vanishes from Eq. (79) and the fact that Δm_* is zero for flat space. It is straightforward but an exceedingly tedious task to substitute the linearized geometry of Sec. II B into the definition of Δm_*. In the later stages the details of the substitution and reduction are similar to that for ΔA_* and ΔJ_*. And the result is that Δm_*, too, vanishes through second order in the deviation from flat space in the weak-field zone.

Thus the effective mass is independent of the location of the surface over which it is evaluated in the weak-field zone through second order in the deviation of the geometry from flat space. At first order the effective mass is the first two terms in Eq. (77) which evaluates to $\mathcal{I}^0$, the mass monopole moment of the linearized geometry

of Sec. II B. Thus we have the following interpretation of the effective mass: In the weak-field zone, at linear order, m_* is the constant mass monopole of the linearized geometry. But, when the second order corrections are included, then the mass monopole has a small contribution, linear in r, from the standing gravitational waves. However, m_* does not change through second order in the deviation from flat space. Thus, to the extent that it is possible to define such a quantity, m_* is the mass monopole of the source alone without a contribution from the standing gravitational waves.

V. CONCLUSIONS

Paper I showed that the periodic solutions of the Einstein equations, with standing waves, give valuable information about physically realistic systems, with outgoing radiation. And the variational principle of Eq. (78) is a valuable tool for studying the periodic solutions. In a preliminary application in paper I, the variational principle has been relatively easy to apply in the study of close orbits of equal mass black holes, without regard to the gravitational radiation. The next paper in this series will use more sophisticated trial geometries which form a complete set via an infinite sequence of possible parameters. In particular these geometries will clearly contain gravitational radiation.

For specific values of J_*, A_*, θ_{lm}^{In}, and θ_{lm}^{Sn} and an approximate solution to the Einstein equations, accurate to order δ, the variational principle directly gives an estimate of m_* which is accurate to order δ^2. But the variational principle can be used to estimate Ω, ω_0, $|I_{lm}^n|^2$, and $|S_{lm}^n|^2$ to order δ^2 as well. For example, to find an accurate estimate of Ω just apply the variational principle a second time with a slightly different choice for the

effective angular momentum, say a value of $J_* + \Delta J_*$. The resulting value of the effective mass will change by an amount, say, Δm_*; then Eq. (79) shows that

$$\Omega = N_0 \Delta m_* / \Delta J_* + O(\delta^2) \tag{86}$$

accurately estimates Ω. The other interesting quantities describing the geometry can be found similarly.

Perhaps the most interesting results of this paper are the definitions of the effective quantities—Eq. (77) for the effective mass, Eq. (75) for the effective angular momentum, and Eq. (??) for the effective M action. These definitions capture the concept of the mass monopole, current dipole, and M action of the source without contribution from the gravitational radiation in the weak-field zone. They are independent of the location of the boundary, through second order in the deviations from flat space, as long as the boundary is in the weak-field zone. A similar analysis has been made of a toy problem of a mass, confined to oscillations along the z axis under the influence of a generic potential, and attached to a semi-infinitely long string stretching out along the x axis. For that case the analogous variational principle is

for the average energy of the mass alone—the energy of the string does not contribute.

To add matter to this analysis it is simple to include matter terms in H_2. And to allow for black hole sources Eq. (78) for m_* still provides the variational principle. But the volume integral has boundaries in the weak-field zone surrounding the spatial infinity within each black hole as discussed in paper I. Then the variations bring in additional surface terms, resembling the ones in Eq. (79), but evaluated at these additional boundaries. These new surface terms correspond to the effective mass and angular momentum and the phase of the radiation in the weak-field zone within the holes.

ACKNOWLEDGMENTS

APPENDIX: BOUNDARY INTEGRALS IN THE WEAK-FIELD ZONE

The detailed analysis leading to Eq. (74) is relegated to this appendix.

