Low energy effective string cosmology

E. J. Copeland,* Amitabha Lahiri,† and David Wands‡

School of Mathematical & Physical Sciences, University of Sussex, Brighton BN1 9QH, United Kingdom

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We give the general analytic solutions derived from the low energy string effective action for four-dimensional Friedmann-Robertson-Walker models with a dilaton and antisymmetric tensor field, considering both long and short wavelength modes of the H field. The presence of a homogeneous H field significantly modifies the evolution of the scale factor and dilaton. In particular it places a lower bound on the allowed value of the dilaton. The scale factor also has a lower bound but our solutions remain singular as they all contain regions where the spacetime curvature diverges signalling a breakdown in the validity of the effective action. We extend our results to the simplest Bianchi type I metric in higher dimensions with only two scale factors. We again give the general analytic solutions for long and short wavelength modes for the H field restricted to the three-dimensional space, which produces an anisotropic expansion. In the case of H field radiation (wavelengths within the Hubble length) we obtain the usual four-dimensional radiation dominated FRW model as the unique late time attractor.

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I. INTRODUCTION

String-inspired cosmology is currently attracting a great deal of attention. The most favored starting point in any analysis is the low energy string effective action from which the lowest order string β-function equations can be derived [1]. These equations, for the closed string, consist of three long range fields, the dilaton ϕ, the Kalb-Ramond field strength $H_{\mu\nu\lambda}$, and the graviton, all arising out of the massless excitation of the string. In addition there is a constant related to the central charge of the string theory which vanishes in the critical number of dimensions 10 or 26. The fact that only the massless excited state is used suggests that the effective action is not a valid description for probing the highest energies associated with string theory. However, we may hope that through the β-function equations we are investigating physics associated with events from say the string scale down to the grand unified theory (GUT) scale. Such an approach has already been adopted by a number of authors [1–12].

Veneziano and his collaborators have emphasized the possible importance of the duality symmetry which characterizes the equations of string cosmology [5,11,13]. They point out the possibility that inflation can occur without relying on a potential energy density. Rather, the duality transforms, which relate the dilaton and the metric, lead to a possible inflationary mechanism. In [13] it is claimed that the inclusion of the H field does not seem to change the underlying properties of these

duality-related cosmologies, although they also point out possible problems with exiting inflation in these models. Cosmological solutions with a dilaton and a nontrivial H field have been obtained by Tseytlin [9], for curved maximally symmetric spaces in string theory, with a nonzero central charge deficit. (See [9] for details and complete references.)

In [12], the authors study the outcome of string-dominated cosmology in a four-dimensional Friedmann-Robertson-Walker (FRW) spacetime including a homogeneous H field ($H_{\mu\nu\lambda}$) as well as a nonzero critical charge deficit V (some of the particular solutions were previously obtained by Tseytlin [8]). Their results are based on a phase-plane analysis. In this paper, we present the general analytic solutions, including both long and short wavelength solutions for the H field, but setting $V = 0$, and then show how this can be extended to simple anisotropic models in higher dimensions.

In Sec. II we introduce the low energy equations of motion in the string frame and demonstrate how they can be related via simple conformal transforms to equations in other frames. Section III concentrates on four-dimensional FRW models and describes the complete solution for a homogeneous H field, reproducing where applicable previous solutions found in the literature [9,12]. We go on to describe radiation solutions in which the H field has a spatial dependence on small scales (i.e., wavelengths much smaller than the Hubble length). Because the metric (and the dilaton) is homogeneous and isotropic we require the H field energy-momentum tensor to be homogeneous and isotropic on average, and demonstrate how at late times we recover the general relativistic results for radiation and curvature dominated models. In Sec. IV we consider the influence of homogeneous and radiation solutions of the H field in $D = 4 + n$ cosmologies where the metric tensor is decomposed into the direct product of a four-dimensional FRW metric

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*Electronic address: edmundjc@central.susx.ac.uk
†Electronic address: a.lahiri@central.susx.ac.uk
‡Electronic address: dwands@star.susx.ac.uk

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and an n-dimensional metric. We give general analytic solutions for the simplest version of the D-dimensional Bianchi type I models where there are just two scale factors. The results indicate how the H field can produce an anisotropic expansion leading to one scale expanding while the other contracts. Finally in Sec. V we summarize our main results.

II. STRING ACTION

We shall take as our starting point the low energy D-dimensional string effective action [1]

$$
S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g} e^{-\phi} \left[R + (\nabla\phi)^2 - V - \frac{1}{12} H^2 \right]
$$

$$
+ \int d^D x \sqrt{-g} \mathcal{L}_{\text{matter}} , \tag{2.1}
$$

where $\kappa_D^2 = 8\pi G_D$ and we adopt the sign conventions denoted $(+++)$ by Misner, Thorne, and Wheeler [14]. ϕ is the dilaton field determining the strength of the gravitational coupling and $H^2 = H_{\mu\nu\lambda} H^{\mu\nu\lambda}$ where $H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$.

The variation of this action with respect to the $g_{\mu\nu}$, $B_{\mu\nu}$, and ϕ, respectively, yields the field equations

$$
R_\mu^\nu - \frac{1}{2} g_\mu^\nu R = \kappa_D^2 e^\phi T_\mu^\nu + \frac{1}{12} \left(3 H_{\mu\lambda\kappa} H^{\nu\lambda\kappa} - \frac{1}{2} g_\mu^\nu H^2 \right)
$$

$$
- \frac{1}{2} g_\mu^\nu V - \frac{1}{2} g_\mu^\nu (\nabla\phi)^2
$$

$$
+ \left(g_\mu^\nu g^{\lambda\kappa} - g_\mu^\lambda g^{\nu\kappa} \right) \nabla_\lambda \nabla_\kappa \phi , \tag{2.2}
$$

$$
\nabla_\mu \left(e^{-\phi} H^{\mu\nu\lambda} \right) = 0 , \tag{2.3}
$$

$$
2 \Box \phi + R - (\nabla\phi)^2 - V - \frac{1}{12} H^2 = 0 , \tag{2.4}
$$

where $T_{\mu\nu}$ is the energy-momentum tensor derived from the matter Lagrangian.

The effect of certain types of "stringy matter" has been considered elsewhere in the literature [11]. Specific schemes of compactification, not to mention the chosen gauge symmetries of the theory, will determine the behavior (and number) of both bosonic and fermionic matter fields. It is not inconceivable that in some "matter-dominated era" of the stringy epoch of the universe these matter fields will play a part in determining the cosmological evolution. In particular, the antisymmetric tensor may be considered a matter field, and as we shall see it plays a significant role in string cosmology. In favor of considering the possible effects of this field on the cosmology we shall ignore all other contributions from the matter Lagrangian.

The charge deficit V is a constant proportional to $D - 26$ for the bosonic string and $D - 10$ in the heterotic or superstring. We will set $V = 0$ in our analysis. This may well be necessary for a consistent theory either by choosing the appropriate number of spatial dimensions or due to cancellation with contributions from other matter fields. Even for nonzero V this should be an increasingly good approximation at early times if the curvature and/or kinetic energy densities are large, $V \ll R, (\nabla\phi)^2, H^2$, but we would need to consider its effect at late times in an expanding universe.

A. Conformal frames

These field equations are similar to those found in Brans-Dicke gravity [15] with the Brans-Dicke parameter $\omega = -1$. This is only strictly true in the absence of the H field as in Brans-Dicke gravity it is assumed that the energy-momentum tensor of all fields (other than the Brans-Dicke field, $\Phi \equiv e^{-\phi}$) are minimally coupled to the metric $g_{\mu\nu}$. While the energy-momentum tensor of other matter fields is assumed to be conserved with respect to this metric (the string metric), so that $\nabla_\mu T^{\mu\nu} = 0$, we cannot define an energy-momentum tensor solely in terms of the H field and the string metric which is conserved independently of the dilaton. This is just a consequence of the equation of motion for H [Eq. (2.3)] which has an explicit dependence upon e^ϕ.

$H_{\mu\nu\lambda}$ is only minimally coupled in the conformally related metric [16]

$$
\bar{g}_{\mu\nu} = \exp\left(\frac{2\phi}{6 - D} \right) g_{\mu\nu} , \tag{2.5}
$$

which we might call the B metric, in which we find[1]

$$
\bar{\nabla}_\mu \bar{H}^{\mu\nu\lambda} = 0 . \tag{2.6}
$$

Another particularly useful metric to introduce is the Einstein metric

$$
\tilde{g}_{\mu\nu} = \exp\left(-\frac{2\phi}{D - 2} \right) g_{\mu\nu} . \tag{2.7}
$$

In this frame the action appears simply as the Einstein-Hilbert action of general relativity in D dimensions, while the dilaton appears simply as a matter field, albeit one interacting with the other matter fields:

$$
S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{1}{D-2} (\tilde{\nabla}\phi)^2 \right.
$$

$$
\left. - V \exp\left(\frac{2\phi}{D-2} \right) - \frac{1}{12} \exp\left(-\frac{4\phi}{D-2} \right) \tilde{H}^2 \right]
$$

$$
+ \int d^D x \sqrt{-\tilde{g}} \exp\left(\frac{D\phi}{D-2} \right) \mathcal{L}_{\text{matter}} . \tag{2.8}
$$

The corresponding field equations are then those for interacting fields in general relativity:

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[1]Note that the three-form $H_{\mu\nu\lambda}$ has a conformally invariant definition in terms of the potential $B_{\mu\nu}$, whereas its covariant form has indices raised by a particular metric.

