Cosmic hoop conjecture?

Takeshi Chiba
Department of Physics, Kyoto University, Kyoto 606-01, Japan

Kei-ichi Maeda
Department of Physics, Waseda University, Tokyo 169, Japan

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We propose a hoop conjecture in the presence of a positive cosmological constant Λ: when an apparent horizon forms in a gravitational collapse, the matter must be sufficiently compactified such that the circumference $\mathcal{C}$ satisfies the condition $\mathcal{C} \lesssim 4\pi M \lesssim 4\pi M_{\rm crit}$, where M is the Abbott-Deser mass of the collapsed body and $M_{\rm crit} = 1/3\sqrt{\Lambda}$. To confirm our conjecture, we investigate two cases; (1) initial data of a prolate or oblate dust spheroid, and (2) the Kastor-Traschen spacetime which describes a black hole collision with Λ. We also discuss a relation between the hoop conjecture and an appearance of a naked singularity.

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I. INTRODUCTION

One of the most important unresolved problems in general relativity is the cosmic censorship hypothesis (CCH) [1]. Many efforts have been devoted to it over the years.

It may be crucial for a resolution of the CCH to understand aspherical collapse. Unfortunately we have at present little knowledge about such an aspherical collapse, but there is one conjecture, the so-called hoop conjecture (HC) [2]. The HC states that a horizon forms if and only if matter with a mass M gets compactified enough such that the circumference in *all* directions satisfies the condition of $\mathcal{C} \lesssim 4\pi M$. Although the HC was proposed two decades ago, it has recently been intensively studied because of the development of the computer [3–6]. There is, so far, no compelling counterexample against the HC.

If the HC is true, then we would have the following picture for a final state of spacetime after a gravitational collapse: The gravitational collapse results in a formation of a black hole if the localized mass satisfies the HC, while if it is not the case, it results in a naked singularity. This picture was already in part confirmed in the numerical simulation of a prolate dust collapse [7]. The verification of the HC would have great importance to the CCH and the investigations so far strongly suggest that the HC is true.

If there is a cosmological constant Λ, however, the condition for the HC is not known and even the HC itself is not formulated. The study of the dynamical behaviors of inhomogeneous spacetimes with Λ and of their final states is, however, a very important subject. The problem of the generality of inflation is related to the study. The present isotropy and homogeneity of the Universe are among the mysteries within the framework of the standard big bang scenario. The inflationary universe scenario is currently the most favorable model to explain the present homogeneity and isotropy of our universe. In this scenario, the vacuum energy of some fundamental field (the so-called "inflaton") behaves as a positive cosmological constant during a period in the early universe. Although some fundamental problems in the standard big bang scenario are resolved by the idea of inflation, there may still be two unsolved problems in the inflationary scenario: One is what is the inflaton field, and the other is the isotropy-homogeneity problem since most inflationary models have been worked within Friedmann-Robertson-Walker spacetime. To answer the former question, we may have to understand the final unified theory of fundamental forces, while the latter question might be solved in the context of the classical Einstein theory. If all or most spacetime with a positive cosmological constant approaches the de Sitter spacetime, we find that the universe is isotropic and homogeneous just after the inflationary era. This argument is closely related to the so-called "cosmic no hair conjecture." This conjecture is true under some restrictions on spacetimes or on those initial conditions, however, we also know black hole spacetime in de Sitter background. Schwarzschild-de Sitter spacetime never approaches de Sitter one. Some dust sphere with a cosmological constant can evolve into the Schwarzschild-de Sitter spacetime. Therefore, in the case of more inhomogeneous spacetimes without any symmetry, it is not clear whether or not the de Sitter-like cosmic expansion is realized even if there exists a positive cosmological constant. In particular, if the inhomogeneities are very "strong" and "localized" enough, some portions may gravitationally collapse into black holes or naked singularities. Therefore, in order to clarify naturalness of inflation, we have to investigate inhomogeneous spacetimes with a cosmological constant and show how plausible inflation is in generic spacetimes.

From the previous considerations, we propose an inflationary scenario, in which many black holes could be formed but they are harmless, for inhomogeneous universes. *There is a critical value for the Abbot-Deser mass of inhomogeneities in an asymptotically de Sitter spacetime, beyond which no black hole is formed* [8]. Since such

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small black holes are diluted away or evaporate in an inflationary era, we naturally find the present isotropic and homogeneous spacetime. A question may, however, arise. That is, "Is there any possibility of any naked singularity which may change the above scenario?" In fact, there is a suggestion that the CCH may be violated in the spacetime with Λ [9]. Therefore, it may be of great importance to investigate a formation mechanism of a naked singularity, in particular, how the cosmological constant should affect the HC.

In this paper, we investigate the HC in the presence of a positive cosmological constant. To describe the HC concretely, in Sec. II we consider three simple models; (1) Newtonian theory with Λ, (2) spherical stars with Λ, and (3) collapse of dust fluid in the asymptotically de Sitter spacetime. To confirm our conjecture, we analyze two more realistic and interesting cases: First, in Sec. III, we solve the initial value problem for a class of axisymmetric prolate or oblate spheroids with the cosmological constant, following Nakamura, Shapiro, and Teukolsky (NST) [3], who considered the similar case but without Λ. We then search for an apparent horizon to know the relations of the circumference $\mathcal{C}$ of the spheroid to the cosmological constant when an apparent horizon will form. As a second example, we consider the Kastor-Traschen solution, which describes a black hole collision. In Sec. V, we then discuss a relation between the CHC and a naked singularity. From the analysis of dynamical evolution for the time $t \ll M$, we find that the spheroid with its eccentricity ~ 1 does collapse when Λ is small as $\Lambda \lesssim M^{-2}/9$, while one with small eccentricity at first does not collapse. Rather it expands due to the background cosmic expansion. This suggests that as in the case of the HC without Λ, if the circumference of a complete collapsed body is $\mathcal{C} > 4\pi M$, a naked singularity may appear. Concluding remarks will follow in Sec. VI.

Our notation and conventions follow Wald [10].

II. COSMIC HOOP CONJECTURE

Taking the following three simple examples, here we propose a cosmic hoop conjecture, i.e., the hoop conjecture in the presence of a cosmological constant Λ. A semiempirical condition for the formation of an apparent horizon that we have obtained is as follows: When an apparent horizon forms, matter must be sufficiently compactified into a region where the circumference $\mathcal{C}$ satisfies

$$\mathcal{C} \lesssim 4\pi M \lesssim 4\pi M_{\text{crit}} = \frac{4\pi}{3\sqrt{\Lambda}} , \tag{2.1}$$

where M is the Abbott-Deser (AD) mass [11] of the collapsed body and $M_{\text{crit}} = 1/3\sqrt{\Lambda}$.