The geometry on the boundary is described at the beginning of Sec. IV A. In particular on the boundary a number of quantities are exactly equal to their linearized counterparts. Thus,

$$\sqrt{\sigma} = \sqrt{\sigma_0}(1 + I - \tfrac{1}{2}h_{rr}), \tag{A1}$$

$$N^2\sqrt{\sigma} = N_0^2\sqrt{\sigma_0}(1 - \tfrac{1}{2}h_{rr}), \tag{A2}$$

$$\sqrt{\sigma}\sigma^{ab} = \sqrt{\sigma_0}[\sigma_0^{ab}(1 - \tfrac{1}{2}h_{rr}) - \sigma_0^{ac}h_{cd}\sigma_0^{db}], \tag{A3}$$

$$N^a = N_0 S^a + \Omega \Phi^a, \tag{A4}$$

$$2N(\sigma_a^c\sigma_b^d D_c r_d - \tfrac{1}{2}\sigma_{ab}D_c r^c) = N_0\left[-\frac{2}{r}\sigma_{0a}^c h_{cd}\sigma_{0b}^d + \frac{1}{r}\sigma_{0ab}\sigma_0^{cd}h_{cd} + \sigma_{0a}^c\sigma_{0b}^d r^e\nabla_e h_{cd} - \tfrac{1}{2}\sigma_{0ab}\sigma_0^{cd}r^e\nabla_e h_{cd}\right.$$
$$\left. -\sigma_{0a}^c\sigma_{0b}^d r^e\nabla_c h_{de} - \sigma_{0a}^c\sigma_{0b}^d r^e\nabla_d h_{ce} + \sigma_{0ab}\sigma_0^{cd}r^e\nabla_c h_{de}\right], \tag{A5}$$

$$N^{-1}D_a r^a = N_0^{-1}\left[\frac{2}{r} + \frac{2h_{rr}}{r} + r_0^a\nabla_a(I + \tfrac{1}{2}h_{rr})\right], \tag{A6}$$

$$r^a D_a N = -\frac{N_0}{2}r_0^a\nabla_a I, \tag{A7}$$

and

$$\gamma^{-1/2}\pi_a{}^b r_b\sqrt{\sigma} = \frac{\sqrt{\sigma_0}}{2N_0}\frac{\partial h_{ar}}{\partial t} - \sqrt{\sigma_0}r_0^b\nabla_{(a}S_{b)}. \tag{A8}$$

Now, consider the term proportional to the stationary δS^a on the left hand side of Eq. (74). It has no contribution from time-dependent terms: All time-dependent terms are multiplied by stationary terms and drop out when

integrated over a full period of τ. With application of Eqs. (3), (42), (46) and use of the periodicity, the terms proportional to δS^a in Eq. (74) contribute

$$\frac{N_0}{2\pi} \int_0^{2\pi} \oint_{r_B} 2\delta S^a \gamma^{-1/2} \pi_a{}^b r_b \sqrt{\sigma}\, d^2x\, d\tau = -\frac{N_0}{2\pi} \int_0^{2\pi} \oint_{r_B} \delta S^i (\nabla_i S_j + \nabla_j S_i) r_0^j \sqrt{\sigma_0}\, d^2x\, d\tau$$

$$= \frac{N_0}{4\pi} \delta \left[\int_0^{2\pi} \oint_{r_B} r^{-1} S^i S_i \sqrt{\sigma_0}\, d^2x\, d\tau \right] - \frac{N_0}{2\pi} \int_0^{2\pi} \oint_{r_B} \delta S^i (\nabla_j S_i) r_0^j \sqrt{\sigma_0}\, d^2x\, d\tau.$$

$$(A9)$$

If there are no stationary sources at infinity, then S^i is of the form given in Eq. (57) and orthogonality of the $\mathcal{Y}_{K_l}^{lm}$ implies that the second term above is symmetric in S^i and δS^i. Thus the right hand side of Eq. (A9) is

$$\frac{N_0}{4\pi} \delta \left[\int_0^{2\pi} \oint_{r_B} (S^i S_i / r - S^i r_0^j \nabla_j S_i) \sqrt{\sigma_0}\, d^2x\, d\tau \right]. \tag{A10}$$

The other surface integral on the left hand side of Eq. (74) is closely related to

$$-\frac{1}{2\pi} \int_0^{2\pi} \oint_{r_B} \left[\delta(\sqrt{\sigma}\sigma^{ab}) N (D_a r_b - \tfrac{1}{2}\sigma_{ab} D_c r^c) + \delta(N^2 \sqrt{\sigma}) N^{-1} D_a r^a + 2\delta(\sqrt{\sigma}) r^a D_a N \right] d^2x\, d\tau$$

$$-\frac{N_0}{2\pi} \int_0^{2\pi} \oint_{r_B} \tfrac{1}{4} \delta(h^{ij} r_0^k \nabla_k h_{ij}) \sqrt{\sigma_0}\, d^2x\, d\tau. \tag{A11}$$

