Black hole relics and inflation: Limits on blue perturbation spectra

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Blue primordial power spectra have spectral index $n > 1$ and arise naturally in the recently proposed hybrid inflationary scenario. An observational upper limit on n is derived by normalizing the spectrum at the quadrupole scale and considering the possible overproduction of Planck mass relics formed in the final stage of primordial black hole evaporation. In the inflationary Universe with the maximum reheating temperature compatible with the observed quadrupole anisotropy, the upper limit is $n = 1.4$, but it is slightly weaker for lower reheat temperatures. This limit applies over 57 decades of mass and is therefore insensitive to cosmic variance and any gravitational wave contribution to the quadrupole anisotropy. It is also independent of the dark matter content of the Universe and therefore the bias parameter. In some circumstances, there may be an extended dustlike phase between the end of inflation and reheating. In this case, primordial black holes form more abundantly and the upper limit is $n = 1.3$.

I. INTRODUCTION

It is now generally accepted that large-scale structure in the Universe arose from the growth of small density fluctuations through gravitational instability. A determination of the power spectrum of these primordial fluctuations is therefore of fundamental importance, and the inflationary scenario [1] provides an attractive, causal mechanism for producing such a spectrum (see [2] for a recent review). During inflation the Universe is dominated by the self-interaction potential $V(\phi)$ of a quantum scalar field ϕ, and this results in a superluminal expansion of the scale factor. Quantum fluctuations in this field are therefore stretched beyond the Hubble radius H^{-1}, where they remain frozen until reentry during the radiation- or matter-dominated eras.

In general, when scalar density fluctuations reenter the Hubble radius, their amplitude is given by

$$\delta \approx \frac{1}{M_{\text{Pl}}^3} \frac{V^{3/2}}{|V'|} , \qquad (1.1)$$

where the quantities on the right-hand side are evaluated when the scale first crossed the Hubble radius during inflation, a prime denotes differentiation with respect to ϕ and M_{Pl} is the Planck mass [3]. The form of these

fluctuations can be parametrized by the spectral index n, which specifies the dependence of the power spectrum on the comoving wave number k (viz., $|\delta_k|^2 \propto k^n$). This means that the density perturbations have a rms amplitude of the form $\delta(M) \propto M^{(1-n)/6}$ when the mass scale M reenters the horizon after inflation.

In the simplest scenarios these horizon-scale fluctuations are expected to be almost scale invariant with $n \approx 1$ [3]. However, observations of large scale structure suggest that scale-invariant fluctuations may not work, at least for the cold dark matter (CDM) model [2,4], and it may be necessary to consider "tilted" spectra with $n \neq 1$ [5]. "Blue" primordial power spectra have $n > 1$ and are currently consistent with the recent anisotropy measurements of the cosmic microwave background (CMB) radiation [6–13]. A best fit of the theoretical and observed autocorrelation functions with the Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) first-year maps implies $n = 1.15^{+0.45}_{-0.65}$ [6]. Smoot et al. [7] and Torres [8] have independently performed a topological analysis of these data and deduce $n = 1.7^{+0.3}_{-0.6}$ and $n = 1.2^{+0.3}_{-0.3}$, respectively. On the other hand, Bond [9] infers $n = 1.8^{+0.6}_{-0.8}$ from FIRS data at 3.8° [10] and $n = 2.0^{+0.4}_{-0.4}$ from the first-year COBE DMR data. The second-year COBE DMR data has now been analyzed by Wright et al. [11] who find $n = 1.46^{+0.41}_{-0.44}$, while a maximum-likelihood analysis of this data by Bennett et al. [12] implies that $n = 1.59^{+0.49}_{-0.55}$. The 53 and 90 GHz second-year data has also been analyzed by Górski et al. [13], who find a maximum likelihood value of $n = 1.22$ (1.02) if the quadrupole is included (excluded). They further obtain a marginal probability distribution with a mean of $n = 1.10 \pm 0.32$

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(0.87 ± 0.36).[1] It is interesting to note that $n > 1$ is deduced by Piran et al. from large-scale structure considerations [14]. If the voids detected in the CfA survey [15] on scales $50h^{-1}$ Mpc arise from an underdensity in the matter distribution and form gravitationally in an $\Omega = 1$ universe, a spectral index of $n \approx 1.25$ is consistent with the COBE DMR detection.

The above results implicitly assume that the spectral index is constant over the scales of interest, but in general n is a function of scale and is determined by the magnitude of the potential and its first and second derivatives [16]. However, scales relevant for large-scale structure lie in the range 1–10^4 Mpc and correspond to only 9 e-foldings of inflationary expansion. Since the scalar field must roll down its potential very slowly during inflation, only a very narrow region of V is relevant for these scales. It is therefore consistent to expand the potential as a Taylor series about a given scale, and this is equivalent to assuming that the spectral index is constant over a sufficiently small range [17,18].

The full class of general potentials leading to spectra with constant spectral index has now been derived [19,20]. The potential leading to $n > 1$ is a combination of trigonometric functions [19] and its Taylor expansion to quadratic order is given by

$$V(\phi) \approx V_0 \left[1 + 2\pi(n-1)\frac{\phi^2}{M_{\rm Pl}^2} \right] , \qquad (1.2)$$

where V_0 is a constant that can be normalized to the COBE DMR quadrupole via Eq. (1.1). It should be emphasized that any potential exhibiting a Taylor expansion of this form over scales corresponding to large-scale structure leads to a blue spectrum. Blue spectra also arise in some variants of the hyperextended scenario [21].

Such potentials form the basis of the hybrid inflationary scenario [22] and can arise in string physics [23]. The COBE DMR detection implies that the last stages of inflation must have occurred at or below the grand unification scale and the superstring is an effective $N = 1$ supergravity theory at these scales. It can be shown that under fairly generic circumstances the one-loop Kähler potential derived from the orbifold compactification of the superstring is given approximately by Eq. (1.2) [23], where the spectral index is determined by fundamental string parameters:

$$n = 1 + \frac{\delta_3^{\rm GS}}{2\pi^2 \langle S + \bar{S} \rangle} . \qquad (1.3)$$

The dilaton field S is the real component of the chiral multiplet, and its vacuum expectation value determines

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[1]It is now recognized, however, that the Doppler peak influences the low multipole harmonics of the temperature anisotropy in such a way that the observed spectral index is tilted slightly to the blue end; i.e., the spectral index as measured by COBE DMR is not the index of the primordial spectrum. For standard CDM the correction is of order 15%.

the string-coupling constant $\langle \mathrm{Re}(S) \rangle \approx g_{\rm GUT}^{-2}$. Results from the CERN e^+e^- collider LEP imply that $g_{\rm GUT}^2 \approx 2\pi/13$ and consequently $\langle \mathrm{Re}(S) \rangle \approx 2$ [24]. The parameter $\delta_3^{\rm GS}$ is a dimensionless coefficient to the one-loop corrections arising in the Green-Schwarz mechanism, and calculations suggest that $\delta_3^{\rm GS} \leq 4\pi^2$ [25]. These values lead to the upper limit $n \leq 1.5$, although the requirement that inflation ends at the correct time leads to a stronger limit $n \leq 1.3$ [23].

We infer that values of $n > 1$ are not excluded at this stage, either observationally or theoretically, and it is therefore important to obtain upper limits on the spectral index. Blue spectra introduce more short-scale power, and this might be problematic for dark matter models of galaxy formation. However, nonlinear effects become more important on larger scales when $n > 1$, and these effects must be adequately accounted for before such spectra can be excluded by large-scale structure arguments. In principle, the extra short-scale power can be significantly reduced by the free streaming of a hot dark matter component and comparison of the mixed dark matter model with observations of large-scale structure above 1 Mpc implies an upper limit of $n \leq 1.35$ [26], with dependence on the current value of the Hubble constant. The limit on the spectral distortion of the CMB from the COBE Far Infrared Absolute Spectrophotometer (FIRAS) experiment leads to a slightly weaker limit of $n < 1.56$ [27].

One of the most interesting constraints on inflationary scenarios, which produce blue spectra, comes from considering the formation and subsequent evaporation of primordial black holes (PBH's) [19]. PBH's are never produced in sufficient numbers to be interesting if $n < 1$, but they could be if $n > 1$. Such limits are interesting because they constrain the power spectrum, and therefore the inflationary potential, over 57 decades of mass, whereas large-scale structure measurements span only 10 decades. This implies that the limits on the spectral index derived from PBH's are extremely insensitive to the precise value of the quadrupole anisotropy and so the problems associated with cosmic variance on these ultralarge scales are evaded. They are also independent of any gravitational wave contribution to the CMB anisotropy, and, when the spectrum is normalized to the COBE detection, they are also independent of the transfer function, the bias parameter, and the value of Hubble's constant. Although such observational limits are essentially independent of observations, they are derived from the assumption that the spectral index is constant over a very large range of scales. In the Appendix it is shown that this is a valid assumption in the hybrid inflationary scenario.