To see that this conjecture may be plausible, in this section, we consider the effect of a cosmological constant Λ for the following three cases: (1) Newtonian theory (Sec. II A), (2) spherical stars (Sec. II B), and (3) collapse of dust fluid (Sec. II C).

A. Newtonian theory

Since the HC was originally proposed in analogy with the Newtonian theory of a nonrotating homogeneous dust spheroid (Lin-Mestel-Shu instability) [12], it is natural to see the effect of Λ first in the Newtonian dynamics [13] (i.e., LMS instability with Λ), following Thorne [2].

First, we consider the oblate spheroidal collapse. The final kinetic energy of the dust particles is roughly equal to their final potential energy:

$$\frac{1}{2}v^2 \sim \frac{M}{\mathcal{C}/2\pi} + \frac{\Lambda}{6}\left(\frac{\mathcal{C}}{2\pi}\right)^2 . \tag{2.2}$$

Hence, as long as

$$\frac{M}{\mathcal{C}/4\pi} + \frac{\Lambda}{3}\left(\frac{\mathcal{C}}{2\pi}\right)^2 \ll 1 \text{ (i.e., } 4\pi M \ll \mathcal{C} \ll 2\pi\sqrt{3/\Lambda}) , \tag{2.3}$$

the collapse is essentially Newtonian and no horizon will be formed. Under the condition of $M \lesssim 1/3\sqrt{\Lambda}$, which guarantees the system to be still Newtonian, when $\mathcal{C} \sim 4\pi M$ or $\mathcal{C} \sim 2\pi\sqrt{3/\Lambda}$, the system gets into the relativistic stage. The latter case may not describe a collapse of a localized system since $1/\sqrt{\Lambda}$ is the horizon scale of the universe. Hence, we expect that Eq. (2.1) is the condition to collapse into relativistic region, where a horizon may be formed. If $M > 1/3\sqrt{\Lambda}$, then $v^2 > 1$. Since there are no Newtonian regions, we have to discuss such a case by a fully relativistic theory.

Second, consider a prolate spheroid with an initial semiminor axis R_0 and eccentricity $e_0 \ll 1$. After a few dynamical time t_d given by

$$t_d = \frac{\pi}{2}\sqrt{\frac{R_0^3}{2(M - \Lambda R_0^3/3)}} \tag{2.4}$$

so long as $\Lambda < 3M/R_0^3$, the spheroid gets deformed into a thin thread of length $l(\ll R_0)$ and of mass per unit length $\lambda = 3M/2l[1-(2z/l)^2]$. Subsequently, the collapse proceeds mainly in its radial direction and we can deal with the thread as a part of an infinite cylinder. The equatorial segment of such a thread implodes with velocity:

$$\frac{dR}{dt} = -2\left[\lambda_{\text{eq}}\ln(l/R) + \frac{\Lambda}{12}R^2\right]^{1/2} , \tag{2.5}$$

where λ_{eq} denotes the value of λ on the equatorial plane. In our ansatz, $\Lambda R^2 < \Lambda l^2 \ll \Lambda R_0^2 < M/R_0 < 1$, while if $l \gg 2M$ so that $\lambda \ll 1$, then the radial inward velocity approaches the speed of light only when the thread has become extremely thin; i.e., $R \sim l\exp(-1/4\lambda_{\text{eq}}) \ll l$, otherwise the collapse is essentially Newtonian and no horizon can form.

From these crude analysis, we expect that the inclusion of Λ makes little difference on the criterion of onset of relativistic deviations from Newtonian collapse, however, there is an important restriction on Λ in order for the collapse to proceed to the relativistic stage, i.e.,

$$\Lambda \lesssim M/R^3 , \quad (2.6)$$

or, since the horizon radius of a body with a mass M is about M, the condition (2.6) yields for a black hole

$$M \lesssim 1/\sqrt{\Lambda} . \quad (2.7)$$

Therefore we expect that Λ brings another inequality for $\mathcal{C}$ in the hoop conjecture as shown in Eq. (2.1).

B. Static star with a cosmological constant

We make a different simple test of our cosmic hoop conjecture. Consider a static uniform spherical star of perfect fluid in the spacetime with Λ. The exterior is the Schwarzschild-de Sitter spacetime and the surface of the star locates at the radius $r = r_0 > r_B = 2/\sqrt{3}H\cos(\frac{\pi}{3} + \tan^{-1}\sqrt{\omega_e})$, where r is the Schwarzschild radial coordinate, $H = \sqrt{\Lambda/3}$, and $\omega_e = (MH)^{-2} - 1 \geq 0$. Let S_a be a surface with a radius $a(> r_0)$. We find that

$$\inf_a \frac{\mathcal{C}(S_a)}{4\pi M} = \frac{1}{\sqrt{3}MH}\cos\left(\frac{\pi}{3} + \tan^{-1}\sqrt{\omega_e}\right) . \quad (2.8)$$

Figure 1 shows the function on the right-hand side of Eq. (2.5). It can be seen that

$$\mathcal{C}(S_a) \geq 4\pi M \quad (2.9)$$

for $\forall a > r_0$, and for $H \leq 1/\sqrt{27}M$, or equivalently for $M \leq 1/3\sqrt{\Lambda}$.

Since such a spacetime is stable and will never collapse into a black hole, the condition (2.1) can be a good criterion for the formation of a horizon.

C. One-dimensional collapse of dust fluid with a cosmological constant

Finally we look at the dynamical behavior of a one-dimensional dust fluid in an asymptotically de Sitter background [14,15]. Here "one-dimensional" means either "spherically symmetric" or "planar or cylindrically symmetric."

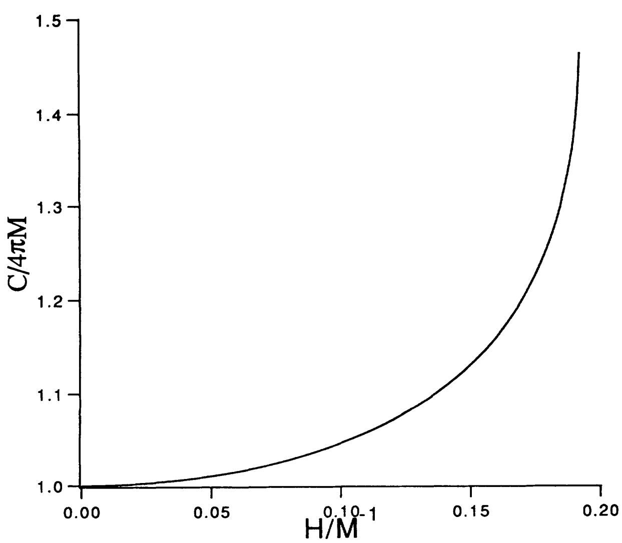

FIG. 1. $\mathcal{C}/4\pi M$ for the Schwarzschild-de Sitter spacetime.