$$\widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{R} = \kappa_D^2 \left(\widetilde{T}_{\mu\nu} +^{(H)}\widetilde{T}_{\mu\nu} + {}^{(\phi)}\widetilde{T}_{\mu\nu} \right.$$

$$\left. +\,{}^{(V)}\widetilde{T}_{\mu\nu} \right) , \qquad (2.9)$$

$$\widetilde{\nabla}_\mu \left[\exp\left(-\frac{4\phi}{D-2}\right) \widetilde{H}^{\mu\nu\lambda} \right] = 0 , \qquad (2.10)$$

$$\widetilde{\Box}\,\phi - V \exp\left(\frac{2\phi}{D-2}\right) + \frac{1}{6}\exp\left(-\frac{4\phi}{D-2}\right)\widetilde{H}^2 + \tfrac{1}{2}\widetilde{T} = 0 , \qquad (2.11)$$

where the terms on the right-hand side of the Einstein equations correspond to

$$\widetilde{T}^\nu_\mu = \exp\left(\frac{D\phi}{D-2}\right) T^\nu_\mu , \qquad (2.12)$$

$$\kappa_D^2 \,{}^{(\phi)}\widetilde{T}^\nu_\mu = \frac{1}{D-2}\left(\widetilde{g}^\lambda_\mu \widetilde{g}^{\nu\kappa} - \frac{1}{2}\widetilde{g}^\nu_\mu \widetilde{g}^{\lambda\kappa}\right)\phi_{,\lambda}\phi_{,\kappa} , \qquad (2.13)$$

$$\kappa_D^2 \,{}^{(H)}\widetilde{T}^\nu_\mu = \frac{1}{12}\exp\left(-\frac{4\phi}{D-2}\right)$$

$$\times \left(3\widetilde{H}_{\mu\lambda\kappa}\widetilde{H}^{\nu\lambda\kappa} - \frac{1}{2}\widetilde{g}^\nu_\mu \widetilde{H}^2\right) , \qquad (2.14)$$

$$\kappa_D^2 \,{}^{(V)}\widetilde{T}^\nu_\mu = -\frac{1}{2}V\exp\left(\frac{2\phi}{D-2}\right)\widetilde{g}^\nu_\mu , \qquad (2.15)$$

the energy-momentum tensors for the matter, dilaton and H fields, and potential V, respectively. The total energy-momentum must be conserved of course by the Ricci identity, but there are interactions between these four components. Henceforth, as remarked earlier, we shall set $T_{\mu\nu}$ and V to be zero.

III. ISOTROPIC $D = 4$ SOLUTIONS

Firstly we will consider the behavior of four-dimensional homogeneous and isotropic cosmologies for which the most general metric is the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) , \qquad (3.1)$$

$$= a^2(\eta)\left(-d\eta^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) , \qquad (3.2)$$

in terms of the conformally invariant time coordinate, η, with $k = +1, 0, -1$ for spatially closed, flat, or open models, respectively. Just as we take the metric to be homogeneous we shall also assume that the dilaton has no spatial dependence, $\phi = \phi(\eta)$.

With $D = 4$ the conformal transform to the Einstein frame gives a rescaled scale factor $\widetilde{a} = e^{-\phi/2}a$. The metric field equations are simplest in terms of the Einstein metric where we have the familiar constraint equation (the 00-component of Eq. (2.9) with $V = 0$ and $\widetilde{T}^\nu_\mu = 0$)

$$-3\left(\frac{\widetilde{a}'^2}{\widetilde{a}^4} + \frac{k}{\widetilde{a}^2}\right) = \kappa^2 \left({}^{(H)}\widetilde{T}^0_0 + {}^{(\phi)}\widetilde{T}^0_0 \right) , \qquad (3.3)$$

where prime denotes differentiation with respect to η.

In four-dimensions the H field equation of motion, Eq. (2.10), is solved by the ansatz

$$\widetilde{H}^{\mu\nu\lambda} = e^{2\phi}\widetilde{\epsilon}^{\mu\nu\lambda\kappa}h_{,\kappa} , \qquad (3.4)$$

where $\widetilde{\epsilon}^{\mu\nu\lambda\kappa}$ is the antisymmetric four-form (obeying $\widetilde{\nabla}_\rho\widetilde{\epsilon}^{\mu\nu\lambda\kappa} = 0$) and the integrability condition, $\partial_{[\mu}\widetilde{H}_{\nu\lambda\kappa]} = 0$, becomes the new equation of motion for h

$$\widetilde{\Box}\,h + 2\widetilde{\nabla}^\mu\phi\widetilde{\nabla}_\mu h = 0 . \qquad (3.5)$$

Thus h evolves as a massless scalar field coupled to the dilaton (except in the B frame where $\overline{\Box}\,h = 0$). The same interaction appears in the dilaton equation of motion (Eq. (2.11), with $V = 0$) as

$$\widetilde{\Box}\,\phi = e^{2\phi}\left(\widetilde{\nabla}h\right)^2 . \qquad (3.6)$$

Thus we have two interacting scalar fields whose energy-momentum tensors are given by

$$\kappa^2 \,{}^{(\phi)}\widetilde{T}^\nu_\mu = \frac{1}{2}\left(\widetilde{g}^\lambda_\mu \widetilde{g}^{\nu\kappa} - \frac{1}{2}\widetilde{g}^\nu_\mu \widetilde{g}^{\lambda\kappa}\right)\phi_{,\lambda}\phi_{,\kappa} , \qquad (3.7)$$

$$\kappa^2 \,{}^{(H)}\widetilde{T}^\nu_\mu = \frac{1}{2}\left(\widetilde{g}^\lambda_\mu \widetilde{g}^{\nu\kappa} - \frac{1}{2}\widetilde{g}^\nu_\mu \widetilde{g}^{\lambda\kappa}\right)e^{2\phi}h_{,\lambda}h_{,\kappa} . \qquad (3.8)$$

These equations simplify considerably in two cases.

A. Homogeneous solution $h(\eta)$

Thus far we have allowed for the possibility of a spatial dependence of the H field. However as we have already restricted ourselves to considering a homogeneous metric and dilaton field, this will only be consistent with choosing a source that is at least homogeneous on average. Indeed as far as we are aware, the only case that has been considered to date [12] is that of a strictly isotropic H field, where $h = h(\eta)$, and thus $\widetilde{H}_{0\mu\nu} = 0$.

In this case the equation of motion for the H field [Eq. (3.5)] becomes

$$h'' + \left(2\frac{\widetilde{a}'}{\widetilde{a}} + 2\phi'\right)h' = 0 , \qquad (3.9)$$

which is easily integrated to give $e^{2\phi}\widetilde{a}^2 h' = \pm L$ where L is a non-negative constant. It is this kinetic energy density of the H field that drives the dilaton. Note that back in the string frame this solution corresponds to $H^2 = 6L^2/a^6$. Thus it will dominate any charge deficit V in the dilaton equation of motion, Eq. (2.4), as $a \to 0$.

Because both h and ϕ are functions only of time, we can define a new scalar field $\psi(t)$ where

$$d\psi^2 = d\phi^2 + e^{2\phi}dh^2 . \qquad (3.10)$$

The energy-momentum tensor in the Einstein frame is then simply that for this single minimally coupled field:

$$\kappa^2 \left(\, {}^{(\phi)}\widetilde{T}^{\nu}_{\mu} + {}^{(H)}\widetilde{T}^{\nu}_{\mu} \right) = \frac{1}{2} \left(\widetilde{g}^{\lambda}_{\mu} \widetilde{g}^{\nu\kappa} - \frac{1}{2} \widetilde{g}^{\nu}_{\mu} \widetilde{g}^{\lambda\kappa} \right) \psi_{,\lambda}\psi_{,\kappa} \ . \tag{3.11}$$

The equation of motion for a homogeneous minimally coupled field is

$$\psi'' + 2\frac{\widetilde{a}'}{\widetilde{a}}\psi' = 0 \ , \tag{3.12}$$

which can be integrated to give $\widetilde{a}^2\psi' = \pm K$, where K is a positive constant. (We will not consider the trivial case where $K = 0$.) Thus in the Einstein frame we have an isotropic perfect stiff fluid whose energy density $\kappa^2\widetilde{\rho} \equiv -\kappa^2(\, {}^{(\phi)}\widetilde{T}^0_0 + {}^{(H)}\widetilde{T}^0_0) = K^2/4\widetilde{a}^6$.

The constraint Eq. (3.3) for the Einstein scale factor, $\widetilde{a}$, can then be integrated to give [17]

$$\widetilde{a}^2 = \frac{K}{\sqrt{3}}\frac{\tau}{1 + k\tau^2} \ , \tag{3.13}$$

where we define

$$\tau(\eta) = \begin{cases} |\tan(\eta - \eta_0)| & \text{for } k = +1 \ , \\ |\eta - \eta_0| & \text{for } k = 0 \ , \\ |\tanh(\eta - \eta_0)| & \text{for } k = -1 \ . \end{cases} \tag{3.14}$$

We emphasize that in the Einstein frame the scale factor evolves in a wholly unremarkable fashion. We have a singularity at $\eta = \eta_0$ with $\widetilde{a} = 0$ and the models expand away from it for $\eta > \eta_0$ or collapse towards it for $\eta < \eta_0$. Only closed models can turn around, and these recollapse at $\eta = \eta_0 \pm \pi/2$. There are no bounce solutions. Notice also that the behavior of the Einstein scale factor is independent of L, and thus is the same in vacuum, i.e., without the presence of the H field, as it is with the H field.