In this work we shall consider the potentially stronger observational constraints, which arise if evaporating PBH's leave stable Planck mass relics. Several people have considered the cosmological consequences of such relics. MacGibbon considered the possibility that they could have around the critical density and thus provide the dark mass required in galactic halos or the cosmological background [28]. She argued that, in the standard noninflationary scenario, this would happen rather naturally if the PBH's formed from scale-invariant fluc-

tuations ($n = 1$) with those of 10^{15} g having the density required to contribute appreciably to 100-MeV cosmic rays. However, the relic density would be much reduced if there were an inflationary period. Barrow, Copeland, and Liddle have studied this problem in more detail, calculating the constraints on the fraction of the Universe going into PBH's in order that their relics do not have *more* than the critical density [29]. They considered the situation in which PBH's form as a result of a first-order phase transition induced by bubble collisions at the end of an extended inflationary period. We go beyond these calculations in several respects. Firstly, we assume that the PBH's form directly from the inflation-induced density fluctuations rather than from bubble collisions. Secondly, we allow for the possibility that the equation of state may go soft in some period between the end of inflation and reheating, as suggested by Khlopov, Malomed, and Zel'dovich [30].

The plan of this paper is as follows. In Sec. II we discuss why PBH's may form after inflation and why stable relics may be left over from the final stages of their evaporation. In Sec. III we derive limits on the fraction of the Universe which goes into PBH's of mass M, and leave relics, both for the standard case in which the Universe is radiation dominated after inflation and for the case in which there is an intermediate dust phase before reheating. In both cases we infer corresponding upper limits on the amplitude $\delta(M)$. In Sec. IV we combine these limits with the observed quadrupole anisotropy to infer upper limits on n. If there were no Planck mass relics, the strongest upper limit would be associated with the photodissociation of primordial deuterium by photons emitted from 10^{10} g PBH's [31] and this would imply $n \leq 1.5$ [19]. If there are Planck mass relics and there is no dust phase, the limit is strengthened to $n < 1.4$ for a reheat temperature of order 10^{16} GeV. If there is an extended dust phase after inflation, the limit on n depends crucially on its duration and could be as strong as $n < 1.3$. In Sec. V we draw some general conclusions.

II. THE FORMATION OF PBH'S AND THEIR RELICS

A. The fraction of the Universe going into PBH's

We are interested in the situation where PBH's form from the density perturbations induced by quantum vacuum fluctuations. We assume spherically symmetric, Gaussian fluctuations with a rms amplitude $\delta(M)$ and a background equation of state $p = \gamma\rho$ with $0 < \gamma < 1$. When an overdense region stops expanding, it must have a size of at least $\sqrt{\gamma}$ times the horizon size in order to collapse against the pressure and this requires that $\delta(M)$ exceed γ [32]. Thus the probability of a region of mass M forming a PBH is

$$\beta_0(M) \approx \delta(M) \exp\left(-\frac{\gamma^2}{2\delta^2(M)}\right) , \qquad (2.1)$$

and this gives the fraction of the Universe expected to go into PBH's on that scale. The mass of a PBH forming at

time t must be at least $\gamma^{3/2}$ times the horizon mass, so

$$M = \eta\frac{t}{t_{\text{Pl}}}M_{\text{Pl}} , \qquad (2.2)$$

where $\eta \approx \gamma^{3/2}$. The value of γ will usually be $\frac{1}{3}$ in the early Universe, corresponding to a radiation equation of state. However, one may have $\gamma = 0$ if the Universe ever passes through a dustlike phase and, in this case, both Eqs. (2.1) and (2.2) are inapplicable [33]. During a dust era, the fraction of the Universe going into PBH's just depends on the probability that regions will be sufficiently spherically symmetric to collapse within their Schwarzschild radius, and this can be shown to be [33]

$$\beta(M) \approx 2 \times 10^{-2}\delta(M)^{13/2} . \qquad (2.3)$$

If this applies for some period $t_1 < t < t_2$, then PBH's are expected to form in the mass range $M_1 \leq M \leq M_{\text{max}}$, where M_1 is given in terms of t_1 by Eq. (2.2) and M_{max} is the mass of a configuration that just detaches itself from the universal expansion at t_2. The latter is calculated as follows. If a region has a fluctuation $\delta(M)$ at the time $t_H(M)$ when it reenters the horizon, then it binds at a time $t_B = t_H\delta^{-3/2}$ with a size $R_B = ct_H\delta^{-1}$. The horizon size at t_B is $ct_H\delta^{-3/2}$, so R_B is $\delta^{1/2}$ times this and the maximal mass is given implicitly by [33]

$$M_{\text{max}} = [\delta(M_{\text{max}})]^{3/2}\left(\frac{t_2}{t_{\text{Pl}}}\right)M_{\text{Pl}} . \qquad (2.4)$$

The last two terms just give the horizon mass at t_2. In order to determine M_{max} explicitly, one needs to know the form of $\delta(M)$.

The possibility of a soft equation of state is particularly relevant to the present work because, in some circumstances, this may occur during the reheating phase at the end of inflation [30], and this is precisely the period in which most PBH's are *expected* to form. If inflation ends by means of a second-order phase transition, the scalar field oscillates in the potential minimum from the time $t_1 \equiv H^{-1}$ and then reheating occurs as friction generated by the coupling of the scalar field to other matter fields turns the kinetic energy of motion into background radiation. The reheating is completed on a time scale determined by the decay width Γ of the scalar field. Providing $\Gamma^{-1} \ll t_1$, the reheating time can be taken to be t_1, so we can assume that the equation of state is hard ($\gamma = \frac{1}{3}$) throughout the period after inflation. In this case, most of the PBH's will form at the epoch t_1 and have the mass $M_1 = \eta(t_1/t_{\text{Pl}})M_{\text{Pl}}$ indicated by Eq. (2.2). This is because Eq. (2.1) implies that, for a blue spectrum, $\beta_0(M)$ decreases exponentially for $M > M_1$, whereas PBH's with $M < M_1$ (which form before t_1) are diluted by inflation. Thus we can regard the PBH's as effectively having a δ-function mass spectrum, as in the Barrow *et al.* bubble collision scenario [29], with a mass M_1. For $\Gamma^{-1} \gg t_1$, the scalar field undergoes coherent oscillations and the equation of state may be soft ($\gamma = 0$) from t_1 until reheating is completed at the time $t_2 \equiv \Gamma^{-1}$, when the scalar field decays rapidly into relativistic particles [34]. In this case, the PBH's would have

an extended mass spectrum, going from M_1 to the mass indicated by Eq. (2.4).

The quantum evaporation of PBH's by thermal emission leads to numerous upper limits on the fraction of the Universe going into PBH's at a given time. In the case of a radiation equation of state, upper limits on $\beta_0(M)$ in the range 10^{10} g $\leq M \leq 10^{17}$ g have been summarized in Refs. [19,32,35] and are shown in Fig. 1(a). The constraints on the probability of PBH formation are modified if there is a dust phase after inflation (between t_1 and t_2) because the ratio of PBH density to radiation density no longer increases in this period. The constraints on the fraction $\beta(M)$ of the Universe going into PBH's during the dust phase are related to $\beta_0(M)$ via the equation [33]

$$\beta(M) = \beta_0(M)\eta^{1/2}\left(\frac{t_2}{t_{\rm Pl}}\right)^{1/2}\left(\frac{M}{M_{\rm Pl}}\right)^{-1/2}. \quad (2.5)$$

This allows the limits on PBH formation during the dust phase to be calculated from the constraints, which apply if there is no dust phase. The limits on $\beta(M)$ are indicated in Figs. 2(a), 3(a), and 4(a), corresponding to three different choices of t_2. Comparison of these observational limits with the predicted values of $\beta_0(M)$ and $\beta(M)$, given by Eqs. (2.1) and (2.3), then leads to constraints on the form of $\delta(M)$, as shown in Figs. 1(b), 2(b), 3(b), and 4(b). The constraints on $\beta_0(M)$, $\beta(M)$, and $\delta(M)$ for $M > 10^{17}$ g are associated with nonevaporating PBH's and correspond to the requirement that they have less than the critical density. We also show the limit on $\delta(M)$ associated with the lack of spectral distortions in the CMB implied by the COBE DMR results [36,37]; this limit is now stronger than indicated in Ref. [19], as pointed out independently by Hu, Scott, and Silk [27]. The constraints for $M < 10^{10}$ g are associated with Planck mass relics and are the focus of the rest of the paper.