Suppose that a dust sphere with Abbott-Deser (AD) mass [11] M is put into an isotropic and homogeneous de Sitter universe with a cosmological constant Λ. We can easily analyze the dynamical behavior of such a dust sphere. We consider the Tolman-Bondi spacetime. The metric is given as

$$ds^2 = -dt^2 + X^2(t,r)dr^2 + Y^2(t,r)d\Omega^2 . \quad (2.10)$$

The Einstein equations give

$$X(t,r) = Y'(t,r)/W(r) , \quad (2.11)$$

and

$$\dot{Y}^2(t,r) - \frac{2M(r)}{Y(t,r)} - \frac{\Lambda}{3}Y^2(t,r) = W^2(r) - 1 . \quad (2.12)$$

Here $M(r)$ is the AD mass of a dust sphere within a comoving radius r and is obtained by integration of the dust energy density $\epsilon(t,r)$ as

$$M(r) = \int 4\pi\epsilon Y^2 Y' dr , \quad (2.13)$$

and $W(r)$ is an arbitral function of r to be regarded as the binding energy of a dust shell at the radius r. We can analyze the behavior of the dust motion from the potential

$$V(Y) \equiv -\frac{2M(r)}{Y} - \frac{\Lambda}{3}Y^2 . \quad (2.14)$$

Since $V(Y)$ has the maximum of $-9M^2\Lambda$ at $Y = (3M/\Lambda)^{1/3}$, the dust shell with $-9M^2\Lambda > -1$ is unbound. If the spacetime is asymptotically expanding, such a dust shell will disperse away into infinity, resulting in a de Sitter universe, rather than collapse into a black hole. When a horizon will appear, the circumference $\mathcal{C}$ must be smaller than $4\pi M$, we find again Eq. (2.1) for formation of a horizon [14].

On the other hand, the dynamical behavior of a dust plane is greatly different [15]. The line element is [16]

$$ds^2 = -dt^2 + A^2(t,z)dz^2 + B^2(t,z)(dx^2 + dy^2) . \quad (2.15)$$

The Einstein equations give

$$A(t,z) = B'(t,z)/K(z) , \quad (2.16)$$

and

$$\dot{B}^2(t,z) - \frac{2M(z)}{B(t,z)} - \frac{\Lambda}{3}B^2(t,z) = K^2(z) . \quad (2.17)$$

Here $K(z)$ is an arbitral function of z and $M(z)$ is related to the dust density $\mu(t,z)$ as

$$M(z) = \int 4\pi\mu B^2 B' dz . \quad (2.18)$$

Note that $M(z)$ is not always positive definite because B' can be negative. However, if $M(z)$ is positive, the right-hand side of Eq. (2.16) is nonnegative, the dust plane is unbound. In any scale of dust fluid, we can easily set up an initial data, from which a naked singularity will be formed. As for a dust cylinder, we find the same results. Those are consistent with our CHC because the circumferences in both cases get infinity in the symmetric direction.

III. INITIAL DATA OF A PROLATE/OBLATE DUST COLLAPSE

We now investigate initial data of a prolate or oblate dust spheroid in the presence of a cosmological constant Λ. This model is more realistic and may help to confirm our cosmic hoop conjecture. We consider conformally flat initial data, which line element is written as

$$dl^2 = \psi^4(dR^2 + dz^2 + R^2 d\phi^2) \ . \tag{3.1}$$

We assume that the dust spheroid has no momentum and no "gravitational waves," which means that the transverse traceless part of the extrinsic curvature K_{ab} vanishes. We also adopt the constant mean curvature slice. Hence the extrinsic curvature K_{ab} is described as

$$K^a{}_b = H\delta^a{}_b \tag{3.2}$$

where $H = \pm\sqrt{\Lambda/3}$. Then the Einstein equation which the initial data must satisfy is only the Hamiltonian constraint;

$$-8\psi^{-5}\Delta\psi = 16\pi\rho \ , \tag{3.3}$$

where Δ is the Laplace operator defined by the flat space metric, and $\rho = T_{ab}n^a n^b$ with n^a being a unit normal to a spacelike initial hypersurface. Here it should be noted that the terms with a cosmological constant and with the extrinsic curvature cancel each other because of our slicing condition, Eq. (3.2). As a result, the Hamiltonian constraint turns out to be the same as that of the time-symmetric slice in the case without Λ, i.e., in an asymptotically flat spacetime.

To find an analytic solution, we set the density profile to

$$\psi^5\rho \equiv \rho_N \ , \tag{3.4}$$

where ρ_N is given as

$$\rho_N = \begin{cases} M/(4\pi a^2 c/3) & \text{for } R^2/a^2 + z^2/c^2 \le 1 \\ 0 & \text{otherwise} \ . \end{cases} \tag{3.5}$$

This ansatz may not be essential in our following results. For oblate spheroids with the eccentricity $e = \sqrt{1 - c^2/a^2}$, the solution ψ is found to be

$$\psi = 1 + \frac{3M\beta}{4ae} + \frac{1}{2}(RK_R + zK_z) \ , \tag{3.6}$$

where K_R and K_z are given by

$$\begin{aligned} K_R &= -\frac{3M}{4(ae)^3}R(\beta - \sin\beta\cos\beta) \ , \\ K_z &= -\frac{3M}{2(ae)^3}z(\tan\beta - \beta) \ , \end{aligned} \tag{3.7}$$

respectively. β is defined as

$$\sin\beta = e$$

for the inside of the spheroid, while

$$R^2\sin^2\beta + z^2\tan^2\beta = a^2e^2$$

for the outside.

As for prolate spheroids with the eccentricity $e = \sqrt{1 - a^2/c^2}$, the analytic solution ψ is

$$\psi = 1 + \frac{3M\beta}{4ce} + \frac{1}{2}(RK_R + zK_z) \ , \tag{3.8}$$

where

$$\begin{aligned} K_R &= \frac{3M}{4(ce)^3}R(\beta - \sinh\beta\cosh\beta) \ , \\ K_z &= \frac{3M}{2(ce)^3}z(\tanh\beta - \beta) \ . \end{aligned} \tag{3.9}$$

Here, β is given by

$$\sinh\beta = \frac{c}{a}e$$

for the inside of the spheroid, while

$$R^2\sinh^2\beta + z^2\tanh^2\beta = c^2e^2$$

for the outside.