Combining the first integrals for ψ and the H field with the definition of $d\psi$ we also have

$$\phi'^2 = \frac{K^2 - e^{-2\phi}L^2}{\widetilde{a}^4} \ . \tag{3.15}$$

This too can be integrated to give

$$e^{\phi} = \begin{cases} \left(\frac{\tau}{\tau_*}\right)^{\pm\sqrt{3}} & \text{for } L = 0 \ , \\ \frac{L}{2K}\left[\left(\frac{\tau_*}{\tau}\right)^{\sqrt{3}} + \left(\frac{\tau}{\tau_*}\right)^{\sqrt{3}}\right] & \text{for } L \neq 0 \ , \end{cases} \tag{3.16}$$

where τ_* is an integration constant. Note that the evolution in the presence of the antisymmetric tensor field is quite distinct from the vacuum behavior. In particular there is a lower bound on the dilaton, $e^{2\phi} \geq K^2/L^2$. By contrast, there are two distinct branches in vacuum where the dilaton is either monotonically increasing or decreasing.

We can use this to recover the scale factor in the string frame:

$$a^2 = \begin{cases} \frac{K\tau_*}{\sqrt{3}}\left(\frac{\tau}{\tau_*}\right)^{1\pm\sqrt{3}}\left(1 + k\tau^2\right)^{-1} \\ \qquad\qquad\qquad\qquad \text{for } L = 0 \ , \\ \frac{L\tau_*}{2\sqrt{3}}\left[\left(\frac{\tau}{\tau_*}\right)^{-\sqrt{3}+1} + \left(\frac{\tau}{\tau_*}\right)^{\sqrt{3}+1}\right]\left(1 + k\tau^2\right)^{-1} \\ \qquad\qquad\qquad\qquad \text{for } L \neq 0 \ . \end{cases} \tag{3.17}$$

The evolution of the scale factor and dilaton in different cases is shown in Figs. 1-4. The singular behavior of the conformal factor, e^{ϕ}, at the initial singularity in the Einstein frame produces cosmologies which have a minimum nonzero value for the scale factor in the string frame. In the presence of the H field, a diverges both as $\eta \to \pm\infty$ (or as $\eta \to \eta_0 \pm \pi/2$ for $k > 0$) and as $\eta \to \eta_0$, so all models "bounce," although they are still singular in the sense that the Ricci curvature diverges.

Note that the vacuum solutions again exhibit two distinct branches corresponding to ϕ and a either monotonically increasing or decreasing. The $k = 0$ models correspond simply to power-law solutions shown in Figs. 1(a) and 1(b). We can write the solutions in terms of the
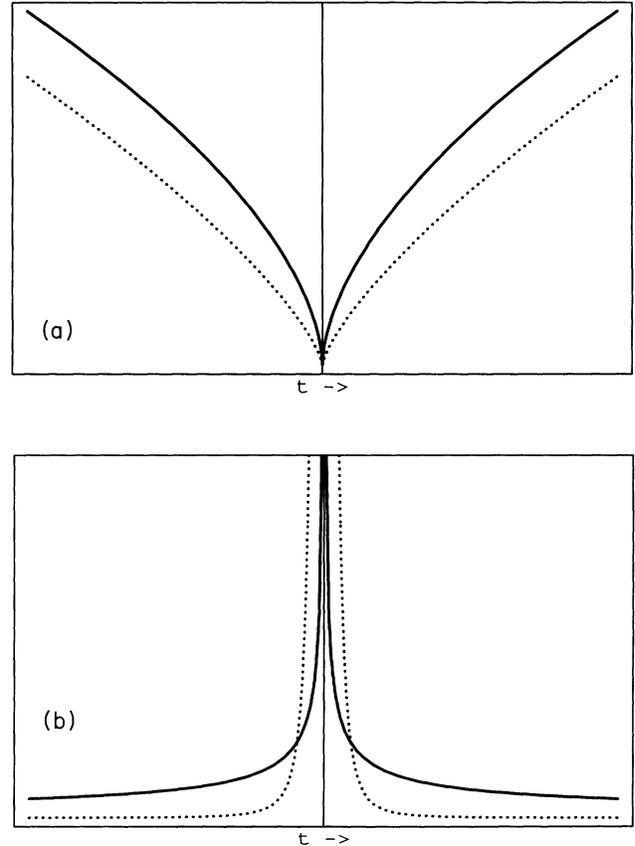
FIG. 1. (a) The decelerated branches of the scale factor a (solid line) and dilaton e^{ϕ} (dotted) in a spatially flat FRW universe with vanishing (or constant) h. (b) The accelerated branches of the scale factor a (solid line) and dilaton e^{ϕ} (dotted) in a spatially flat FRW universe with vanishing (or constant) h.
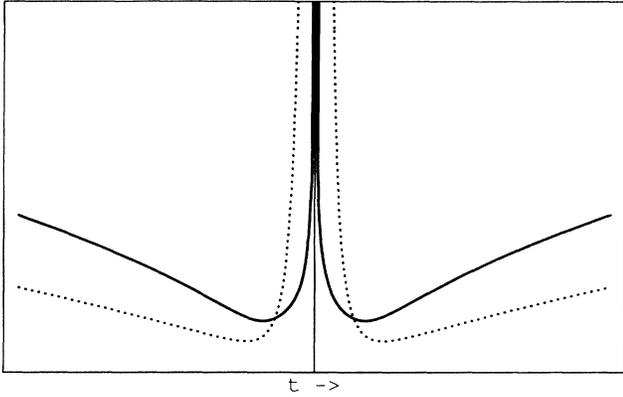
FIG. 2. Scale factor a (solid line) and dilaton e^ϕ (dotted) in a spatially flat FRW universe with homogeneous h.

proper time in the string frame by integrating $dt = a\,d\eta$. The two solutions are then

$$e^\phi \propto |t - t_0|^{+\sqrt{3}-1} \text{ and } a \propto |t - t_0|^{+1/\sqrt{3}};$$

$$e^\phi \propto |t - t_0|^{-\sqrt{3}-1} \text{ and } a \propto |t - t_0|^{-1/\sqrt{3}}.$$

These two branches, corresponding to a decelerated or accelerated scale factor, are related by the duality transformation [18] $a \to 1/a$ and $e^\phi \to e^\phi/a^6$.

For $L \neq 0$ the $k = 0$ solution corresponds to the power-law vacuum solutions at early and late times and we find that it smoothly interpolates between the accelerated and decelerated branches. (See Fig. 2.) In some sense then it could be described as a "self-dual" solution.

As $\eta \to \eta_0$, we have $e^\phi \propto |t - t_0|^{-\sqrt{3}-1}$ and $a \propto |t - t_0|^{-1/\sqrt{3}}$ as $t \to t_0$.

As $\eta \to \pm\infty$, we have $e^\phi \propto |t|^{\sqrt{3}-1}$ and $a \propto |t|^{1/\sqrt{3}}$ as $t \to \pm\infty$.

This is precisely the behavior found by Goldwirth and Perry [12] in their phase-plane analysis. The vacuum solutions correspond to particular solutions (here corre-
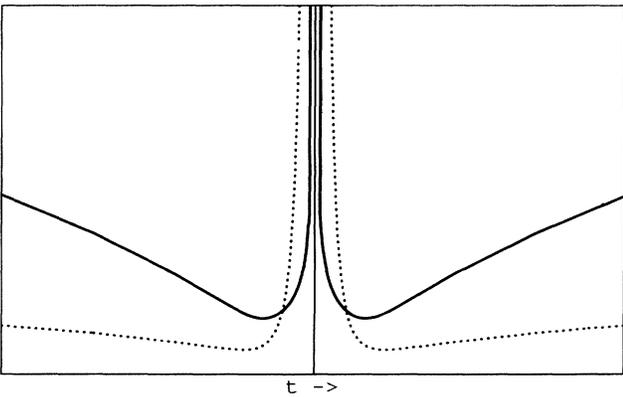
FIG. 3. Scale factor a (solid line) and dilaton e^ϕ (dotted) in an open ($k = -1$) FRW universe with homogeneous h.
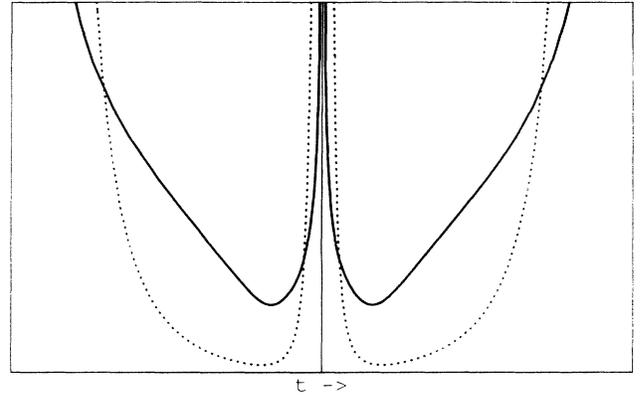
FIG. 4. Scale factor a (solid line) and dilaton e^ϕ (dotted) in a closed ($k = +1$) FRW universe with homogeneous h.

sponding to the limiting behavior where τ_* is either infinite or zero) found previously [8].