B. Formation of relics

The usual assumption is that evaporation proceeds until the PBH vanishes completely [38], but there are various arguments against this. For example, the uncertainty principle implies that the uncertainty in the mass M of a Schwarzschild black hole of radius $r_S = 2GM/c^2$ must
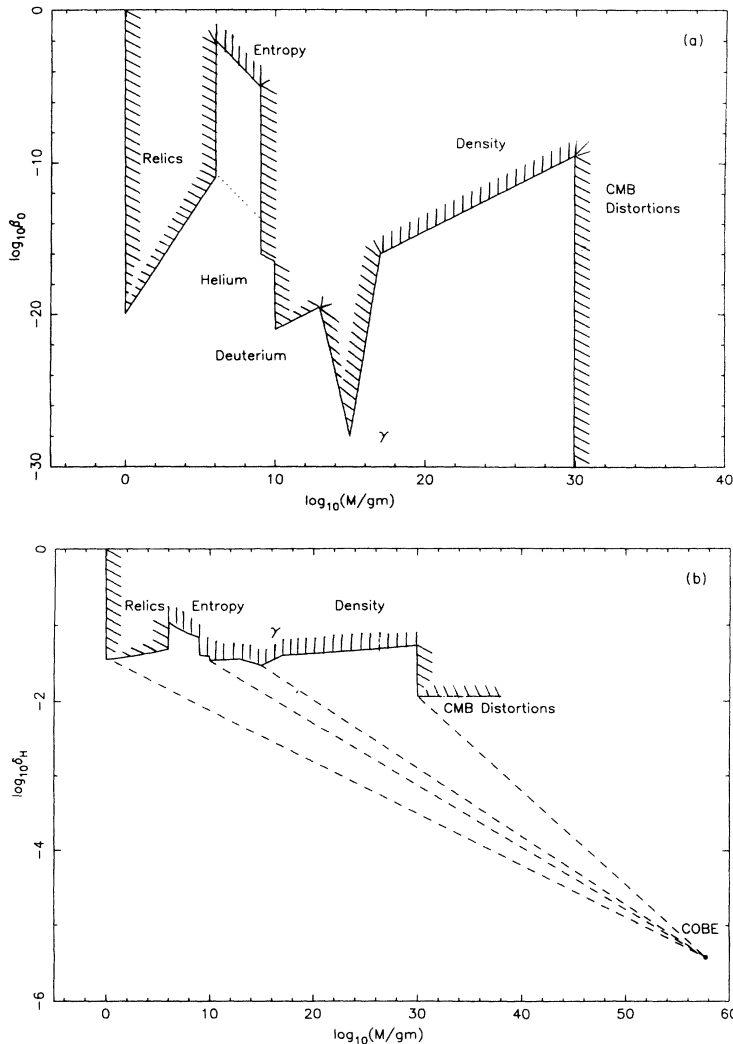

FIG. 1. (a) The constraints on the fraction $\beta_0(M)$ of the early Universe going into PBH's with mass M if the equation of state is radiationlike ($\gamma = \frac{1}{3}$). The origin of the constraints above 10^{10} g is summarized in Ref. [19]. The entropy constraint arises from the requirement that the PBH evaporations do not generate more than the observed photon-to-baryon ratio. There is potentially a stronger but less secure constraint in this mass range derived from the assumption that the observed baryon asymmetry must be generated above the electroweak scale, and this is shown dotted. Below 10^6 g the strongest limit is due to the relics left over from PBH evaporations. (b) The corresponding constraints on the rms amplitude of the density fluctuations. If PBH's do not leave behind stable relics after evaporation, the strongest upper bound on the spectral index is given by the dashed line, which joins the COBE DMR point and the deuterium constraint at 10^{10} g. This limit applies for reheat temperatures above 10^9 GeV. If relics are formed, the limit is strengthened at higher reheat temperatures as indicated.

satisfy $c\Delta M\Delta r_S > \hbar/2$, and a black hole evaporating below the Planck mass $M_{\rm Pl} = (\hbar c/G)^{1/2}$ would violate this [39]. Also the expected energy of the emitted particle exceeds the rest mass of the black hole for $M < M_{\rm Pl}$. For these reasons Zel'dovich proposed that black holes smaller than the Planck mass should be associated with stable elementary particles [40]. Another argument [29] is that quadratic curvature corrections to the gravitational Lagrangian would change the Hawking temperature to the form $T_{\rm BH} = k_1 M^{-1} - k_n M^{-n}$, where k_1 and k_n are constants, and this becomes zero (implying that evaporation stops) once the mass gets down to

$$M_{\rm rel} = \left(\frac{k_n}{k_1}\right)^{1/(n-1)} M_{\rm Pl} . \qquad (2.6)$$

Although we do not know what form quantum gravity corrections should take as one approaches the Planck mass, it would be surprising if there were none at all. The formation of relics is also related to the paradox of information loss [38,41]. The evaporation of a black hole involves an initially pure quantum state evolving into a mixed one and the basic principle of unitarity is thereby

violated. To avoid such a conclusion, one must either suppose that the information is contained in the evaporated particles [42] or that the evaporation terminates when the black hole reaches the Planck mass, the information being stored in a stable or very long-lived relic [43]. The above arguments are too vague to be very convincing, but more specific arguments for stable relics have been given and we now review some of these.

Bowick et al. suggest that stable relics may form because black holes can carry axionic charge [44]. The axionic field does not gravitate, and this means that a black hole may have arbitrarily large charge without forming a naked singularity. On the other hand, it does have a nonzero potential, and this would be relevant for string interactions (cf. the Aharonov-Bohm effect). This suggests that axion charge will become important once a black hole has evaporated down to of order of the Planck mass, since string effects will then apply. Causality and energy conservation presumably limit the amount of axion charge that a black hole can radiate within the age of the Universe, so Bowick et al. infer that it cannot evaporate completely.
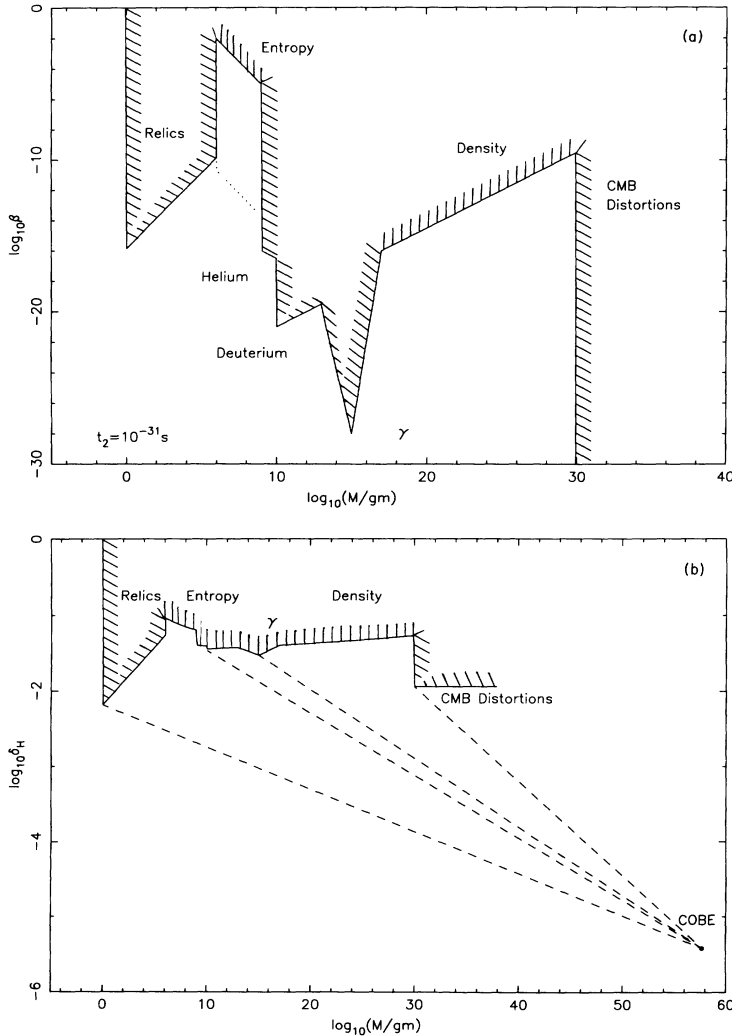
Coleman, Preskill, and Wilczek also argue for a mini-