Thus we obtain the solution ψ analytically, which is of great advantage to determine the apparent horizon numerically. Remember that the above solutions are the same ones which are already obtained in an asymptotically flat spacetime, but the equation to determine apparent horizons becomes different because of the existence of Λ. Since at large distance $\psi \to 1 + M/2r, M$ is interpreted as a gravitational mass of the spheroid in the presence of Λ, which is called the Abbott-Deser (AD) mass. It becomes the same as the ADM mass when there is no momentum density.

For a given ψ, we now search for an apparent horizon. In the three-metric Eq. (3.1) and the slicing Eq. (3.2), an apparent horizon is given by a surface of $r = r(\theta)$ (where the radial coordinate r is defined by $R = r\sin\theta, z = r\cos\theta$) satisfying the differential equation [17],

$$r_{,\theta\theta} + \left(\frac{r_{,\theta}^3}{r^2} + r_{,\theta}\right)\left(\frac{4\psi_{,\theta}}{\psi} + \cot\theta\right) - r_{,\theta}^2\left(\frac{3}{r} + \frac{4\psi_{,r}}{\psi}\right)$$
$$-2r - r^2\frac{4\psi_{,r}}{\psi} - \frac{2H\psi^2(r^2 + r_{,\theta}^2)^{(3/2)}}{r} = 0 \tag{3.10}$$

with the following boundary conditions;

$$r_{,\theta} = 0 \quad \text{at } \theta = 0,\ \frac{\pi}{2}\ . \tag{3.11}$$

Since we have ψ analytically, Eq. (3.10) is an ordinary differential equation. We solve it by the 4th order Runge-Kutta method.

In order to assess the hoop conjecture, we have to calculate the minimum circumference C outside the matter. In a previous paper [6], one of the present authors has proposed a definition of C by using closed geodesics. In axisymmetric space, C is given by [4,6]

$$C = \max(C_{\text{eq}}^{\text{min}}, C_{\text{pol}}^{\text{min}})\ , \tag{3.12}$$

where $C_{\text{eq}}^{\text{min}}$ and $C_{\text{pol}}^{\text{min}}$ are the minimum values of the circumferences on the equatorial plane and on the meridianal plane, respectively, both of which are evaluated by closed geodesics. Clearly, $C_{\text{eq}}^{\text{min}}$ is evaluated by circles and $C_{\text{pol}}^{\text{min}}$ is by the geodesics which satisfy the following geodesic equation:

$$r_{,\theta\theta} - 2\frac{r_{,\theta}^2}{r} - r + 2\left(1 + \frac{r_{,\theta}^2}{r^2}\right)\left(\frac{\psi_{,\theta}}{\psi}r_{,\theta} - \frac{\psi_{,r}}{\psi}r^2\right) = 0\ . \tag{3.13}$$

It is to be noted that C is independent of H, because ψ does not contain H.

TABLE I. The existence of apparent horizons and the circumference for various values of H and c. [(a): prolate spheroids; (b): oblate spheroids.] We use Y when the apparent horizon exists, otherwise N. M denotes the AD mass. We also listed H/H_{crit}, where $H_{\text{crit}} = 1/3\sqrt{3}M$ beyond which no apparent horizon appears (however, see text for the details).

(a)

c/M	H/M^{-1}	H/H_{crit}	AH?	$C_{\text{pol}}^{\text{min}}/4\pi M$	$C_{\text{pol}}^{\text{min}}/4\pi M_{\text{crit}}$
0.10	0.000 00	0.000 00	Y	1.001 9	0.0000
	0.100 00	0.519 62	Y		0.5206
	0.191 00	0.992 47	Y		0.9943
	0.192 00	0.997 66	N		0.9995
	−0.100 00	−0.519 62	Y		0.5206
	−0.192 45	−1.000 00	Y		1.0019
	−0.192 46	−1.000 05	N		1.0019
0.40	0.000 00	0.000 00	Y	1.0281	0.0000
	0.050 00	0.259 81	Y		0.2671
	0.110 00	0.571 58	Y		0.5876
	0.120 00	0.623 54	N		0.6411
	−0.100 00	−0.519 62	Y		0.5342
	−0.192 45	−1.000 00	Y		1.0281
	−0.192 46	−1.000 05	N		1.0282
1.0	0.000 00	0.000 00	N	1.1414	0.0000
	−0.100 00	0.519 62	Y		0.5931
	−0.150 00	0.779 42	Y		0.8896
	−0.192 52	−1.00 036	Y		1.1418
	−0.192 53	−1.000 42	N		1.1419

(b)

a/M	H/M^{-1}	H/H_{crit}	AH?	$C_{\text{eq}}^{\text{min}}/4\pi M$	$C_{\text{eq}}^{\text{min}}/4\pi M_{\text{crit}}$
0.10	0.000 00	0.000 00	Y	1.0039	0.0000
	0.100 00	0.519 62	Y		0.5216
	0.191 80	0.996 62	Y		1.0005
	0.191 90	0.997 14	N		1.0010
	−0.100 00	−0.519 62	Y		0.5216
	−0.192 45	−1.000 00	Y		1.0039
	−0.192 46	−1.000 05	N		1.0040
0.40	0.000 00	0.000 00	Y	1.0549	0.0000
	0.050 00	0.259 81	Y		0.2741
	0.074 00	0.384 52	Y		0.4056
	0.075 00	0.389 71	N		0.4111
	−0.100 00	−0.519 62	Y		0.5481
	−0.192 45	−1.000 00	Y		1.0549
	−0.192 46	−1.000 05	N		1.0550
1.0	0.000 00	0.000 00	N	1.2483	0.0000
	−0.100 00	−0.519 62	N		0.6486
	−0.150 00	−0.779 42	Y		0.9730
	−0.192 52	−1.000 36	Y		1.2487
	−0.192 53	−1.000 42	N		1.2488

Fixing the AD mass M, we have done a large survey of parameter space of $(H/M^{-1}, a/M, c/M)$ and searched for apparent horizons. The results are summarized in Table I and the typical examples are shown in Fig. 2. Comparing numerical results with analytical ones for a spherical source, we are confirmed that our calculation has the accuracy of six digits at least.

As is shown in the figure, the location of an apparent horizon shrinks inward in coordinate space as H increases, and sometimes the apparent horizon goes inside the matter.