The solutions in spatially curved models approach the flat space results only near η_0. At late times $\tau \to 1$ in open ($k = -1$) models (Fig. 3) and thus the dilaton becomes frozen-in at a fixed value as the curvature dominates the evolution and and we approach the Einstein result. In closed models (Fig. 4) where $\tau \to \infty$ as $\eta - \eta_0 \to \pm\pi/2$, the scale factor diverges in a finite proper time. Thus although these models bounce, and therefore must undergo a period of inflation ($\ddot{a} > 0$) in the string frame, they still become curvature dominated at late times.

B. Radiation solution, $(\nabla h)^2 = 0$

It is also possible to consider cases where the H field does have a spatial dependence. Because our metric (and dilaton) is homogeneous and isotropic we will require that the H field energy-momentum tensor is also homogeneous and isotropic on average. It is natural (in flat space) to decompose any field into Fourier modes, $h = \sum_q h_q(\eta) \exp(iq_i x^i)$ where q_i is a spatial comoving three-vector. We see then that the preceding case corresponds to the long wavelength mode where $q^2 = \sum_i q_i^2 \to 0$. The other case in which we can solve the equation of motion is where $q \to \infty$ where effects of spacetime curvature can be neglected and the usual Minkowski spacetime result holds.

Specifically we find that, in the B frame, for $q^2 \gg \bar{a}''/\bar{a}$, $k/\bar{a}^2$, the equation of motion for $h_q(\eta)$ reduces to

$$(\bar{a}h_q)'' + q^2(\bar{a}h_q) \simeq 0. \tag{3.18}$$

Thus, for a single short wavelength mode, in the limit $q \to \infty$, we have $h_q = m_q e^{iq\eta}/\bar{a}$ where m_q is a constant. This corresponds to an energy-momentum tensor for the H field in the Einstein frame:

$$\kappa^2 \, ^{(H)}\tilde{T}_\mu^\nu = q_\mu q^\nu \frac{m_q^2}{\bar{a}^2}, \tag{3.19}$$

where q_μ is a null four-vector with $q_0 = q$. Clearly this is not isotropic for a single wave vector q_μ, but if we consider an isotropic distribution of wave vectors we find

$$-\kappa^2 \langle {}^{(H)}\widetilde{T}^0_0 \rangle = 3\kappa^2 \langle {}^{(H)}\widetilde{T}^i_i \rangle \quad \text{(no sum)} , \tag{3.20}$$

$$= \frac{M}{\widetilde{a}^4} , \tag{3.21}$$

the usual result for radiation in a FRW universe, where $M \equiv \int m_q^2 q^2 dq$.

Note that the fluid is trace-free, and thus conformally invariant. This means that the energy-momentum of the radiation is conserved in all conformal frames and so the H field is decoupled from the dilaton which appears as a minimally coupled scalar field in the Einstein frame. Thus the first integral of its equation of motion, for a homogeneous $\phi(\eta)$, gives

$$\phi' = \frac{A}{\widetilde{a}^2} . \tag{3.22}$$

Just like the ψ field for homogeneous $h(\eta)$, the energy-momentum of the ϕ field behaves like a stiff fluid, with

$$-\kappa^2\,{}^{(\phi)}\widetilde{T}^0_0 = \kappa^2\,{}^{(\phi)}\widetilde{T}^i_i \quad \text{(no sum)} , \tag{3.23}$$

$$= \frac{K^2}{4\widetilde{a}^6} . \tag{3.24}$$

Once again the constraint Eq. (3.3), now for two noninteracting fluids, one radiation, one stiff, can be integrated [17] to give

$$\widetilde{a}^2 = \frac{K + M\tau}{\sqrt{3}}\,\frac{\tau}{1 + k\tau^2} , \tag{3.25}$$

using the time coordinate defined in Eq. (3.14). (As in the homogeneous case, this is just the familiar behavior for a FRW universe in general relativity with matter obeying the strong energy condition and thus singular for all models at $\tau = 0$.) This in turn allows the equation for ϕ', Eq. (3.22), to be integrated, yielding e^ϕ and thus, via the conformal transformation, the scale factor in the string frame:

$$e^\phi = \left(\frac{s}{s_*}\right)^{\pm\sqrt{3}} , \tag{3.26}$$

$$a^2 = \left(\frac{s}{s_*}\right)^{\pm\sqrt{3}} \frac{K + M\tau}{\sqrt{3}}\,\frac{\tau}{1 + k\tau^2} , \tag{3.27}$$

where we have introduced

$$s(\eta) = \frac{M\tau(\eta)}{K + M\tau(\eta)} . \tag{3.28}$$

Notice how the evolution of the dilaton field is now similar to that in the vacuum case except it is determined by the function $s(\eta)$ rather than $\tau(\eta)$. At early times ($\eta \sim \eta_0$) s is proportional to τ but, unlike the vacuum case, $s \to 1$ as $\tau \to \infty$ and so the field becomes frozen-in at late times in the flat model as well as the open model [where $\tau \to 1$ and thus $s \to M/(K + M)$]. Thus we recover the late time general relativistic results for radiation and curvature dominated models, respectively.
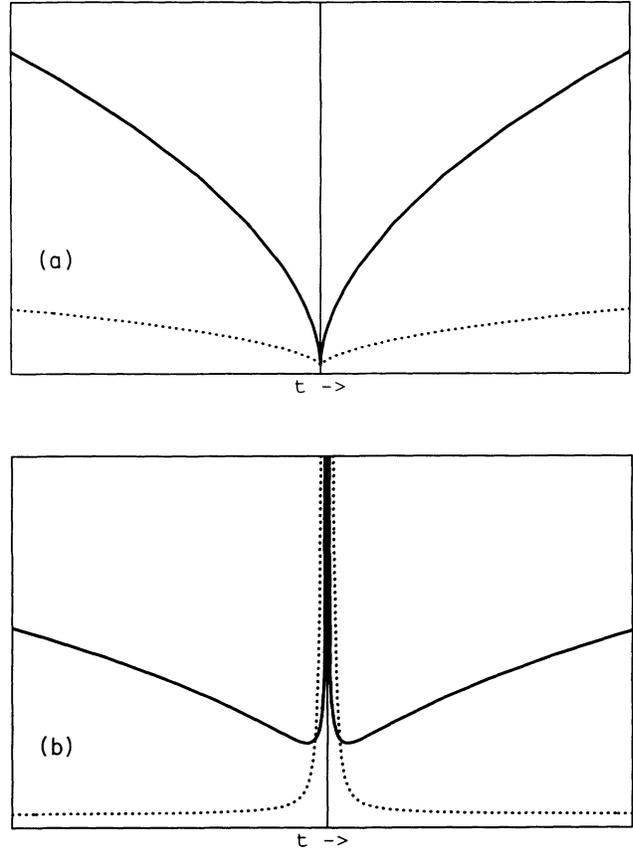
FIG. 5. (a) The decelerated branches of the scale factor a (solid line) and dilaton e^ϕ (dotted) in a spatially flat FRW universe with short wavelength h. (b) The accelerated branches of the scale factor a (solid line) and dilaton e^ϕ (dotted) in a spatially flat FRW universe with short wavelength h.

In common with the vacuum solutions, there are two distinct branches [Figs. 5(a) and 5(b)] with the dilaton monotonically increasing from zero (the decelerated branch) or decreasing from infinity (the accelerated branch) at $\eta = \eta_0$. For $M\tau \ll |K|$ the evolution is essentially identical to that in the vacuum case with two branches where $a \to \infty$ as $\tau \to 0$ when $\phi \to \infty$, but $a \to 0$ when $\phi \to 0$. Thus we have no solution interpolating between the vacuum branches.

IV. ANISOTROPIC $D = 4 + N$ SOLUTIONS

Having investigated the homogeneous and radiation solutions for the H field in four dimensions, we return to Eqs. (2.8)–(2.14) in the Einstein frame where we will consider a metric tensor in $D = 4 + n$ dimensions that can be decomposed into the direct product form

$$d\widetilde{s}^2 = -d\widetilde{t}^2 + \widetilde{a}^2(\widetilde{t})dx_i dx^i + \widetilde{b}^2(\widetilde{t})dx_J dx^J , \tag{4.1}$$

where we let i run from 1 to 3 and J run from 4 to $n + 3$. The scale factors $\widetilde{a}$ and $\widetilde{b}$ thus refer to the three-

space and n-space, respectively. We have chosen these spaces to be spatially flat, thus this is a Bianchi type I metric. The procedure we outline here could also be applied to the general D-dimensional Bianchi type I metric, but we restrict our discussion here to a two-scale factor model to avoid introducing too many degrees of freedom. Equation (4.1) is, of course, related to the original string metric through the conformal transformation Eq. (2.7).

The Einstein evolution equations, Eq. (2.9), for the two scale factors written in terms of $\dot{\alpha} \equiv (d\widetilde{a}/d\widetilde{t})/\widetilde{a}$ and $\dot{\beta} \equiv (d\widetilde{b}/d\widetilde{t})/\widetilde{b}$ are then

$$\ddot{\alpha} + (3\dot{\alpha} + n\dot{\beta})\dot{\alpha} = \frac{\kappa_D^2}{n+2}\left[\widetilde{\rho} + (n-1)\widetilde{p} - n\widetilde{q}\right] , \quad (4.2)$$

$$\ddot{\beta} + (3\dot{\alpha} + n\dot{\beta})\dot{\beta} = \frac{\kappa_D^2}{n+2}\left(\widetilde{\rho} + 2\widetilde{q} - 3\widetilde{p}\right) , \quad (4.3)$$

plus the constraint equation

$$3\dot{\alpha}^2 + 3n\dot{\alpha}\dot{\beta} + \frac{n(n-1)}{2}\dot{\beta}^2 = \kappa_D^2\widetilde{\rho} , \quad (4.4)$$

where $\widetilde{\rho} = -\widetilde{T}_0^0$, $\widetilde{p} = \widetilde{T}_i^i$, and $\widetilde{q} = \widetilde{T}_J^J$ (no sum over i or J). Note that any isotropic stiff fluid for which $\widetilde{\rho} = \widetilde{p} = \widetilde{q}$ makes no contribution to the right-hand-side of the evolution equations, entering only into the constraint equation.