FIG. 2. (a) The constraints on $\beta(M)$, the fraction of the Universe going into PBH's of mass M during a post-inflationary dust phase. The dust phase is assumed to last until $t_2 \approx 10^{-31}$ s, long enough for 10^6 g configurations to detach themselves from the cosmological expansion before the radiation era begins. (b) The corresponding constraints on $\delta(M)$, with the lines having the same significance as in Fig. 1(b). Although the limit on the probability of PBH formation is reduced during the dust phase, the upper limit on the rms amplitude of the fluctuation at a given scale is increased because the fraction of the Universe going into PBH's is no longer exponentially damped.

mum black-hole mass on the basis that a black hole can have quantum hair, which affects its temperature even though it has no classical effects [45]. They focus on hair associated with a Z_N gauge symmetry. Although a hole with Z_N electric charge has no *classical* hair, quantum effects (associated with virtual strings that wrap around the event horizon) generate a nonvanishing electric field, which decays exponentially with distance from the hole. In fact, this has a negligible effect on the temperature, and much more important are the effects of *magnetic* charge. This possibility arises if one considers $SU(N)/Z_N$ gauge theories, which permit monopoles, since a black hole can then have classical magnetic hair. In this case, the Hawking process stops when the temperature reaches zero, and this occurs at a mass

$$M_{\rm rel} = [n(N-n)/2N]^{1/2}e^{-1}M_{\rm Pl} \approx 10^3 M_{\rm Pl} , \qquad (2.7)$$

where $n = 1, 2, \ldots, N - 1$ is the Z_N charge and e is the coupling constant. These relics can still decay to elementary monopoles if the monopoles are light enough. Indeed the elementary monopoles may themselves be $n = 1$ black holes, in which case the extreme magnetically

charged black holes for other values of n are kinematically forbidden to decay to lighter objects of the same charge [46]. Thus the relics are stable.

Gibbons and Maeda consider scale-invariant theories in which Maxwell fields and antisymmetric tensor fields are coupled to gravity with a dilaton [47]. The solutions in four dimensions are of the Reissner-Nordström type, in which the black-hole temperature goes to zero when the mass equals the charge. For a general dimensionality D, they find that the thermodynamic behavior of an electrically and/or magnetically charged black hole depends on g, the coupling of the dilaton to the Maxwell fields. For $g > \sqrt{D-3}$ the black hole evolves to a naked singularity with infinite temperature as in the standard picture. For $g < \sqrt{D-3}$ a singular zero-temperature endpoint is reached with a mass dependent on the scalar charge and of order $g^{-1}M_{\rm Pl}$. For $g = \sqrt{D-3}$ the hole evolves to a finite temperature final state. It is interesting that string theory predicts $g = 1$ which corresponds to $D = 4$ in this case.

Torii and Maeda consider a theory that couples a scalar dilaton field to a Yang-Mills field [48]. It arises as the four-dimensional effective theory corresponding to higher
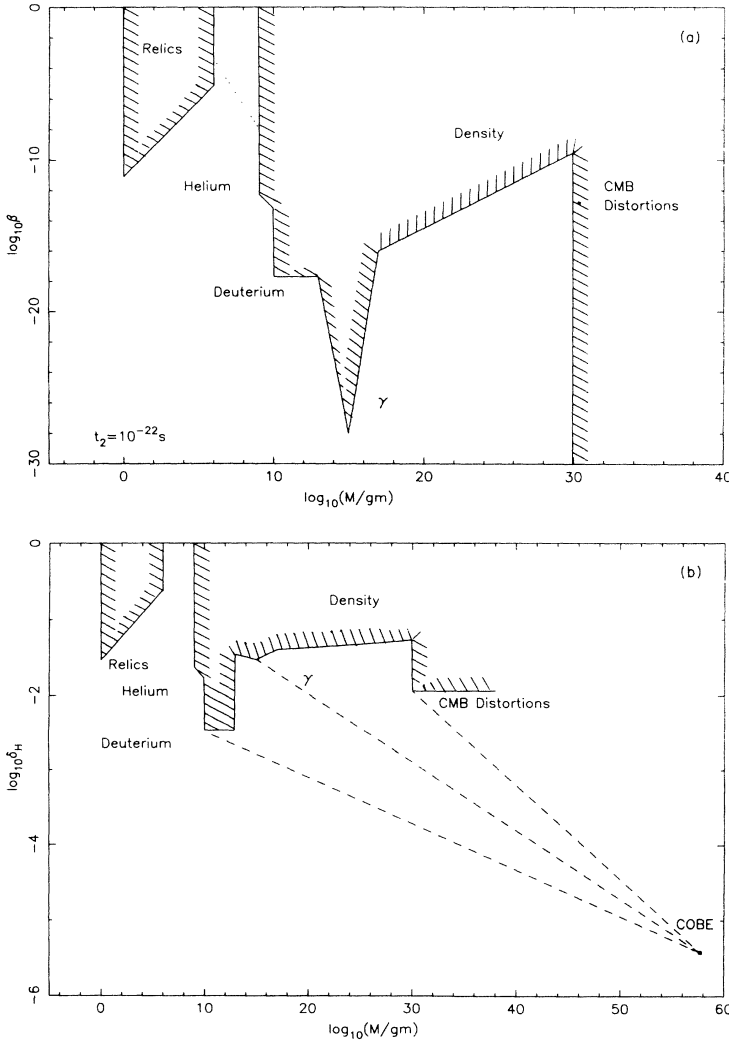

FIG. 3. (a) and (b): The same as Figs. 2(a) and 2(b) with the duration of the dust phase extended until $t_2 \approx 10^{-22}$ s to allow 10^{10} g PBH's to form.

dimensional unified theories. For colored black holes with zero dilaton coupling they find the effective Yang-Mills charge increases as the mass decreases. The temperature also increases at low mass and becomes infinite at a mass of $0.83 M_{\text{Pl}}/g_c$, where g_c is the coupling constant for the Yang-Mills field. At this point the event horizon disappears leading to a stable particlelike solution. Similar behavior occurs for a sufficiently small dilaton coupling. They also consider Skyrme black holes and find a similar effect. The evaporation shrinks the event horizon until it disappears, leaving a non-Abelian particlelike solution. The skyrmion mass is determined by the coupling constants and is of order $f_s g_c$, where f_s relates to the mass of the Yang-Mills field [49].

Callan et al. examine field equations, which arise from low-energy string theory [50]. The action includes terms quadratic in the curvature tensor, in addition to the usual Einstein action, so Eq. (2.6) is applicable. Solving the equations of motion by perturbation expansion, they find that close to the black hole the dilaton field decreases and the string interactions become weaker. The stringy black-hole temperature is unchanged for $D = 4$, but it is lower than the usual Hawking temperature for $D \geq 5$

by a factor proportional to the inverse string tension. As such black holes lose mass, their temperature reaches a maximum before falling to zero at a finite mass.

Myers and Simon study black-hole thermodynamics in second-order theories in higher dimensions [51]. Here the Lagrangian is the sum of k combinations of the Riemann invariants that give second-order field equations. When $D = 2k + 1 = 5$, they find that the temperature can vanish at finite mass. Although the horizon also vanishes at this point, it requires an infinite amount of time to evaporate to this limit and a naked singularity does not form. Whitt also considers second-order gravity theories and obtains similar results [52].