We have searched for the critical value of $\mathcal{C}$ above which no apparent horizon forms for a given H. The results are depicted in Fig. 3. It shows that the maximum value of the circumference $\mathcal{C}$ depends on H, and no apparent horizon forms when $|H| > H_{\rm crit} = M^{-1}/\sqrt{27}$. At $H \simeq H_{\rm crit}$, the maximum value of $\mathcal{C}/4\pi M$ nearly equals to unity. We should, however, note that for $H < 0$, the exact critical value is slightly moved. Although the deviation from $-H_{\rm crit}$ is very little, we find an apparent horizon for $H < 0$ even beyond $-H_{\rm crit}$ (see Table I).

From Fig. 3, we find that if $\mathcal{C} < 4\pi M$ the apparent

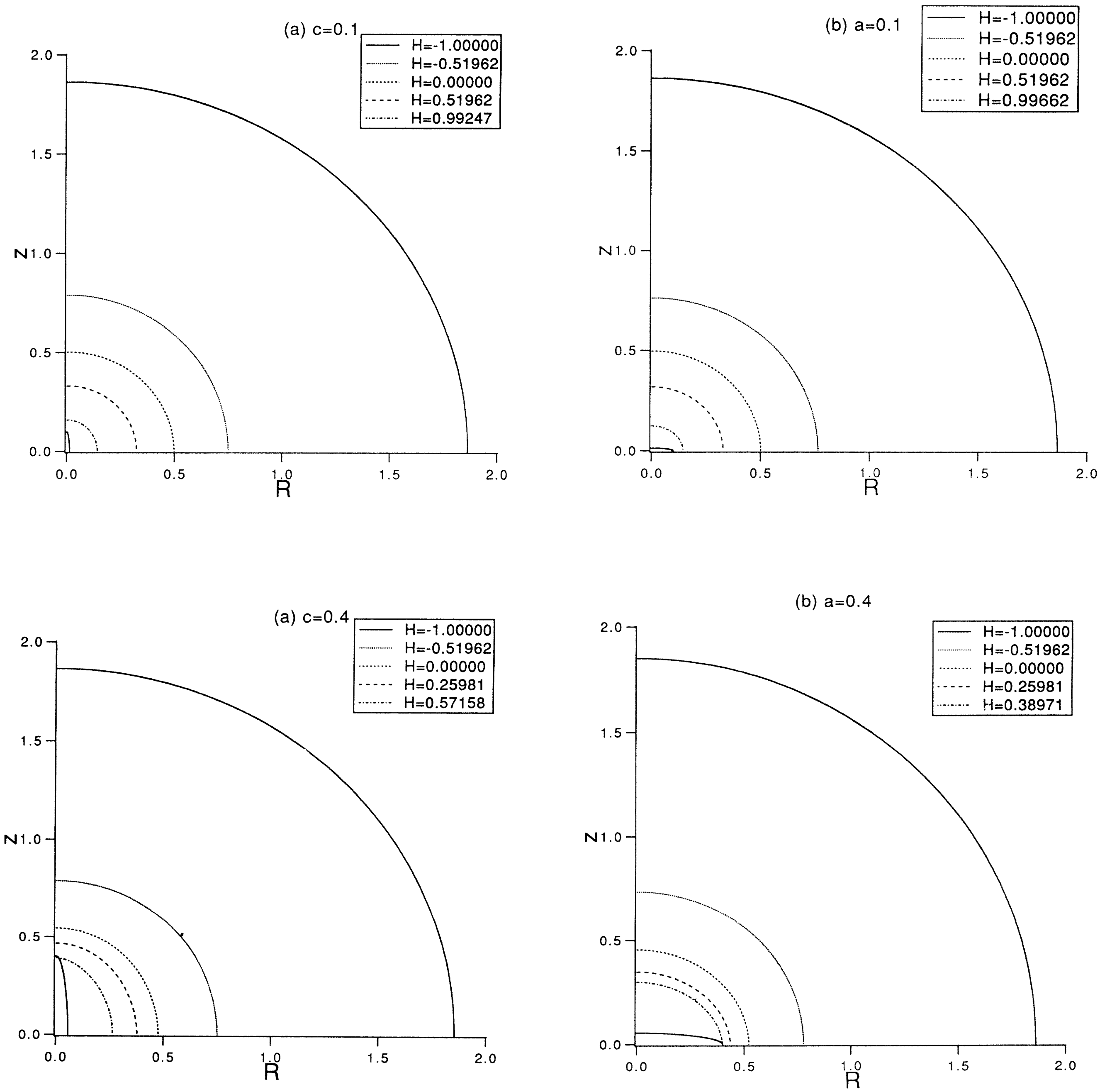

FIG. 2. (a) Shapes and location of apparent horizons for the prolate dust of $c/M = 0.1$, 0.4, 1.0 and (b) the oblate one of $a/M = 0.1$, 0.4, 1.0 for each $H/H_{\rm crit}$. The innermost solid line represents the surface of dust spheroid.

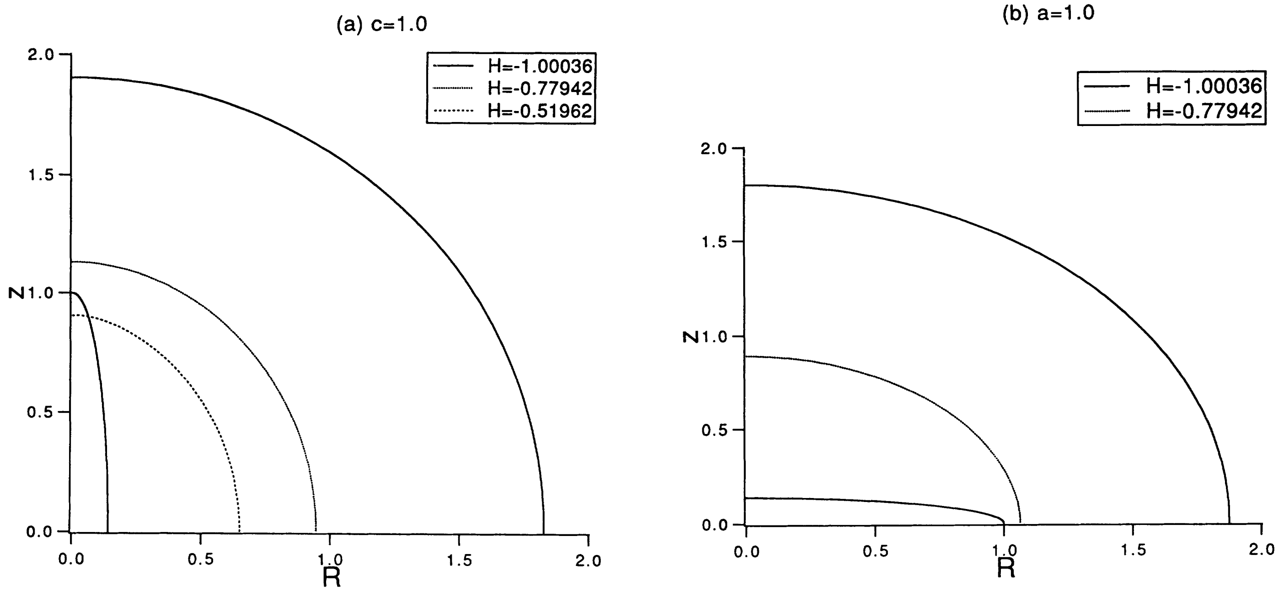

FIG. 2. (*Continued*).