The part of expression (A11) depending upon the stationary moments is

$$\frac{N_0}{2\pi} \int_0^{2\pi} \oint_{r_B} \delta I r_0^i \nabla_i I \sqrt{\sigma_0}\, d^2x\, d\tau. \tag{A12}$$

And if there are no sources at infinity, then Eq. (56) and the orthogonality of the $\mathcal{Y}_{K_l}^{lm}$ imply that the first term is symmetric in I and δI and expression (A12) is

$$\frac{N_0}{4\pi} \delta \left[\int_0^{2\pi} \oint_{r_B} I r_0^i \nabla_i I \sqrt{\sigma_0}\, d^2x\, d\tau \right]. \tag{A13}$$

Expressions (A10) and (A13) are the contributions from the stationary moments to the right hand side of Eq. (74).

The radiative terms in expression (A11) follow with the substitutions from Eqs. (A1) to (A8) along with liberal use of Stoke's theorem, Eqs. (43), (45) and the orthogonality of the $\mathcal{Y}_{K_l}^{lm}$; the result is

$$\frac{N_0}{4\pi} \int_0^{2\pi} \oint_{r_B} (\tfrac{1}{2}\delta h^{ij} r_0^k \nabla_k h_{ij} - \tfrac{1}{2} h^{ij} r_0^k \nabla_k \delta h_{ij}) \sqrt{\sigma_0}\, d^2x\, d\tau$$

$$+ \frac{N_0}{4\pi} \delta \left\{ \int_0^{2\pi} \oint_{r_B} [-r_{0p} \nabla_i (h^{ij} \sigma_{0jq} h^{pq}) + \tfrac{5}{2} h_{rr} h_{rr}/r - h^{ik} h_{ki}/r] \sqrt{\sigma_0}\, d^2x\, d\tau \right\}. \tag{A14}$$

The focus remains on the first integral of expression (A14); the second integral appears directly in Eq. (74).

The quantity h_{ij} is decomposed into its m, n components as in Eq. (62), where h_{ij}^{mn} is an exact solution to the flat space wave equation (63) with frequency ω_{mn} and dependence upon I_{lm}^n, θ_{lm}^{In}, S_{lm}^n, θ_{lm}^{Sn} and also upon Ω and ω_0 through the dependence upon ω_{mn}. Thus

$$\delta h_{ij} = \sum_{lmn} \left[\frac{\delta h_{ij}^{mn}}{\delta I_{lm}^n} \delta I_{lm}^n + \frac{\delta h_{ij}^{mn}}{\delta \theta_{lm}^{In}} \delta \theta_{lm}^{In} + \frac{\delta h_{ij}^{mn}}{\delta S_{lm}^n} \delta S_{lm}^n + \frac{\delta h_{ij}^{mn}}{\delta \theta_{lm}^{Sn}} \delta \theta_{lm}^{Sn} + \frac{\delta h_{ij}^{mn}}{\delta \omega_{mn}} \delta \omega_{mn} \right]. \tag{A15}$$

Now, the substitution of Eq. (A15) into the first integral of expression (A14) and the use of orthogonality properties of the $\mathcal{Y}_{K_l}^{lm}$ lead to a sum of terms each involving a single choice of l, m, n and either I_{lm}^n or S_{lm}^n. The resulting terms that are proportional to δI_{lm}^n and to $\delta \theta_{lm}^{In}$ are easy to evaluate because the coefficients of δI_{lm}^n and $\delta \theta_{lm}^{In}$ in Eq. (A15) are also solutions to Eq. (63); and the terms in the first integral of expression (A14), which involve these, form the Wronskian of two solutions to Eq. (63). In fact, the term involving δI_{lm}^n is just proportional to the Wronskian of two dependent solutions of the same linear equation and, hence, vanishes. The surface integral of the Wronskian involving $\delta \theta_{lm}^{In}$ is independent of the surface over which it is evaluated; and it is easiest to evaluate in the wave zone. The terms

in the first integral of expression (A14) involving δS_{lm}^n and $\delta \theta_{lm}^{Sn}$ may be handled in a completely analogous manner. This results in

$$N_0 \sum_{lmn} \left[\frac{8(\omega_{mn})^{2l+1}(l+1)(l+2)}{(l!)^2 l(l-1)} I_{lm}^n I_{lm}^{n*} \delta \theta_{lm}^{In} + \frac{32(\omega_{mn})^{2l+1}l(l+1)(l+2)}{[(l+1)!]^2(l-1)} S_{lm}^n S_{lm}^{n*} \delta \theta_{lm}^{Sn} \right]$$
(A16)

for the δ-phase contribution to the first integral of expression (A14); this appears directly in Eq. (74).