III. OBSERVATIONAL CONSTRAINTS FROM PBH RELICS

In this section we will derive the constraints, which can be placed on the fraction of the Universe going into PBH's of mass M in order to avoid their relics having more than the critical density. We will assume that the relics have a mass κM_{Pl}, where the parameter κ is in the
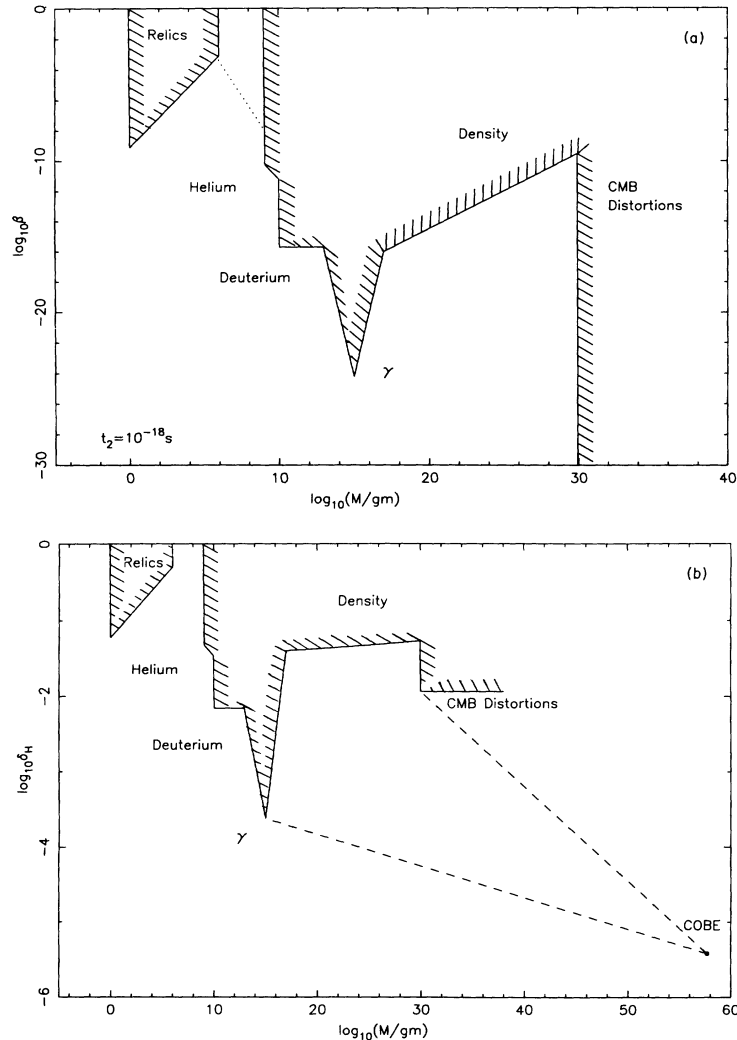

FIG. 4. (a) and (b): The same as for Figs. 2(a) and 2(b) with the duration of the dust phase extended until $t_2 \approx 10^{-18}$ s to allow 10^{15} g PBH's to form.

range 1–10^3 for the scenarios discussed above. We note at the outset that one can place no constraints on PBH's, which form before the end of inflation at t_1 since they will have been diluted away. Thus there are no limits for PBH's with mass below $M < M_1 = \eta(t_1/t_{Pl})M_{Pl}$. Although we do not know the value of t_1 a priori, we can place an upper limit T_{max} on the reheat temperature and thus T_1 from the observed CMB quadrupole anisotropy. Thus there are no constraints on the mass range

$$M < M_{min} = \eta M_{Pl}(T_{max}/T_{Pl})^{-2} . \qquad (3.1)$$

If one assumes that the quadrupole anisotropy is due entirely to tensor (gravitational wave) fluctuations and that the spectral index n does not deviate significantly from unity, the expansion rate of the Universe during inflation cannot exceed $H = 2.9 \times 10^{-5} M_{PL}$ [53,54]. This leads to a maximum reheat temperature of 1.6×10^{16} GeV and a minimum mass $M_{min} \approx 2$ g. A more careful determination of M_{min}, allowing for the contribution of the scalar fluctuations to the quadrupole anisotropy, is given in Sec. IV [cf., Eq. (4.9)].

A. Constraints for a hard equation of state

Let us first consider the $\Gamma^{-1} \ll t_1$ situation, in which the early universe is radiation-dominated immediately after inflation. The form of the relic constraint depends crucially on whether the PBH's dominate the density before they evaporate at [38]

$$t_{evap}(M) = (M/M_{Pl})^3 t_{Pl} . \qquad (3.2)$$

Since the ratio of the PBH density to radiation density increases as $t^{1/2}$, the condition for the radiation to dominate at evaporation is

$$\frac{\beta_0}{1-\beta_0} < \left(\frac{t}{t_{evap}}\right)^{1/2} = \eta^{-1/2}\left(\frac{M}{M}\right)^{-1} , \qquad (3.3)$$

where t is the time at which PBH's of mass M form and is given by Eq. (2.2). This assumes that there are no PBH's in a different mass range; we are essentially taking the PBH's to have a δ-function mass spectrum. In this case, the ratio of the relic density to the critical density at the present epoch is

$$\Omega_{rel} = \left(\frac{\beta_0}{1-\beta_0}\right)\left(\frac{\kappa M_{Pl}}{M}\right)\left(\frac{t_{eq}}{t}\right)^{1/2} , \qquad (3.4)$$

where $t_{eq} = t_0 \Omega_{rad}^{3/2}$ is the time at which the matter and radiation densities are equal and $t_0 = 6.5 h^{-1}$ Gyr is the age of the Universe for $\Omega_0 = 1$. Here $\Omega_{rad} = 2 \times 10^{-5} h^{-2}$ is the current radiation density parameter, and h is the Hubble parameter in units of 100 km s^{-1} Mpc^{-1}. Using Eq. (2.2), the constraint $\Omega_{rel} < 1$ becomes

$$\frac{\beta_0}{1-\beta_0} < \eta^{-1/2}\kappa^{-1}\left(\frac{t_0}{t_{Pl}}\right)^{-1/2}\Omega_{rad}^{-3/4}\left(\frac{M}{M_{Pl}}\right)^{3/2}$$

$$= 10^{-27}\eta^{-1/2}\kappa^{-1}h^2\left(\frac{M}{M_{Pl}}\right)^{3/2} . \qquad (3.5)$$

The PBH's dominate the density at t_{evap} if condition (3.3) is not satisfied. In this case, most of the background photons derive from the PBH's, so Eq. (3.4) is replaced by

$$\Omega_{rel} = \kappa\left(\frac{M_{Pl}}{M}\right)\left(\frac{t_{eq}}{t_{evap}}\right)^{1/2}$$

$$= \kappa\left(\frac{M}{M_{Pl}}\right)^{-5/2}\left(\frac{t_{eq}}{t_{Pl}}\right)^{1/2} \qquad (3.6)$$

and the constraint $\Omega_{rel} < 1$ becomes

$$M > \kappa^{2/5}\left(\frac{t_0}{t_{Pl}}\right)^{1/5}\Omega_{rad}^{3/10}M_{Pl}$$

$$= 10^{11}\kappa^{2/5}h^{-4/5}M_{Pl} . \qquad (3.7)$$

Note that this limiting mass also corresponds to the intersect of conditions (3.3) and (3.5). Equations (3.1), (3.5), and (3.7) define the shaded lines on the left-hand-side of Fig. 1(a). These equations were implicitly derived by Barrow, Copeland, and Liddle [29], although they expressed them in terms of T rather than M. There is also a mistake in their Eq. (5.20), and this leads to an error in their Eq. (5.31), which is equivalent to our Eq. (3.7).

The relics have the critical density only on the boundary specified by Eqs. (3.5) and (3.7). It is interesting that any value of β above 10^{-12} will suffice to provide the critical density if M is fine tuned to have the value given by Eq. (3.7). Recall that most of the PBH's actually form at the end of inflation, i.e., with the mass given by Eq. (3.1), so this corresponds to fine tuning the reheat temperature to $T = 10^{-5}\eta^{1/2}\kappa^{-1/5}T_{Pl} \approx 10^{14}$ GeV. Although one might regard this as unlikely, all the present-day radiation originates from PBH evaporations in this situation, so in this case one might regard Ω_{rad} as a free parameter determined by the reheat temperature.

Two other constraints are shown on the left of Fig. 1(a). The stronger limit (shown dotted) comes from the fact that the observed baryon asymmetry of the Universe could not be generated below the critical temperature $T_{min} \sim 10^3$ GeV associated with the electroweak phase transition. PBH's that form with a mass in excess of

$$M_{Pl}(T_{min}/T_{Pl})^{-2/3} = 10^6 \text{ g} \qquad (3.8)$$

evaporate after this, so, if such holes come to dominate the Universe before evaporation, the evaporated radiation is not sufficiently hot to allow baryogenesis to proceed. This implies that condition (3.3) must apply for $M \geq 10^6$ g, and this corresponds to the dotted line in Fig. 1(a). Coincidentally, the masses given by Eqs. (3.7) and (3.8) are nearly the same, so this would exclude the critical density condition associated with Eq. (3.7). However, this limit is not completely secure, since there may be other mechanisms (including black-hole evaporations themselves [55]) for generating baryon asymmetry. A weaker but more reliable constraint corresponds to the requirement that evaporating PBH's do not generate a photon-to-baryon ratio exceeding the current value $S_0 = 10^9$ [56]. One can show that this just corresponds

to the condition

$$\frac{\beta_0}{1 - \beta_0} < 10^9 \left(\frac{M}{M_{\rm Pl}}\right)^{-1} \quad (M < 10^{11} \text{ g}) , \qquad (3.9)$$

where the upper limit on M arises because the PBH's must evaporate early enough for the photons to be thermalized. This is weaker than condition (3.3) by the factor S_0 and is labeled "entropy" in Fig. 1(a).