horizon is always formed and more easily forms as the value of H decreases, i.e., the condition for formation of apparent horizon gets loose for the larger contraction of the background spacetime.

The above condition for the apparent horizon can be read as

$$M < M_{\rm crit} \equiv 1/3\sqrt{\Lambda} \ ,$$

when Λ is fixed and M is regarded as a dynamical variable. Hence, these results support our CHC, which we presented in the previous section, i.e., Eq. (2.1).

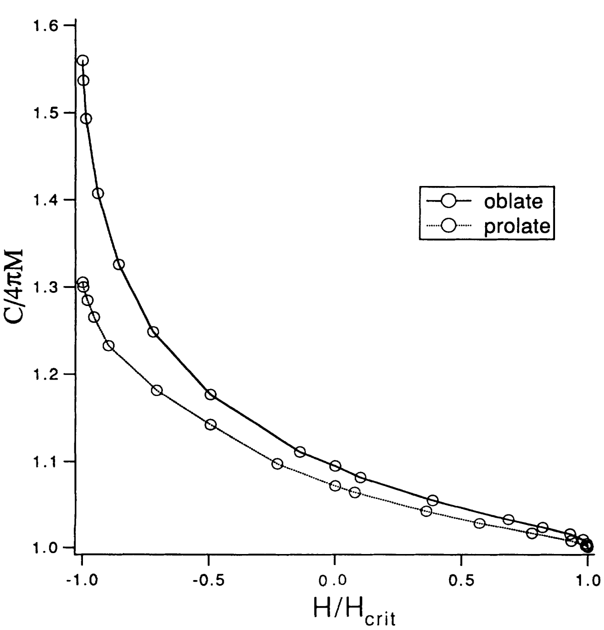

FIG. 3. Maximum value of C of dust spheroid for an apparent horizon to be formed for each H.

IV. BLACK HOLE COLLISIONS

Next, we claim that our cosmic hoop conjecture may be valid not only for initial data but also in the dynamical process. As an example, we consider the Kastor-Traschen (KT) solution [18], which represents a collision of maximally charged balck holes with Λ.

The line element is given as

$$ds^2 = -\Omega^{-2}dt^2 + a^2(t)\Omega^2(dx^2 + dy^2 + dz^2) \ , \quad (4.1)$$

where

$$a(t) = e^{Ht}, \quad \Omega = 1 + \sum_{i=1}^{N} \frac{m_i}{a(t)r_i} \ , \quad (4.2)$$

and the vector potential of the electromagnetic field is

$$A_t^{-2} = \Omega^{-1} \ . \quad (4.3)$$

Here $r_i (i = 1, \ldots, N)$ denotes the location of N black holes and $m_i (i = 1, \ldots, N)$ is each mass (=charge because of the extreme case). When $H < 0$, the solution describes a collision of N black holes.

We consider two black hole collisions for simplicity. Each black hole is located at $z = \pm d$ at rest and has the same mass m. Then the line element in cylindrical coordinate (R, z, ϕ) is

$$ds^2 = -\Omega^{-2}dt^2 + a^2(t)\Omega^2(dR^2 + dz^2 + R^2 d\phi^2) \ , \quad (4.4)$$

$$\Omega = 1 + \frac{m}{a(t)\sqrt{R^2 + (z-d)^2}} + \frac{m}{a(t)\sqrt{R^2 + (z+d)^2}} \ . \quad (4.5)$$

Since the three-metric is analytic, we can again determine

TABLE II. The existence of apparent horizons for extreme ($Q = M$) Reissner-Nordström-de Sitter black hole.

d/M	H/M^{-1}	$H/H_{\rm crit}$	AH?	$A/16\pi M^2$	$A/16\pi M^2$ (analytic)
0.0	−0.100	−0.400	Y	0.3175	0.3175
	−0.200	−0.800	Y	0.4775	0.4775
	−0.250	−1.000	Y	1.0000	1.0000
	−0.251	−1.004	N	–	–

an apparent horizon with good accuracy by the method described in Sec. III.

The results are summarized in Tables II ($d = 0$) and III ($d \neq 0$). Physical variables are normalized by the total mass $M = 2m$. Without loss of generality, we can set $t = 0$. $d = 0$ corresponds to extreme ($Q = M$) Reissner-Nordström-de Sitter black hole. For a code check, the result with $d = 0$ is compared with the analytic value of an area of black hole A, which is now

$$A = \pi \left(\frac{1 - \sqrt{1 - 4|H|M}}{|H|} \right)^2 . \tag{4.6}$$

Both numerical and analytical values are in good agreement (see Table II). We cannot find any apparent horizon for $H = 0$ numerically, because it is located at $R = z = 0$ in the coordinate (4.4).

We also have searched for the critical value of $\mathcal{C}$ above which no apparent horizon forms for a given H. The results are shown in Fig. 4. For $d \lesssim 0.4$, which corresponds to $\mathcal{C}/4\pi M \lesssim 1.06$, we find apparent horizons. For $d \gtrsim 0.5$, which corresponds to $\mathcal{C}/4\pi M \gtrsim 1.07$, we cannot find apparent horizons whatever H might be. Note that in the extreme Reissner-Nordström-deSitter spacetime a black hole exists only for $M|H| \leq 1/4$, i.e., $M_{\rm crit} = 1/4|H| = \sqrt{3}/4\sqrt{\Lambda}$.

We can conclude that the apparent horizon is formed only when $\mathcal{C} \lesssim 4\pi M \lesssim 4\pi M_{\rm crit} = \sqrt{3}\pi/\sqrt{\Lambda}$. Our cosmic hoop conjecture Eq. (2.1) is consistent with the KT solution.