We will adopt the simplest extension of our $D = 4$ ansatz for the H field,

$$\widetilde{H}^{\mu\nu\lambda} = e^{4\phi/(n+2)}\widetilde{\epsilon}^{\mu\nu\lambda\kappa}h_{,\kappa} , \quad (4.5)$$

where

$$\widetilde{\epsilon}^{\mu\nu\lambda\kappa} = \frac{4!}{\sqrt{-\widetilde{g}}}\delta_{[0}^\mu\delta_1^\nu\delta_2^\lambda\delta_{3]}^\kappa . \quad (4.6)$$

Note that whereas this ansatz included all the solutions in four dimensions, this represents only one of many degrees of freedom for the H field in the $D = 4+n$ case. An antisymmetric field in the higher-dimensional space corresponds to a whole range of fields in the four-dimensional reduced theory [19]. At a classical level we can obtain self-consistent results if these other fields are set to zero initially, though in a full quantum treatment we would have to consider precisely how the massive modes of these fields are excited in the extra dimensions. Our choice of ansatz makes any dependence of the function h on the n-space coordinates, x^J, irrelevant as this cannot affect $\widetilde{H}^{\mu\nu\lambda}$ and so we need consider only $h = h(\widetilde{t}, x^i)$. A similar approach of demanding the H field live only in three space dimensions was considered in [4] who found numerically that in the Einstein frame the effect was to drive that space to a large size while the other spaces remained of order the Planck scale.

Using (4.5) the integrability condition becomes the equation of motion for $h(\widetilde{t}, x^i)$:

$$\sum_{\mu,\nu=0}^{3}\widetilde{g}^{\mu\nu}\partial_\mu\left(\widetilde{a}^3\widetilde{b}^{-n}e^{4\phi/(n+2)}h_{,\nu}\right) = 0 . \quad (4.7)$$

We can decompose the field h into its Fourier components, $h(\widetilde{t}, x^i) = h_q(\widetilde{t})\exp(iq_ix^i)$ so that the equation of motion for the homogeneous function h_q is

$$\ddot{h}_q + \left(3\dot{\alpha} - n\dot{\beta} + \frac{4}{n+2}\dot{\phi}\right)\dot{h}_q + \frac{q^2}{\widetilde{a}^2}h_q = 0 . \quad (4.8)$$

The final equation of motion is that for the dilaton in the Einstein frame, Eq. (2.11), driven by $\widetilde{H}^2 = -6\exp(8\phi/(n+2))\widetilde{b}^{-2n}(\widetilde{\nabla}h)^2$ giving

$$\ddot{\phi} + (3\dot{\alpha} + n\dot{\beta})\dot{\phi} = -\frac{e^{4\phi/(n+2)}}{\widetilde{b}^{2n}}\left(\widetilde{\nabla}h\right)^2 . \quad (4.9)$$

Note that having chosen all the fields to be independent of the coordinates on the n-space we could replace the D-dimensional action by an effective theory written in terms of the four-dimensional part of the metric in the string frame:

$$S_4 = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g_4}\, e^{-\varphi}\left[R_4 + (\nabla\varphi)^2 - n[\nabla(\ln b)]^2\right.$$
$$\left. -\frac{1}{12}H^2\right] , \quad (4.10)$$

where the effective $D = 4$ dilaton is given by

$$e^\varphi \equiv \frac{e^\phi}{b^n} , \quad (4.11)$$

and the scale factors of the extra dimensions just act as massless (moduli) fields. However to emphasize the dynamical evolution of the n-space we shall treat this scale factor on an equal footing with that of the three-space. Note that the effective $D = 4$ Einstein frame would not be the same as the $(D = 4+n)$-dimensional Einstein frame due to the modified dilaton.

A. Homogeneous solution $h(\widetilde{t})$

The equation of motion for h for long-wavelength modes (where $q^2 \to 0$) becomes

$$\ddot{h} + \left(3\dot{\alpha} - n\dot{\beta} + \frac{4}{n+2}\dot{\phi}\right)\dot{h} = 0 , \quad (4.12)$$

or

$$\dot{h} = \frac{\pm L\widetilde{b}^n}{\widetilde{a}^3}e^{-4\phi/(n+2)} , \quad (4.13)$$

where L is a positive constant. From Eq. (2.14) and making use of Eqs. (4.5) and (4.13) we have

$$\widetilde{H}_{\mu\lambda\kappa}\widetilde{H}^{\nu\lambda\kappa} = 0 \quad \text{if } \mu \text{ or } \nu = \{0, 4, 5, \ldots, n+3\} ,$$

$$\widetilde{H}_{i\lambda\kappa}\widetilde{H}^{j\lambda\kappa} = \frac{2L^2}{\widetilde{a}^6}\delta_i^j \quad \text{for } i \text{ and } j = \{1, 2, 3\} .$$

Hence in Eq. (2.14) we obtain

$$\kappa_D^2\, {}^{(H)}\widetilde{T}_0^0 \equiv -\kappa_D^2\widetilde{\rho}_H = -\frac{L^2}{4\widetilde{a}^6}e^{-4\phi/(n+2)} , \quad (4.14)$$

$$\kappa_D^2 \, {}^{(H)}\widetilde{T}_i^i \equiv \kappa_D^2 \widetilde{p}_H = \frac{L^2}{4\widetilde{a}^6} e^{-4\phi/(n+2)} \quad \text{(no sum)} , \quad (4.15)$$

$$\kappa_D^2 \, {}^{(H)}\widetilde{T}_J^J \equiv \kappa_D^2 \widetilde{q}_H = -\frac{L^2}{4\widetilde{a}^6} e^{-4\phi/(n+2)} \quad \text{(no sum)} , \quad (4.16)$$

which means that the $\widetilde{H}$ field acts like an anisotropic fluid satisfying

$$\widetilde{\rho}_H = \widetilde{p}_H = -\widetilde{q}_H . \quad (4.17)$$

Note that, just as in the $D = 4$ case, H^2 in the string frame is inversely proportional to the square of the volume of the three-space ($H^2 \propto a^{-6}$).

There is also a contribution to the energy-momentum from the dilaton field given by Eq. (2.13):

$$\kappa_D^2 \, {}^{(\phi)}\widetilde{T}_0^0 \equiv -\kappa_D^2 \widetilde{\rho}_\phi = -\frac{\dot{\phi}^2}{2(n+2)} , \quad (4.18)$$

$$\kappa_D^2 \, {}^{(\phi)}\widetilde{T}_i^i = \kappa_D^2 \, {}^{(\phi)}\widetilde{T}_J^J = \frac{\dot{\phi}^2}{2(n+2)} \quad \text{(no sum)} , \quad (4.19)$$

which means that the ϕ field acts like an isotropic stiff fluid,

$$\widetilde{\rho}_\phi = \widetilde{p}_\phi = \widetilde{q}_\phi , \quad (4.20)$$

and thus, as remarked in the preceding section, drops out of the evolution equations for α and β.

We can substitute the energy-momentum tensors into Eqs. (2.11), (4.2), and (4.3) to obtain the equations of motion

$$\ddot{\phi} + (3\dot{\alpha} + n\dot{\beta})\dot{\phi} = \frac{L^2}{\widetilde{a}^6} e^{-4\phi/(n+2)} ,$$

$$\ddot{\alpha} + (3\dot{\alpha} + n\dot{\beta})\dot{\alpha} = \frac{n}{2(n+2)} \frac{L^2}{\widetilde{a}^6} e^{-4\phi/(n+2)} ,$$

$$\ddot{\beta} + (3\dot{\alpha} + n\dot{\beta})\dot{\beta} = -\frac{1}{(n+2)} \frac{L^2}{\widetilde{a}^6} e^{-4\phi/(n+2)} ,$$

while the constraint equation becomes

$$3\dot{\alpha}^2 + 3\dot{\alpha}\dot{\beta} + \frac{n(n-1)}{2}\dot{\beta}^2 = \frac{\dot{\phi}^2}{2(n+2)} + \frac{L^2}{4\widetilde{a}^6} e^{-4\phi/(n+2)} . \quad (4.21)$$

Notice how the presence of the H field on the right-hand-side of these evolution equations tends to drive ϕ and α in a positive direction, but drives β negative. Thus it produces shear and an anisotropic expansion.