The term in the first integral of expression (A14) [after substitution from Eq. (A15)] which involves $\delta \omega_{mn}$ is rather more complicated. From Eq. (63) it is clear that the product $r h_{ij}^{mn}$ may be written so that r and ω_{mn} only appear in the combination $r \omega_{mn}$, except for a possible overall amplitude dependence upon ω_{mn}. Thus

$$\delta \omega_{mn} \frac{\delta h_{ij}^{mn}}{\delta \omega_{mn}} = \delta \omega_{mn} \frac{\delta (r h_{ij}^{mn})}{\delta (r \omega_{mn})},$$
(A17)

and $\delta(r h_{ij}^{mn})/\delta(r\omega_{mn})$ is closely related to just the r derivative of $r h_{ij}^{mn}$, so that

$$\delta \omega_{mn} \frac{\delta h_{ij}^{mn}}{\delta \omega_{mn}} = \frac{\delta \omega_{mn}}{\omega_{mn}} \frac{\partial}{\partial r} (r h_{ij}^{mn}) = \frac{\delta \omega_{mn}}{\omega_{mn}} r_0^k \nabla_k (r h_{ij}^{mn}).$$
(A18)

Now, the $\delta \omega_{mn}$ contribution to the first integral of expression (A14) is [let $h_{ij} \rightarrow h_{ij}^{mn*}$ and $\delta h_{ij} \rightarrow \delta \omega_{mn} r_0^k \nabla_k (r h_{ij}^{mn})/\omega_{mn}$, sum over m and n and do the integral over τ]

$$\frac{N_0}{4} \sum_{mn} \oint_{r_B} \frac{\delta \omega_{mn}}{\omega_{mn}} \{ r_0^p \nabla_p (r h_{ij}^{mn}) r_0^q \nabla_q h_{ij}^{mn*} - h_{ij}^{mn*} r_0^p \nabla_p [r_0^q \nabla_q (r h_{ij}^{mn})] \} \sqrt{\sigma_0} \, d^2 x.$$
(A19)

The term in the curly brackets simplifies, and a δ variation by parts and average over τ result in

$$\frac{N_0}{8\pi} \delta \left\{ \int_0^{2\pi} \oint_{r_B} [r r_0^p \nabla_p (h_{ij}) r_0^q \nabla_q h^{ij} - h_{ij} r_0^p \nabla_p (r r_0^q \nabla_q h^{ij})] \sqrt{\sigma_0} \, d^2 x \, d\tau \right\}$$

$$- \frac{N_0}{8\pi} \sum_{mn} \left(n \omega_0 - \frac{m\Omega}{N_0} \right) \delta \left\{ \int_0^{2\pi} \oint_{r_B} \omega_{mn}^{-1} [r r_0^p \nabla_p (h_{ij}^{mn}) r_0^q \nabla_q h_{ij}^{mn*} \right.$$

$$\left. - h_{ij}^{mn*} r_0^p \nabla_p (r r_0^q \nabla_q h_{ij}^{mn})] \sqrt{\sigma_0} \, d^2 x \, d\tau \right\};$$
(A20)

this appears directly in Eq. (74).

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[1] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[2] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
[3] R. Price and K. S. Thorne, Astrophys. J. **155**, 163 (1969).
[4] G. W. Gibbons and J. M. Stewart, in *Classical General Relativity*, edited by W. B. Bonner, J. N. Islam, and M. A. H. MacCallum (Cambridge University Press, Cambridge, England, 1984), pp. 77–94.
[5] J. K. Blackburn and S. Detweiler, Phys. Rev. D **46**, 2318 (1992).
[6] K. S. Thorne, Astrophys. J. **158**, 1 (1969).

[7] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
[8] J. W. York, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979), pp. 83–126.
[9] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 7, pp. 227–265.
[10] G. Hayward, Phys. Rev. D **47**, 3275 (1993).
[11] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).