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V. NAKED SINGULARITY AND COSMOLOGICAL CONSTANT

In the rest of the present paper, we discuss a relationship between the hoop conjecture and an appearance of naked singularities.

It should be noted that in the presence of a positive H, the absence of an event horizon does not necessarily imply a naked singularity. Because there exists a background cosmic expansion and a localized matter may expand with the cosmic expansion rather than contract and may disperse away, it is not clear whether or not a gravitational collapse really proceeds.

In order to determine whether the spheroid really collapses or expands, we may observe the dynamical behavior of the mass density. In the present approach, we can just deal with it for a short time period of $t \ll M$. As a coordinate condition, we set the lapse function $N = 1$ and the shift vector $N^a = 0$. Then the evolution of energy density ρ is determined by the dynamical equations;

$$\frac{\partial(\sqrt{h}\rho)}{\partial t} = 0 , \tag{5.1}$$

TABLE III. The existence of apparent horizons in the Kastor-Traschen spacetime and the circumference for various values of H and d, where d is a separation parameter of two black holes. Y and N are the same as in Table I. M is the total mass of two black holes. We also listed $H/H_{\rm crit}$, where $H_{\rm crit} = 1/4M$.

d/M	H/M^{-1}	$H/H_{\rm crit}$	AH?	$\mathcal{C}_{\rm pol}^{\rm min}/4\pi M$	$\mathcal{C}_{\rm pol}^{\rm min}/4\pi M_{\rm crit}$
0.10	0.000	0.000	N	1.0284	0.0000
	−0.100	−0.400	N		0.4114
	−0.150	−0.600	Y		0.6170
	−0.200	−0.800	Y		0.8227
	−0.250	−1.000	Y		1.0284
	−0.251	−1.004	N		1.0325
0.20	0.000	0.000	N	1.0353	0.0000
	−0.150	−0.400	N		0.6212
	−0.200	−0.800	Y		0.8282
	−0.250	−1.000	Y		1.0353
	−0.251	−1.004	N		1.0394
0.40	−0.200	−0.800	N	1.0591	0.8473
	−0.250	−1.000	Y		1.0591
	−0.251	−1.004	Y		1.0633
	−0.252	−1.008	N		1.0676

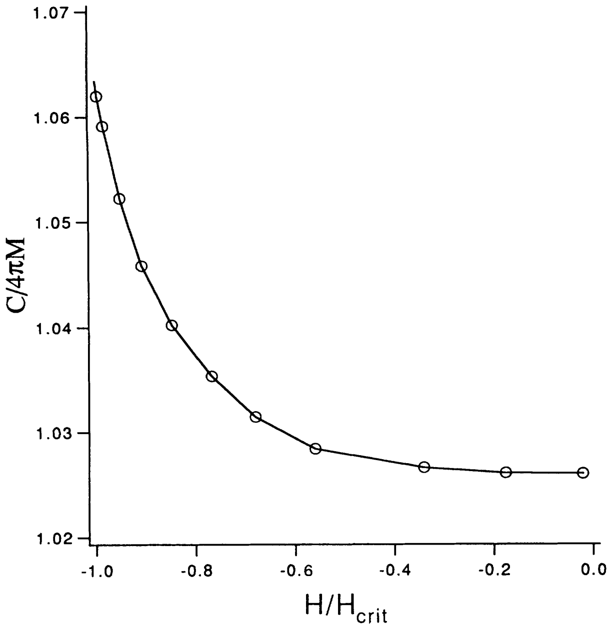

FIG. 4. Maximum value of C in the Kastor-Traschen spacetime for an apparent horizon to be formed for each H.

$$\frac{\partial K_{ab}}{\partial t} = -{}^3R_{ab} - 3H^2 h_{ab} \ , \tag{5.2}$$

and

$$\frac{\partial h_{ab}}{\partial t} = 2K_{ab} \ . \tag{5.3}$$

Here h is the determinant of the three metric h_{ab} and ${}^3R_{ab}$ is the Ricci tensor of h_{ab}. We expand K_{ab} and h_{ab} in a power series of $t(\ll M)$ as [19]

$$K_{ab} = K^{(0)}_{ab} + K^{(1)}_{ab} t + K^{(2)}_{ab} t^2 + \dots \ , \tag{5.4}$$

and

$$h_{ab} = h^{(0)}_{ab} + h^{(1)}_{ab} t + h^{(2)}_{ab} t^2 + \dots \ . \tag{5.5}$$

To zeroth order, K_{ab} and h_{ab} are given as

$$\begin{aligned} K^{(0)}_{ab} &= H\psi^4 \delta_{ab} \ , \\ h^{(0)}_{ab} &= \psi^4 \delta_{ab} \ , \end{aligned} \tag{5.6}$$

Inserting Eqs. (5.4)–(5.6) into Eqs. (5.2) and (5.3), we solve K_{ab} and h_{ab} order by order. We find that

$$\begin{aligned} K^{(1)}_{ab} = & -6\psi^{-2}\psi_{,a}\psi_{,b} + 2\psi^{-1}\psi_{,ab} + 2\psi^{-2}\delta_{ab}\psi^{,c}\psi_{,c} \\ & +2\psi^{-1}\delta_{ab}\psi^{,c}{}_{,c} - 3H^2\delta_{ab}\psi^4 \ , \end{aligned} \tag{5.7}$$

$$h^{(1)}_{ab} = 2H\delta_{ab}\psi^4 \ , \tag{5.8}$$

$$h^{(2)}_{ab} = K^{(1)}_{ab} \ . \tag{5.9}$$

From Eqs. (5.1) and (5.7)–(5.9), $\rho(t)$ is obtained as

$$\begin{aligned} \rho(t) &= \sqrt{h(0)/h(t)}\rho(0) \\ &= \rho(0)\Big[1 - 3Ht + \frac{15H^2t^2}{2} - 4\psi^{-5}\psi^{,a}{}_{,a}t^2 \\ &\quad +O(t^3/M^3)\Big] \ . \end{aligned} \tag{5.10}$$

Figure 5 shows $\rho(t)/\rho(0)$ at $z = c, R = 0$ until $t = 0.01M$ for each e and H. From these, it is seen that when the eccentricity approaches to unity and $H \lesssim H_{\rm crit}$ then the collapse proceeds with or without Λ, while when the eccentricity is small and Λ is nonzero the spheroid expands. Although the detail of the collapse or expansion may de-