Introducing a new time coordinate ξ through

$$d\xi \equiv \widetilde{a}^{-3}\widetilde{b}^n e^{-4\phi/(n+2)} d\widetilde{t} , \quad (4.22)$$

we obtain first integrals for the equations of motion:

$$\phi' = \frac{L^2 e^{4\phi/(n+2)}}{\widetilde{b}^{2n}} (\xi + \xi_\phi) , \quad (4.23)$$

$$\alpha' = \frac{n}{2(n+2)} \frac{L^2 e^{4\phi/(n+2)}}{\widetilde{b}^{2n}} (\xi + \xi_\alpha) , \quad (4.24)$$

$$\beta' = -\frac{1}{(n+2)} \frac{L^2 e^{4\phi/(n+2)}}{\widetilde{b}^{2n}} (\xi + \xi_\beta) , \quad (4.25)$$

and

$$h = \pm L(\xi + \xi_h) , \quad (4.26)$$

where $\phi' \equiv d\phi/d\xi$, etc., and ξ_ϕ, ξ_α, ξ_β, and ξ_h are constants of integration. In fact at least one of these is redundant as the origin of the variable ξ is clearly arbitrary as ξ is only defined by the differential relation in Eq. (4.22). Henceforth we shall take $\xi_\beta = 0$ so that $\beta' = 0$ at $\xi = 0$. We can solve for β' by differentiating Eq. (4.25) and substituting in for ϕ' from Eq. (4.23) to obtain

$$\beta' \equiv \frac{\widetilde{b}'}{\widetilde{b}} = \frac{\xi}{(n+2)\xi^2 + 4\xi_\phi\xi + C} , \quad (4.27)$$

where C is another integration constant related to the others via the constraint Eq. (4.21).

$$(n+2)C = 2(n+2)\xi_\phi^2 - 3n^2\xi_\alpha^2 . \quad (4.28)$$

There is another important constraint which emerges. From Eqs. (4.25) and (4.27), by demanding that the combination $\widetilde{b}^{2n} e^{-4\phi/(n+2)}$ remains non-negative we obtain

$$(n+2)\xi^2 + 4\xi_\phi\xi + C \leq 0 , \quad (4.29)$$

which means that the allowed range of ξ is bounded by $m_- \leq \xi \leq m_+$, where $m_\pm$ are the roots of the above expression,

$$m_\pm = -\frac{2\xi_\phi}{(n+2)} \pm \frac{\Delta}{n+2} , \quad (4.30)$$

and we have introduced $\Delta \equiv [3n^2\xi_\alpha^2 - 2n\xi_\phi^2]^{1/2}$. Clearly solutions only exist for $3n\xi_\alpha^2 > 2\xi_\phi^2$.

We solve Eq. (4.27) to obtain

$$\widetilde{b}(\xi) = \widetilde{b}_0(\xi - m_-)^{q_-} (m_+ - \xi)^{q_+} , \quad (4.31)$$

where $\widetilde{b}_0$ is a constant and

$$q_\pm = \frac{1}{2(n+2)} \left[1 \mp \frac{2\xi_\phi}{\Delta} \right] . \quad (4.32)$$

We obtain similar solutions for $\widetilde{a}$ and ϕ using Eqs. (4.24), (4.23), and (4.31):

$$\widetilde{a}(\xi) = \widetilde{a}_0(\xi - m_-)^{p_-} (m_+ - \xi)^{p_+} , \quad (4.33)$$

$$e^{\phi(\xi)} = e^{\phi_0} (\xi - m_-)^{s_-} (m_+ - \xi)^{s_+} , \quad (4.34)$$

where $\widetilde{a}_0$ and ϕ_0 are constants (with $\widetilde{b}_0^n e^{-2\phi_0/(n+2)} = L$), $p_\pm$ and $s_\pm$ being given by

$$p_\pm = -\frac{n}{4(n+2)} \left[1 \mp \frac{2\xi_\phi - (n+2)\xi_\alpha}{\Delta} \right] , \quad (4.35)$$

$$s_\pm = -\frac{1}{2} \left[1 \pm \frac{n\xi_\phi}{\Delta} \right] . \quad (4.36)$$

It is clear from Eqs. (4.31), (4.33), and (4.34) that the behavior of $\tilde{a}$, $\tilde{b}$, and e^{ϕ} is similar in each case, the differences emerging in the exponents of the $(\xi - m_{\pm})$. We need to know how these solutions appear in the string frame as this is where the original theory has emerged from. This is straightforward to do. Using Eqs. (2.7) we see that the scale factors in the string frame are given by

$$a(\xi) = a_0(\xi - m_-)^{u_-}(m_+ - \xi)^{u_+} , \qquad (4.37)$$

$$b(\xi) = b_0(\xi - m_-)^{v_-}(m_+ - \xi)^{v_+} , \qquad (4.38)$$

where the constants $u_{\pm}$ and $v_{\pm}$ are

$$u_{\pm} = -\frac{1}{4}\left[1 \pm \frac{n\xi_{\alpha}}{\Delta}\right] ,$$

$$v_{\pm} = \mp\frac{\xi_{\phi}}{2\Delta} .$$

Thus the qualitative behavior of the solutions is quite simple, approaching power laws as $\xi \to m_{\pm}$ with the exponents $u_{\pm}$ and $v_{\pm}$.

Using Eqs. (2.7) and (4.22) we see that the time (dt) in the string frame is related to $(d\xi)$ by

$$dt = a^3 b^{-n} e^{\phi} d\xi , \qquad (4.39)$$

$$\propto (\xi - m_-)^{-1+w_-}(m_+ - \xi)^{-1+w_+} d\xi , \qquad (4.40)$$

where

$$w_{\pm} = 1 + 3u_{\pm} - nv_{\pm} + s_{\pm} ,$$

$$= -\frac{1}{4}\left[1 \pm \frac{3n\xi_{\alpha}}{4\Delta}\right] . \qquad (4.41)$$

Although we cannot in general integrate this relation to obtain $t(\xi)$ in closed form, we can solve for the time in the string frame in the limits $\xi \to m_{\pm}$ to give

$$t \propto |\xi - m_{\pm}|^{w_{\pm}} . \qquad (4.42)$$

We see that $\xi \to m_{\pm}$ in finite proper time if $w_{\pm} > 0$, which means that $\xi \to m_+$ in finite proper time if $\xi_{\alpha} < 0$, and $\xi \to m_-$ in a proper time if $\xi_{\alpha} > 0$. Thus the proper time interval is always semi-infinite. (The case $\xi_{\alpha} = 0$ is excluded by the requirement that $\Delta \neq 0$.)

From Eqs. (4.42), (4.37), (4.38), and (4.34) we have the limiting behavior of the solutions as power laws

$$a(t) \propto |t - t_0|^{u_{\pm}/w_{\pm}} , \qquad (4.43)$$

$$b(t) \propto |t - t_0|^{v_{\pm}/w_{\pm}} , \qquad (4.44)$$

$$e^{\phi(t)} \propto |t - t_0|^{s_{\pm}/w_{\pm}} . \qquad (4.45)$$

Thus we need to know the behavior of the ratio $u_{\pm}/w_{\pm}$ in order to understand the late time behavior of the solutions for $a(t)$. Assuming that we are dealing with events at late proper time as $\xi \to m_+$ and $t \to \infty$ (i.e., $\xi_{\alpha} > 0$) then we have

$$\frac{u_+}{w_+} = \frac{n\xi_{\alpha} + \Delta}{3n\xi_{\alpha} + \Delta} , \qquad (4.46)$$

which is bounded by

$$\frac{1}{3} < \frac{u_+}{w_+} \leq \frac{1}{\sqrt{3}} . \qquad (4.47)$$

The upper bound corresponds to the case $\xi_{\phi} = 0$ where $b = $const and we recover the late-time behavior of the isotropic $D = 4$ solutions. This implies that in the string frame we do not obtain inflationary solutions (by which we mean $d^2a/dt^2 > 0$) at late times. This of course does not imply that we do not obtain inflationary solutions at earlier times. On the contrary, all the solutions which initially contract $(da/dt < 0)$ but then expand at late times $(da/dt > 0)$ must undergo a period of accelerated expansion.

Comparison with the isotropic $D = 4$ solutions discussed in Sec. III A is easiest if we use the conformally invariant time coordinate defined by $d\eta = dt/a$, which is related to ξ via

$$\left|\frac{\eta - \eta_0}{\eta_* - \eta_0}\right| = \left(\frac{\xi - m_-}{m_+ - \xi}\right)^{n\xi_{\alpha}/2\Delta} , \qquad (4.48)$$

where $\eta_* - \eta_0 = a_0^2/n\xi_{\alpha}L$. Note that for $\xi_{\alpha} < 0$ the coordinate η runs $-\infty$ to η_0 and for $\xi_{\alpha} > 0$ we must have $\eta_0 \leq \eta < \infty$, i.e., the range for η is always semi-infinite, coinciding with the semi-infinite range for the proper time. To avoid any ambiguity we introduce the non-negative variable τ defined in Eq. (3.14) but here restricted to the spatially flat $(k = 0)$ case

$$\tau = |\eta - \eta_0| , \qquad (4.49)$$

which runs from ∞ to 0 for $\xi_{\alpha} < 0$ and from 0 to ∞ for $\xi_{\alpha} > 0$. We then have

$$a = \frac{a_*}{\sqrt{2}}\left\{\left(\frac{\tau}{\tau_*}\right)^{1-\Delta/n\xi_{\alpha}} + \left(\frac{\tau}{\tau_*}\right)^{1+\Delta/n\xi_{\alpha}}\right\}^{1/2} , \qquad (4.50)$$

$$b = b_*\left(\frac{\tau}{\tau_*}\right)^{\xi_{\phi}/n\xi_{\alpha}} , \qquad (4.51)$$

$$e^{\phi} = \frac{e^{\phi_*}}{2}\left\{\left(\frac{\tau}{\tau_*}\right)^{(n\xi_{\phi}-\Delta)/n\xi_{\alpha}} + \left(\frac{\tau}{\tau_*}\right)^{(n\xi_{\phi}+\Delta)/n\xi_{\alpha}}\right\} . \qquad (4.52)$$

Note that b is a monotonic function of τ. However the qualitative behavior of a depends on the value of $n\xi_{\alpha}/\xi_{\phi}$ and can be separated into two cases.