In order to convert these constraints on $\beta_0(M)$ into constraints on the horizon-scale density fluctuations $\delta(M)$, we use Eq. (2.1) to obtain

$$\delta(M) < 0.16[\log_{10}\delta(M) - \log_{10}\beta_0(M)]^{-1/2} , \qquad (3.10)$$

where the value of β_0 at a given M is determined by the boundary of the shaded regions in Fig. 1(a). The relic constraint then becomes

$$\delta(M) < 0.16\left[27 - \log_{10}(\kappa\eta^{1/2}) + 1.5 \log_{10}\left(\frac{M}{M_{\rm Pl}}\right) + \log_{10}\delta(M)\right]^{-1/2}$$

$$= 0.13\left[17 - \log_{10}\left(\frac{M}{M_{\rm Pl}}\right)\right]^{-1/2} \quad \text{for } M < 10^{11}\kappa^{2/5}h^{-4/5}M_{\rm Pl} , \qquad (3.11)$$

where we assume $\kappa = \eta = 1$ and take $\log_{10}\delta(M) \approx -1.5$ in the second expression. The relic limit on $\delta(M)$ is shown on the left of Fig. 1(b).

B. Constraints for a soft equation of state

We now consider the case in which the Universe has a dust era between t_1 and t_2. We first assume that this ends before the PBH's evaporate, so that

$$\frac{M}{M_{\rm Pl}} > \left(\frac{t_2}{t_{\rm Pl}}\right)^{1/3} \qquad (3.12)$$

from Eq. (3.2). The condition that the radiation dominates the density when the PBH's evaporate now becomes

$$\frac{\beta}{1 - \beta} < \left(\frac{t_2}{t_{\rm evap}}\right)^{1/2} = \left(\frac{t_2}{t_{\rm Pl}}\right)^{1/2}\left(\frac{M}{M_{\rm Pl}}\right)^{-3/2} \qquad (3.13)$$

with no dependence on the time at which the PBH's form and the condition $\Omega_{\rm rel} < 1$ corresponds to the upper limit

$$\frac{\beta}{1 - \beta} < 10^{-27}\kappa^{-1}h^2\left(\frac{M}{M_{\rm Pl}}\right)\left(\frac{t_2}{t_{\rm Pl}}\right)^{1/2} . \qquad (3.14)$$

Equations (3.13) and (3.14) just come from Eqs. (3.3) and (3.5) together with Eq. (2.5). If the PBH's start to dominate the density before $t_{\rm evap}$, then Eqs. (3.6) and (3.7) still pertain.

Next we assume that the dust era ends after $t_{\rm evap}$, so that Eq. (3.12) is violated and the density of the post-inflationary Universe is never dominated by radiation before t_2. The current relic-to-radiation ratio therefore becomes

$$\Omega_{\rm rel} = \beta\kappa\left(\frac{M_{\rm Pl}}{M}\right)\left(\frac{t_{\rm eq}}{t_2}\right)^{1/2}$$

$$\times \left[(1 - \beta) + \beta\left(\frac{t_{\rm evap}}{t_2}\right)^{2/3}\right]^{-1} , \qquad (3.15)$$

where the two terms in square brackets give the contributions to the radiation density from the reheating and the PBH evaporations. Since the latter can be neglected, the condition $\Omega_{\rm rel} < 1$ is again given by Eq. (3.14).

In order to determine the general constraints on $\beta(M)$ and $\delta(M)$, one needs to specify the epoch t_2 at which the dust phase ends. Equation (2.4) then determines the mass range of PBH's forming during the dust era. For $M_{\rm min} \leq M \leq M_{\rm max}$ the constraint on $\beta(M)$ is given by Eq. (2.5), where the value of $\beta_0(M)$ is determined by the shaded boundary in Fig. 1(a). Equation (2.3) then implies the upper limit

$$\delta(M) < 1.8[\beta_0(M)]^{2/13}\left(\frac{t_2}{t_{\rm Pl}}\right)^{1/13}$$

$$\times \left(\frac{M}{M_{\rm Pl}}\right)^{-1/13}\eta^{1/13} \qquad (3.16)$$

and, in particular, the relic limit given by Eq. (3.5) implies

$$\delta(M) < 1.3 \times 10^{-4}\kappa^{-2/13}h^{4/13}\left(\frac{M}{M_{\rm Pl}}\right)^{2/13}\left(\frac{t_2}{t_{\rm Pl}}\right)^{1/13} . \qquad (3.17)$$

The relic limits on $\beta(M)$ and $\delta(M)$, together with the previously known constraints, are shown in Figs. 2, 3, and 4. Figure 2 applies if the dust era ends with the formation of 10^6 g PBH's ($t_2 = 10^{-31}$ s), corresponding to the mass given by Eq. (3.7). Figure 3 applies if it extends until 10^{-22} s, which is just after the 10^{10} g PBH's form. Figure 4 applies if it extends until 10^{-18} s, which is just after the 10^{15} g PBH's form. [Equation (2.4) implies that the condition for this is $t_2 > 10^{-23}\delta(10^{15}$ g$)^{-3/2}$ s; this exceeds the usual time of 10^{-23} s because the PBH's are now smaller than the particle horizon at formation.] Equations (3.1), (3.7), and (3.14) define the shaded lines on the left-hand side of Figs. 2(a), 3(a), and 4(a). Qualitatively, the effect of softening is to bring down the constraints relative to the hard case. For low values of t_2, only the relic constraint is brought down, as indicated in

Fig. 2. As t_2 increases, the deuterium and γ-ray limits are also brought down, as indicated in Figs. 3 and 4.

IV. RESULTS FOR A CONSTANT SPECTRAL INDEX

A. Normalization to the quadrupole

So far the analysis has been completely general and has made no assumptions about the form of the power spectrum of the density fluctuations. In this section we shall consider the special case in which the spectral index n is constant and consider the corresponding upper limit on its value. For a general inflaton potential, n will not be *exactly* constant but we show in the Appendix that this is always a good approximation in the hybrid inflationary scenario. The strongest upper limit on n is derived by normalizing at the COBE DMR quadrupole scale and finding the steepest straight line, which avoids all the shaded areas in Figs. 1(b), 2(b), 3(b), and 4(b). In case (1), the strongest constraint is associated with the relics from the PBH's of mass $M_{\min}$. This means that relics can have the critical density but that PBH's cannot contribute appreciably to 100-MeV cosmic rays. In case (2), the constraint on n associated with the relics is even stronger because only the relic dip is brought down by the dust era. However, in case (3), the deuterium limit gives the strongest constraint on n because the 10^{10} g dip has also been pulled down. In case (4) the γ-ray limit gives the strongest constraint because the 10^{15} g dip has been pulled down, so PBH's can generate the cosmic rays but not a critical density of relics. Note that it is not possible to produce *both* the cosmic rays and a critical relic density, i.e., there is no situation in which a single straight line passed through both the $M_{\min}$ and 10^{15} g points. Either the 10^{15}-g PBH's form during the dust era, in which case the γ-ray limit implies the relics are unimportant, or they form after the dust era, in which case the relic limit implies the γ-ray limits are unimportant.

In order to determine the constraint on n, we need to normalize the density spectrum at the quadrupole scale. To do this, we must first convert the observed temperature fluctuation into a corresponding density fluctuation. Inflation also produces primordial tensor fluctuations (gravity waves) and these can be significant in some cases. If no reionization occurs, the surface of last scattering is located at a redshift $z_{\rm LS} \approx 1100$ for $\Omega_0 = 1$ and the angle subtended by the horizon at that redshift is approximately $\theta \approx z_{\rm LS}^{-1/2} \approx 2°$. Once gravity waves reenter the horizon, they redshift as relativistic matter and soon become negligible. However, experiments probing angular scales larger than $2°$ measure superhorizon-sized perturbations and are therefore sensitive to gravitational wave effects.

The CMB anisotropies can be expanded into spherical harmonics with coefficients $a_{lm}(\mathbf{x})$, which are stochastic random variables. The observed multipoles detected from a single point in space are defined in terms of these coefficients: $Q_l^2 = (1/4\pi)\sum_{m=-l}^{l}|a_{lm}|^2$ [57]. A given inflationary model predicts values for the averaged quantities $\Theta_l^2 = (1/4\pi)(2l+1)\langle|a_{lm}(\mathbf{x})|^2\rangle$, where the average is taken over all observer points. If the $a_{lm}(\mathbf{x})$ are uncorrelated, the expected contributions from the scalar ($\Theta_l^2[S]$) and tensor ($\Theta_l^2[T]$) fluctuations add in quadrature, so $\Theta_l^2 = \Theta_l^2[S] + \Theta_l^2[T]$.