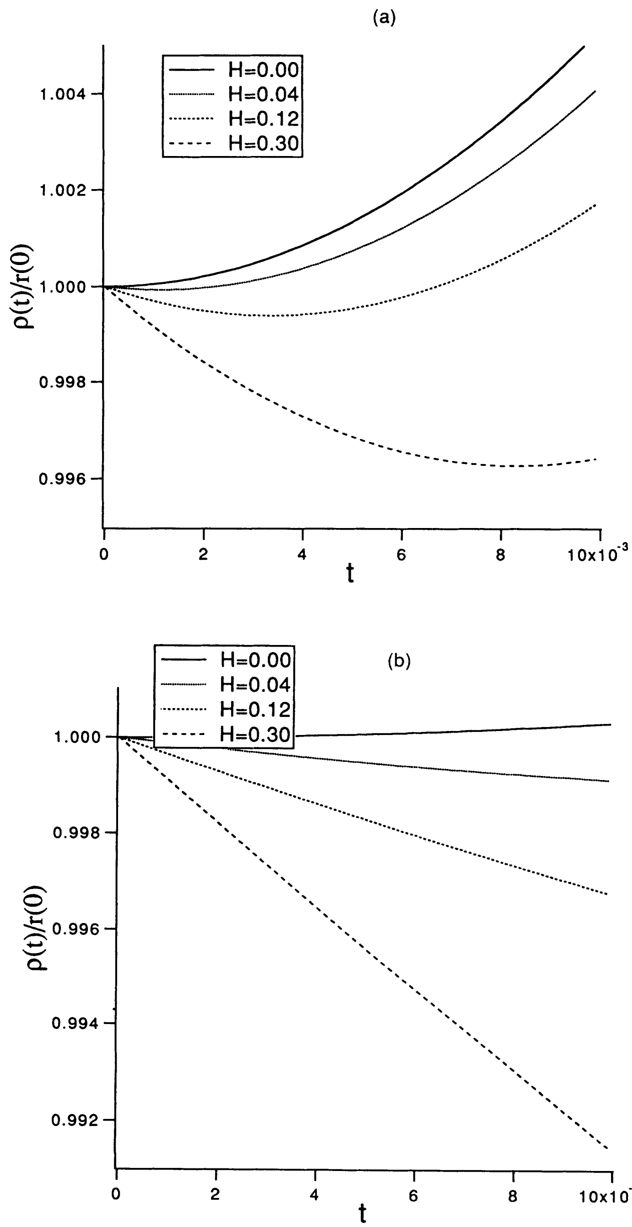

FIG. 5. The dynamical behavior of dust spheroid in the initial state. $\rho(t)/\rho(0)$ at $z = c = 0.4M, R = 0$ until $t = 0.01M$ for (a) $e = 0.99$ and (b) $e = 0.1$ for each H.

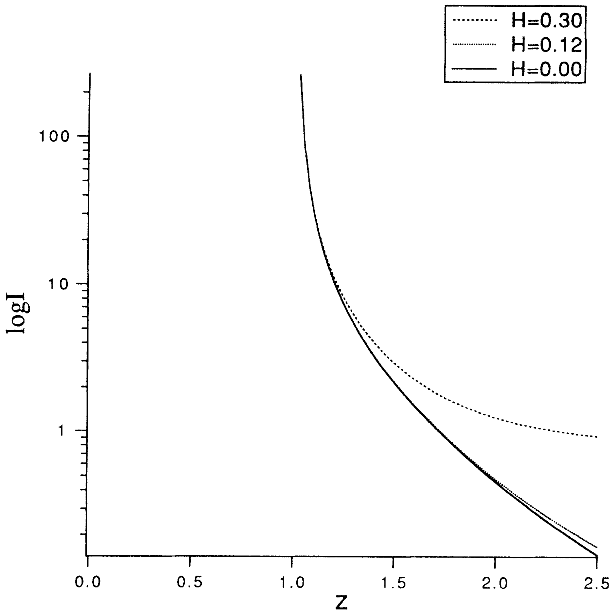

FIG. 6. The Riemann invariant I on the z axis ($R = 0$) at $t = 0$ for the prolate of $c/M = 1.0$ and $e = 0.9999$ for each H.

pend on our coordinate choice, i.e., the proper-time slicing ($N = 1, N^a = 0$), we expect that our result may be generic since dust fluid follows the geodesic.

We also calculate the Riemann invariant:

$$I = R_{abcd}R^{abcd} = 8({}^3R_{ab} + 2H^2 h_{ab})({}^3R^{ab} + 2H^2 h^{ab})$$
$$= 192\frac{(\psi_{,a}\psi^{,a})^2}{\psi^{12}} - 192\frac{\psi_{,ab}\psi^{,a}\psi^{,b}}{\psi^{11}} + 32\frac{\psi_{,ab}\psi^{,ab}}{\psi^{10}} + 96H^4$$

to estimate the behavior of the singularity with Λ at $t = 0$. As shown in Fig. 6, the singularity appears at some point on the z axis. The behavior of this singularity is not affected by the inclusion of Λ. Thus we may conclude that once the collapse proceeds, a singularity will be formed with or without Λ.

VI. CONCLUDING REMARKS

We have investigated how $\Lambda(> 0)$ should affect the hoop conjecture. As estimated in Sec. II and shown for initial data of dust spheroid in Sec. III, besides an existence of the maximum mass of a black hole, the same inequality holds as a criterion of horizon formation. This was also confirmed by the Kastor-Traschen solution. There is no significant counterexample against the cosmic hoop conjecture. It seems that the hoop conjecture contains a large patch of truth of aspherical collapse, even in the presence of a cosmological constant.

From the analysis of initial data and dynamical evolution for a short time period of $t \ll M$, we expect that a naked or "seminude" singularity, which is called by NST [3] and means that an apparent horizon runs into the inside of matter, might be easily formed in the spacetime with Λ. The time evolution from these initial data would be of great interest.

Finally, we should comment on the hoop conjecture and a quasilocal mass in the spacetime with Λ. There is a suggestion that the apparent failure of the hoop conjecture can be avoided by using a quasilocal mass [4]. However, the notion of quasilocal mass in the spacetime with Λ is quite ambiguous. For example in the Schwarzschild-de Sitter spacetime the Hawking mass [20] (or equivalently the Hayward mass [21]) is

$$M_H = M + \frac{\Lambda}{6}r^3 . \tag{6.1}$$

It does not approach to the AD mass (the asymptotic mass) in the limit $r \to \infty$, although M_H satisfies criterions of quasilocal energy in the spacetime without Λ [21]. At present there is no appropriate definition of quasilocal mass in the presence of Λ. Hence we have not adopted any quasilocal mass in Eq. (2.1).

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