(1) $\Delta \leq |n\xi_{\alpha}|$ (requiring $|n\xi_{\alpha}| \leq |\xi_{\phi}|$). $a(\tau)$ is monotonically increasing function.

(2) $\Delta > |n\xi_{\alpha}|$ (requiring $|n\xi_{\alpha}| > |\xi_{\phi}|$). $a(\tau)$ contracts initially, bounces and then expands.

The latter case includes the $D = 4$ isotropic case (shown in Fig. 2) where $\xi_{\phi} = 0$ and b remains constant. We then have $\Delta/n\xi_{\alpha} = \sqrt{3}$ and we recover the solutions given in Eqs. (3.16) and (3.17) independent of n, the number of extra dimensions while they remain static. Another example is when we start with an isotropic contraction $(\dot{a}/a = \dot{b}/b < 0)$ at $\tau = 0$ as shown in Fig. 6. The size of the n-space continues to decrease whereas the H field prevents the three-dimensional space from collapsing, and it then expands at late times.
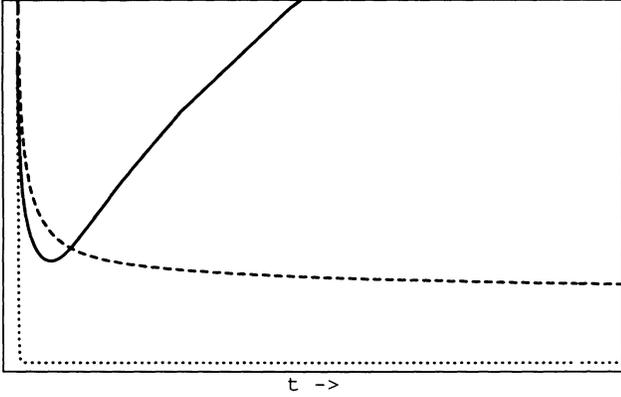
FIG. 6. Scale factors a (solid), b (dashed), and dilaton e^ϕ (dotted) in a $D = 4 + 6$ Bianchi I universe with homogeneous h, starting from an approximately isotropic state at $t = 0$.

The critical case where $\Delta = |n\xi_\alpha| = |\xi_\phi|$ corresponds to the case where the scale factor a remains finite but nonzero at $\tau = 0$. Thus the curvature of the four-dimensional spacetime is nonsingular, although it must be emphasized that both the dilaton and n-space scale factor remain singular. This "wormhole" solution in the $D = 5$ case has been discussed recently by Behrndt and Förste [20].

Although the $(4 + n)$-dimensional dilaton ϕ displays the same range of qualitative behavior as the scale factor a, the behavior of the effective four-dimensional dilation φ defined in Eq. (4.11) is much more restrictive, evolving as

$$e^\varphi = \frac{e^{\phi_*}}{2b_*^n}\left\{\left(\frac{\tau}{\tau_*}\right)^{-\Delta/n\xi_\alpha} + \left(\frac{\tau}{\tau_*}\right)^{\Delta/n\xi_\alpha}\right\} . \quad (4.53)$$

Thus it always has a minimum value at $\tau = \tau_*$. As in the isotropic $D = 4$ case, the presence of the homogeneous antisymmetric tensor field introduces a minimum allowed value for the effective gravitational coupling constant.

B. Radiation solutions, $(\nabla h)^2 = 0$

As in the $D = 4$ isotropic case we can seek solutions corresponding to the short wavelength limit of the Fourier decomposition of $h(\eta, x^i) = h_q(\eta)\exp(iq_i x^i)$, where it is again convenient to write the equation of motion for a Fourier mode h_q using the conformally invariant time coordinate, η, defined such that $d\eta = dt/a = d\tilde{t}/\tilde{a}$. We then have, in the extreme short wavelength limit, $q \to \infty$,

$$\left(\widetilde{ab}^{n/2}e^{2\phi/(n+2)}h_q\right)'' + q^2\left(\widetilde{ab}^{n/2}e^{2\phi/(n+2)}h_q\right) = 0 , \quad (4.54)$$

and thus

$$h(\eta, x^i) = m_q\frac{\widetilde{b}^{n/2}\exp[-2\phi/(n+2)]}{\widetilde{a}}e^{iq_\mu x^\mu} , \quad (4.55)$$

where m_q is a constant and q_μ is a null four-vector restricted to the four-dimensional metric $d\tilde{s}_4^2 = \tilde{a}^2(-d\eta^2 + dx_i^2)$.

The corresponding energy-momentum tensor in the Einstein frame is

$$\kappa_D^2 {}^{(H)}\widetilde{T}_\mu^\nu \propto \frac{q_\mu q^\nu}{\tilde{a}^2\tilde{b}^n} , \quad (4.56)$$

which when averaged over an isotropic distribution (with respect to the three-space) gives the anisotropic [with respect to the whole $(n + 3)$-space] radiation fluid

$$-\kappa_D^2 {}^{(H)}\widetilde{T}_0^0 \equiv \kappa_D^2\widetilde{\rho}_H = \frac{M}{\tilde{a}^4\tilde{b}^n} , \quad (4.57)$$

$$\kappa_D^2\langle {}^{(H)}\widetilde{T}_i^i\rangle \equiv \kappa_D^2\widetilde{p}_H = \frac{1}{3}\frac{M}{\tilde{a}^4\tilde{b}^n} \quad \text{(no sum)} , \quad (4.58)$$

$$\kappa_D^2 {}^{(H)}\widetilde{T}_J^J \equiv \kappa_D^2\widetilde{q}_H = 0 \quad \text{(no sum)} . \quad (4.59)$$

The dilaton field again acts as an isotropic stiff fluid with

$$\widetilde{\rho}_\phi = \widetilde{p}_\phi = \widetilde{q}_\phi = \frac{\dot{\phi}^2}{2(n+2)\kappa_D^2} , \quad (4.60)$$

which does not appear in the Einstein evolution equations.

Substituting this H field solution into the evolution Eqs. (4.9), (4.2), and (4.3) we have

$$\ddot{\phi} + \left(3\dot{\alpha} + n\dot{\beta}\right)\dot{\phi} = 0 , \quad (4.61)$$

$$\ddot{\alpha} + \left(3\dot{\alpha} + n\dot{\beta}\right)\dot{\alpha} = \frac{M}{3\tilde{a}^4\tilde{b}^n} , \quad (4.62)$$

$$\ddot{\beta} + \left(3\dot{\alpha} + n\dot{\beta}\right)\dot{\beta} = 0 , \quad (4.63)$$

while the constraint Eq. (4.4) becomes

$$3\dot{\alpha}^2 + 3n\dot{\alpha}\dot{\beta} + \frac{n(n-1)}{2}\dot{\beta}^2 = \frac{\dot{\phi}^2}{2(n+2)} + \frac{M}{\tilde{a}^4\tilde{b}^n} . \quad (4.64)$$

We can again obtain first integrals of all three evolution equations this time using the conformally invariant time, η, so that

$$\phi' = \frac{1}{3}\frac{M}{\tilde{a}^2\tilde{b}^n}\eta_\phi , \quad (4.65)$$

$$\alpha' = \frac{M}{\tilde{a}^2\tilde{b}^n}(\eta + \eta_\alpha) , \quad (4.66)$$

$$\beta' = \frac{M}{\tilde{a}^2\tilde{b}^n}\eta_\beta , \quad (4.67)$$

where η_ϕ, η_α, and η_β are constants of integration.

Using the same technique as in the homogeneous case, differentiating Eq. (4.66) and substituting in for β' using Eq. (4.67) we obtain a second-order equation for $\alpha(\eta)$ whose first integral gives

$$\alpha' = \frac{\eta + \eta_\alpha}{\eta^2 + (2\eta_\alpha + 3n\eta_\beta)\eta + C} . \quad (4.68)$$

The constant C is given in terms of the other constants from the constraint Eq. (4.64) as

$$C = \eta_\alpha^2 + 3n\eta_\alpha\eta_\beta + \frac{3n(n-1)}{2}\eta_\beta - \frac{3}{2(n+2)}\eta_\phi^2 \ . \quad (4.69)$$

Comparing Eq. (4.68) with Eq. (4.66) we see that

$$\tilde{a}^2\tilde{b}^n = \frac{M}{3}\left[\eta^2 + (2\eta_\alpha + 3n\eta_\beta)\eta + C\right] \ , \quad (4.70)$$

For each choice of integration constants we have two distinct solutions. We have two semi-infinite intervals: one where η approaches m_- from $-\infty$, and the other for $\eta \geq m_+$ approaching $+\infty$. $m_\pm$ are the roots of the above expression:

$$m_\pm = -\eta_\alpha - \frac{3}{2}n\eta_\beta \pm \Delta \ , \quad (4.71)$$

and

$$\Delta = \frac{1}{2}\sqrt{3n(n+2)\eta_\beta^2 + \frac{6}{n+2}\eta_\phi^2} \ . \quad (4.72)$$