We follow the notation of Ref. [18] and define the scalar and tensor contributions to Θ_2^2 as

$$S \equiv \Theta_2^2[S] = \frac{5\langle|a_{2m}[S]|^2\rangle}{4\pi} ,$$

$$T \equiv \Theta_2^2[T] = \frac{5\langle|a_{2m}[T]|^2\rangle}{4\pi} ,$$

(4.1)

respectively. The observable quantities S and T have been calculated numerically in terms of the corresponding values of the potential and its derivatives [18]. The relevant expressions for the present work are

$$\frac{V_C}{M_{\rm Pl}^4} \approx 1.65\left(1 + 0.20\frac{T}{S}\right)T ,$$

(4.2)

$$\frac{|V_C'|}{M_{\rm Pl}^3} \approx 3.14\frac{T^{3/2}}{S^{1/2}} ,$$

(4.3)

where a subscript C refers to the quadrupole scale. Substituting these expressions into Eq. (1.1) yields the magnitude of the density fluctuation at the quadrupole scale:

$$\delta_C \approx 0.67S^{1/2}\left(1 + 0.3\frac{T}{S}\right) .$$

(4.4)

B. Constraints on the spectral index for no dust phase

If the Universe has the critical density, the comoving rest mass within a sphere of radius $\lambda/2$ is $M = 1.5 \times 10^{11}M_\odot h^2(\lambda/{\rm Mpc})^3$. The quadrupole corresponds to a scale $\lambda_C \approx 6000h^{-1}$ Mpc and has an associated mass $M_C = 10^{57}h^{-1}$ g. For a power-law spectrum we require $\delta_C(M/M_C)^{(1-n)/6}$ be less than the value of $\delta(M)$ indicated in Figs. 1(b)–4(b). Taking logarithms and substituting Eq. (4.4) then yields an upper limit of n in terms of the S and T:

$$n - 1 \leq 6\left[\log_{10}\left(\frac{10^{57}h^{-1}\ {\rm g}}{M}\right)\right]^{-1}\left[0.17 + \log_{10}\left(\frac{\delta(M)}{\sqrt{S}}\right) - \log_{10}\left(1 + 0.3\frac{T}{S}\right)\right] .$$

(4.5)

The last logarithmic term is a correction accounting for the gravitational wave contribution to the observed quadrupole anisotropy. Since $\log_{10}(\delta/S^{1/2}) \geq 4$, we conclude that this correction term is negligible. Furthermore, although the COBE DMR experiment measures the quantity $S + T$, it is easily seen that limit (4.5) is not altered significantly if it is assumed that the entire COBE DMR signal is due to the scalars. This shows that a separate determination of the gravitational wave contribution is not required in order to derive a limit on n.

A second feature of this limit is its relative insensitivity to cosmic variance. Cosmic variance arises because the fluctuations predicted by theory are stochastic in nature and have a Gaussian probability distribution. A set of observations can only measure a finite number of realizations of this distribution, and this is never sufficient to specify it completely. Thus, there always exists an intrinsic uncertainty in the analysis. Although the effect is most significant at the quadrupole scale and can be problematic when normalizing large-scale structure observations, it should not significantly affect the PBH limit, since this spans such a large range of scales.

It only remains to substitute the numerical value of $S^{1/2}$ into Eq. (4.5). A maximum likelihood estimation made by Seljak and Bertschinger [58] with the first-year COBE DMR data implies a central value of

$$S^{1/2} = 5.8 \times 10^{-6} e^{0.46(1-n)} \qquad (4.6)$$

if the blackbody temperature of the CMB radiation is $T_{rad} = 2.736$ K. Smoot et al. [7] and Bennett et al. [12] arrive at similar expressions, although we emphasize that the precise form of the relationship between S and n is not important for the PBH constraints. Hence, substituting Eq. (4.6) into Eq. (4.5) implies

$$n - 1 \leq \frac{6}{56 - \log_{10}(M/g)}[5.4 + \log_{10}\delta(M)]$$

$$(M \geq M_{min}) . \quad (4.7)$$

An expression for M_{min} may be derived in terms of quantities that are, in principle, observable. During reheating the potential energy of the scalar-field driving inflation is converted into relativistic particles with an energy density $\rho_{rad} = \pi^2 \epsilon T_1^4/30$, where ϵ is the number of relativistic degrees of freedom. Typically $\epsilon \sim 10^2$ and $\epsilon \approx 160$ in the simplest SU(5) grand unified theory. An observable upper limit on the reheat temperature may be calculated by assuming that the available energy is given by V_C. For an efficient reheating process this gives

$$T_1 \leq \left(\frac{30}{\pi^2 \epsilon} V_C\right)^{1/4} = \left[\frac{5}{\epsilon}\left(1 + 0.2\frac{T}{S}\right)T\right]^{1/4} T_{Pl} , \qquad (4.8)$$

where the second expression uses the lowest-order terms in Eq. (4.2). The smallest black holes formed after inflation must therefore have a mass exceeding

$$M_{min} = 1.7 \times 10^{-6}\left(\frac{\epsilon}{T}\right)^{1/2}\left(1 - 0.1\frac{T}{S}\right) g . \qquad (4.9)$$

In principle, one requires a knowledge of the tensor contribution to determine M_{min}, but such information is not currently available.

In the efficient reheating case, the strongest upper bound on the spectral index n depends on the value of M_{min} or equivalently t_1. For $M_{min} < 10^6$ g, the strongest limit is associated with the relics and the relevant mass is M_{min} itself. Substituting Eq. (3.11) into Eq. (4.7) then gives

$$n - 1 \leq \frac{6}{56 - \log_{10}(M_{min}/g)}$$
$$\times \left\{4.5 - \frac{1}{2}\log_{10}\left[12 - \log_{10}\left(\frac{M_{min}}{g}\right)\right]\right\} . \quad (4.10)$$

Thus the limit on the spectral index is $n \leq 1.4$ for $M_{min} \approx 1$ g, corresponding to $T_1 \sim 10^{16}$ GeV, and $n \leq 1.5$ for $M_{min} \approx 10^6$ g, corresponding to $T_1 \sim 10^{14}$ GeV. The limits on n for other ranges of M_{min} are shown by the upper shaded curve in Fig. 5. Here we are regarding n as a continuous function of T_1 (or equivalently t_1). The closure density constraint applies for $M_{min} \geq 10^{17}$ g $(t_1/t_{Pl} \geq 10^{22})$, the γ-ray constraint for 10^{13} g $\leq M_{min} \leq 10^{17}$ g $(10^{18} \leq t_1/t_{Pl} \lesssim 10^{22})$ and the deuterium constraint for 10^{10} g $\leq M_{min} \leq 10^{13}$ g $(10^{15} \leq t_1/t_{Pl} \leq 10^{18})$. Since the helium and entropy constraints are associated with higher values of δ_H than the deuterium constraint, the limit on n does not improve until M_{min} gets down to 10^6 g when the relic constraint (4.10) becomes relevant. This is why the part of the upper curve in Fig. 5 between "relics" and "deuterium" is flat, with a discontinuity at 10^6 g. Note that the CMB distortion gives a better limit on n than the closure density limit for $M_{min} > 10^{24}$ g (or $t_1 > 10^{-14}$ s) and this is why the limit flattens off to $n = 1.76$. Hu, Scott, and Silk [27] have calculated this limit somewhat more carefully and find that it flattens off at $n = 1.54$, as indicated by the dotted line in Fig. 5.