Integrating the first-order equation for $\alpha(\eta)$, Eq.(4.68), and similar expressions for β and ϕ finally yields

$$\tilde{a}(\eta) = \tilde{a}_0|\eta - m_-|^{p_-}|\eta - m_+|^{p_+} \ , \quad (4.73)$$

$$\tilde{b}(\eta) = \tilde{b}_0|\eta - m_-|^{q_-}|\eta - m_+|^{q_+} \ , \quad (4.74)$$

$$e^{\phi(\eta)} = e^{\phi_0}|\eta - m_-|^{s_-}|\eta - m_+|^{s_+} \ , \quad (4.75)$$

with the exponents

$$p_\pm = \frac{1}{2}\left(1 \mp \frac{3n\eta_\beta}{2\Delta}\right) \ , \quad (4.76)$$

$$q_\pm = \pm\frac{3\eta_\beta}{2\Delta} \ , \quad (4.77)$$

$$s_\pm = \pm\frac{3\eta_\phi}{2\Delta} \ . \quad (4.78)$$

These are related to the solutions for the two scale factors in the original string frame by the conformal transformation, Eq. (2.7), which gives

$$a(\eta) = a_0|\eta - m_-|^{u_-}|\eta - m_+|^{u_+} \ , \quad (4.79)$$

$$b(\eta) = b_0|\eta - m_-|^{v_-}|\eta - m_+|^{v_+} \ , \quad (4.80)$$

with the exponents

$$u_\pm = \frac{1}{2}\left(1 \mp \frac{3n\eta_\beta - \frac{6}{n+2}\eta_\phi}{2\Delta}\right) \ , \quad (4.81)$$

$$v_\pm = \pm\frac{3\eta_\beta - \frac{3}{n+2}\eta_\phi}{2\Delta} \ . \quad (4.82)$$

As in the homogeneous case the conformal time η cannot in general be given in closed form as a function of the proper time in the string frame, and can be given instead only by the differential relation $dt = a\,d\eta$. However, unlike the homogeneous case, we can identify a unique late/early time behavior as $\eta \to \pm\infty$. We then find $dt \propto \eta^{u_- + u_+}d\eta$ and thus $|t| \propto \eta^2$ giving

$$a \propto |\eta| \propto |t|^{1/2} \ , \quad (4.83)$$

$$b, e^\phi \to \text{const} \ . \quad (4.84)$$

Thus the unique late time attractor for $\eta > m_+$ is the familiar solution for an expanding $D = 4$ radiation dominated universe. For $\eta < m_-$ the asymptotic solution at early times represents the usual solution for a collapsing radiation dominated universe. Both these solutions are singular as $\eta \to m_\pm$ with $a \to 0$ for $u_\pm > 0$ or $a \to \infty$ for $u_\pm < 0$.

To compare these solutions with the isotropic case discussed in Sec. III B we can write them in terms of the non-negative time variable introduced for the spatially flat $D = 4$ metric in Eq. (3.14):

$$\tau = \begin{cases} m_- - \eta & \text{for } \eta < m_- \ , \\ \eta - m_+ & \text{for } \eta > m_+ \ . \end{cases} \quad (4.85)$$

When $\eta \leq m_-$, τ decreases as the solutions approach the singularity at $\tau = 0$, while when $\eta \geq m_+$, τ increases away from the singularity. Then $s(\eta)$, defined in Eq. (3.28), is given by

$$s(\eta) = \frac{\tau(\eta)}{\tau(\eta) + \Delta} \ , \quad (4.86)$$

and we can write

$$a = a_0\tau^{1/2}(\tau + \Delta)^{1/2}s^{(2A-nB)/2} \ , \quad (4.87)$$

$$b = b_0 s^{A+B} \ , \quad (4.88)$$

$$e^\phi = e^{\phi_0}s^{(n+2)A} \ , \quad (4.89)$$

where

$$A = \frac{3\eta_\phi}{2(n+2)\Delta} \ , \quad (4.90)$$

$$B = \frac{3\eta_\beta}{2\Delta} \ , \quad (4.91)$$

so that $(n+2)(2A^2 + nB^2) = 3$. We can now identify the isotropic case with b remaining constant and $A = -B = \pm\sqrt{3}/(n+2)$, which yields the results of Sec. III B.

V. CONCLUSIONS

We have written down the general solutions to the low-energy equations of motion derived from string theory for a four-dimensional Friedmann-Robertson-Walker cosmology including a homogeneous dilaton field and antisymmetric tensor field.

In the absence of the antisymmetric tensor field ($H_{\mu\nu\lambda} = 0$) we recover the known vacuum results for the evolution of the dilaton. In spatially flat models we have two distinct branches; the decelerated branch where both the scale factor a and the dilaton ϕ grow monotonically from zero at the curvature singularity, and the accelerated branch where a and ϕ decrease from infinity at the singularity. These solutions are invoked in the so-called "pre-big-bang" cosmologies based on the accelerated branch expanding from a and ϕ zero with low curvature, and approaching the high curvature regime, where it must evolve into the expanding decelerated branch

with a and $\phi \to \infty$ as the curvature again vanishes. But achieving such a smooth transition in the context of the low energy effective theory has proved difficult even with the inclusion of a dilaton potential [13].

We have shown that the presence of the homogeneous H field introduces a minimum value for both the FRW scale factor and the dilaton in $D = 4$ models. Spatially flat models have $a \to \infty$ both at the spacetime curvature singularity at $\eta = \eta_0$ and in the low curvature limit ($\eta \to \pm\infty$). We no longer find two distinct branches but rather a solution which smoothly interpolates between the accelerated vacuum branch at high curvatures and the decelerated branch at low curvatures. Our analytic solutions confirm some of the results of the phase-plane analysis by Goldwirth and Perry [12]. Thus the extreme weak coupling limit, $\phi \to 0$, is attained only when $H_{\mu\nu\lambda} = 0$. This is in contrast to the results of [11] because our H field does not obey the requirement for $O(d,d)$ symmetry in their solutions in that our $B_{\mu\nu}$ is not homogeneous. Rather, we consider solutions where it is the field $H_{\mu\nu\lambda} \equiv \partial_{[\mu}B_{\nu\lambda]}$ which is homogeneous. These solutions are instead related to the homogeneous vacuum solutions by an $SL(2,R)$ duality transform [21] where the dilaton and H field are considered to be the real and imaginary parts of a single complex field.[2]

We find qualitatively similar solutions when we consider $4 + n$ dimensions. We give analytic solutions for a Bianchi type I model with two scale factors where the antisymmetric tensor field acts as a scalar field on only three of the spatial dimensions. It tends to accelerate their expansion with respect to the other n dimensions producing an anisotropic expansion even from isotropic initial conditions. The effective four-dimensional dilaton always has a minimum value. We recover the isotropic $D = 4$ solutions in the limit that the second scale factor b =const.

By treating the antisymmetric tensor field as a scalar field on the four-dimensional subspace we can consider the short wavelength modes which act like radiation as well as the long wavelength homogeneous modes. We have also given analytic solutions in the case where this radiation is isotropic on three spatial dimensions. The dynamical effect of the H field is then negligible near the curvature singularity, and we see both the accelerated and decelerated branches seen in vacuum. However we find a unique late time attractor solution in spatially flat models corresponding to the usual $D = 4$ radiation dominated solution in general relativity with $a \propto t^{1/2}$ and $\phi \to$const (and $b \to$const in the two scale factor anisotropic model). The dividing line between the short and long wavelength regimes corresponds to wavelengths within or outside the Hubble length. Although this is a frame dependent quantity, for power-law evolu-

tion, $a \propto |\eta - \eta_0|^p$, (which we find in all asymptotic limits) any given wavelength must become "long" as $\eta \to \eta_0$ and conversely any wavelength eventually becomes "short" as $|\eta| \to \infty$.

In models with nonzero spatial curvature (considered only in the isotropic $D = 4$ case) we find that although our solutions approach the flat solutions near the spacetime curvature singularity they become dominated by the spatial curvature as $\eta \to \pm\infty$ where the spacetime curvature becomes small. This is hardly surprising if one considers the evolution of the metric in the Einstein frame, of which we have made extensive use throughout, where these string models correspond simply to a universe with a massless scalar field (plus radiation for the short wavelength H field), and so there can be no inflation in this frame.

It is important to emphasize the limited cosmological era in which the results presented here may be valid. While the appearance of a minimum value for the scale factor of three-dimensional space is intriguing, the string metric is still in general singular at $\eta = \eta_0$. By solving the field equations only to lowest order in the string coupling constant α' we are neglecting terms of order $\alpha' R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa}$ with respect to terms such as R, H^2, $(\nabla\phi)^2$, etc. in the field equations. As all our solutions approach power-law evolution ($a_i \propto |\eta - \eta_0|^{p_i}$) as $\eta \to \eta_0$, then these terms inevitably become divergent, so our solutions will only be good approximations to the true evolution when the spacetime curvature is sufficiently small. On the other hand we have neglected any potential for the dilaton, assuming $V \ll (\nabla\phi)^2$. If we hope to recover the standard hot big bang cosmology at late times then we will have to include matter fields which presumably provide a potential to fix the present day value of the dilaton (and thus satisfy observational limits on the allowed variation of the gravitational coupling strength within the solar system today). Indeed if this potential for the dilaton or for any other fields produces an inflationary era any memory of a preceding stringy era would be all but erased. Neglecting the dilaton potential is only likely to be valid while kinetic terms dominate. As we find $H^2 \propto a^{-6}$ (for homogeneous modes) this will only to be valid at sufficiently small a.

In summary, we have given general analytic solutions for the evolution of an early, but sufficiently low-energy, stringy era where the massless bosonic fields dominate the dynamics. We have shown that the presence of the antisymmetric tensor field has a dramatic effect on the evolution of the dilaton in four dimensions and can also produce an anisotropic expansion in higher-dimensional models preferentially expanding three spatial dimensions.

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