C. Constraints for an early dust phase

If there is a dust phase after inflation, the constraints on the index n depend on both the value of t_1 and t_2, with the latter itself being determined by the decay width Γ. We first assume that t_1 is fixed at the value associated with Eq. (4.8) and allow Γ to vary. An upper limit on Γ follows from the fact that the reheat temperature at the end of the dust phase is $T_{RH} \approx (\Gamma t_{Pl})^{1/2} M_{Pl}$. Therefore, if efficient conversion of the vacuum energy to relativistic particles occurs, the assumption that the COBE DMR quadrupole signal is due entirely to gravitational waves implies $\Gamma t_{Pl} \leq 10^{-6}$ as shown by the shaded line on the left of Fig. 5. The more general limit is $\Gamma t_{Pl} < (T_1/T_{Pl})^2$, where T_1 is given by Eq. (4.8). A lower limit on T_{RH} follows from the requirement that baryogenesis must proceed after reheating. If the lowest temperature for which the observed baryon asymmetry may be generated is the electroweak scale, $\sim 10^3$ GeV, we require $\Gamma t_{Pl} \geq 10^{-30}$. As discussed in Sec. III A, there is some uncertainty in the argument, so we show this limit by a broken limit in Fig. 5. For intermediate values of Γ, one can merely ex-
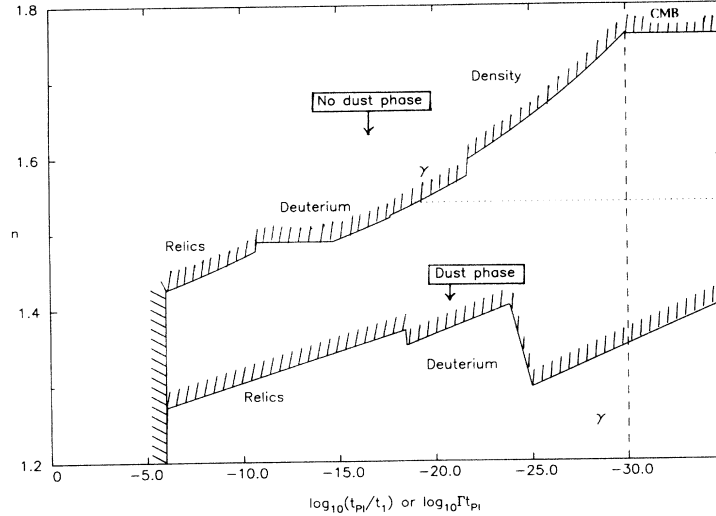
FIG. 5. Illustrating the constraints on the spectral index arising from the overproduction of primordial black holes, the shaded area being excluded. The lower line applies if there is a dust phase immediately after inflation, in which case the ordinate is $\log_{10} \Gamma t_{Pl}$, the upper line if there is no dust phase in which case it is $\log_{10}(t_{Pl}/t_1)$. The constraints depend on the reheat temperature $T_{RH} \approx 10^{18}(\Gamma t_P)^{1/2}$ GeV, where Γ is the decay width of the scalar field that decays into relativistic particles. The n-independent upper and lower limits on the decay width arise from assuming the COBE DMR detection is due entirely to gravitational waves (shaded line) and from requiring that baryogenesis can only proceed above the electroweak scale (dashed line). The dotted horizontal line indicates the CMB distortion limit of Hu, Scott, and Silk [27]. For reheat temperatures above $T_{RH} \approx 10^{9.5}$ GeV the most important constraint arises from the requirement that any Planck mass relics left over from the final stages of PBH evaporation should have less than the critical density at the present epoch. For lower reheat temperatures, more massive PBH's may form and the strongest constraints then arise from the photodissociation of deuterium by evaporating 10^{10}-g PBH's, from the observed γ-ray background in the energy range 0.1–1 GeV, or from the distortions of the CMB. It is clear from this figure that the region $1.3 \leq n \leq 1.4$ has a number of interesting astrophysical consequences if there was an extended dust phase immediately after the inflationary expansion.

clude certain areas in the (Γ, n) plane as we now demonstrate.

If we normalize on the COBE DMR quadrupole scale M_C, the rms amplitude on a smaller scale M is $\delta(M) = \delta_C(M/M_C)^{(1-n)/6}$, where $\delta_C \approx 3.8 \times 10^{-6}$ from Eqs. (4.4) and (4.6). Substituting this expression into Eq. (2.4) implies that the maximum mass of a PBH formed during the dust phase is

$$M_{\max} \approx \left(\frac{\delta_C^{3/2}}{\Gamma t_{Pl}}\right)^{4/(n+3)} \left(\frac{M_C}{M_{Pl}}\right)^{(n-1)/(n+3)} M_{Pl} . \quad (4.11)$$

It follows that PBH's with mass M are formed during the dust phase only if

$$\log_{10} \Gamma t_{Pl} \leq -23.6 + 15.5n$$
$$- \left(\frac{3+n}{4}\right) \log_{10}\left(\frac{M}{M_{Pl}}\right) . \quad (4.12)$$

Another upper limit on Γt_{Pl} follows from Eq. (3.16) with t_2 identified with Γ^{-1}:

$$\log_{10} \Gamma t_{Pl} \leq 208 - 134n + \tfrac{1}{6}(13n - 19)\log_{10}\left(\frac{M}{M_{Pl}}\right)$$
$$+2 \log_{10} \beta_0(M) + \log_{10} \eta , \quad (4.13)$$

where the value of $\beta_0(M)$ is indicated in Fig. 1(a). For a given value of Γ, Eqs. (4.12) and (4.13) can also be interpreted as giving lower and upper limits on n.

The strongest limit on n or Γ is associated with the relic constraint for $M_{\max} < 10^{10}$ g, the deuterium constraint for 10^{10} g$< M_{\max} < 10^{15}$ g and the γ-ray constraint for $M_{\max} > 10^{15}$ g. Equation (4.11) implies that the relic constraint applies for

$$\log_{10} \Gamma t_{Pl} > -35 + 12n , \quad (4.14)$$

and Eqs. (3.5) and (4.13) with $M = M_{\min} \approx 1$ g then imply

$$\log_{10} \Gamma t_{Pl} < 153 - 125n . \quad (4.15)$$

Similarly the deuterium constraint, $\beta_0(10^{10} \text{ g})< 10^{-21}$, applies for

$$-35 + 12n > \log_{10}(\Gamma t_{Pl}) < -38.6 + 10.5n , \quad (4.16)$$

and Eq. (4.13) with $M = 10^{10}$ g gives

$$\log_{10}(\Gamma t_{Pl}) < 118 - 101n . \quad (4.17)$$

The γ-ray constraint, $\beta_0(10^{15} \text{ g})< 10^{-26}$, applies for

$$\log_{10}(\Gamma t_{Pl}) < -38.6 + 10.5n \quad (4.18)$$

and Eq. (4.13) with $M = 10^{15}$ g gives

$$\log_{10}(\Gamma t_{Pl}) < 93 - 91n . \quad (4.19)$$

These constraints on (Γ, n) are shown by the lower

shaded line in Fig. 5. The relic constraint applies for $10^{-8} \geq \Gamma t_{\text{Pl}} \geq 10^{-17}$, the deuterium constraint for $10^{-17} \geq \Gamma t_{\text{Pl}} \geq 10^{-23}$, and the γ-ray limit for $10^{-23} \geq \Gamma t_{\text{Pl}} \geq 10^{-30}$. We therefore arrive at an upper limit of $n = 1.4$ if the PBH's form relics, and there is an extended dust phase immediately before the standard radiation epoch. This limit is independent of the form of the dark matter, the bias parameter, cosmic variance, and any gravitational wave contribution to the COBE DMR signal. It is clear from Fig. 5 that the range $1.3 < n < 1.4$ is astrophysically very interesting. For $T_{\text{RH}} < 10^6$ GeV the origin of the observed γ-ray and cosmic-ray spectra can, in principle, be explained in terms of evaporating 10^{15} g PBH's formed during the early dust phase. This corresponds to the limit (4.19) becoming an equality. For $T_{\text{RH}} \geq 10^{9.5}$ GeV, however, the Planck mass relics of 1-g PBH's may be a natural candidate for the cold dark matter in the Universe. Note that if the time t_1 associated with Eq. (4.8) is allowed to increase, then the limits on (n, Γ) are somewhat weaker than indicated by the lower curve. The lower curve therefore gives the most stringent limit.

V. CONCLUSIONS AND DISCUSSION

In this paper we have derived upper limits on the fraction of the Universe going into PBH's at various mass scales and thereby inferred upper limits on the rms amplitude of the initial density perturbations. In the inflationary scenario the spectrum of density fluctuations arising from quantum fluctuations in the vacuum is uniquely determined by the functional form of the self-interaction potential of the inflaton field, so it is possible to constrain the inflaton potential with the PBH constraints.

One of our main purposes has been to explore the consequences of the suggestion that evaporating PBH's may leave behind stable relics with masses of order the Planck mass. The requirement that such relics have less than the critical energy density leads to constraints that are stronger than those previously derived for masses smaller than 10^6 g. We have also investigated to what extent the PBH constraints are altered if there was an early dust phase in the history of the Universe. Generically inflation ends via a phase transition, which can be either first or second order. In the latter case, the scalar field rolls down to the global minimum of its potential and begins to undergo coherent oscillations. If the decay of the field to relativistic particles is rapid relative to the expansion rate of the Universe, reheating occurs almost instantaneously. However, these oscillations can be prolonged if the coupling constants are sufficiently small and, in this case, the Universe becomes dustlike for some period before the standard radiation epoch is recovered. This is very important for the PBH constraints because this is precisely the period in which the PBH's are expected to form. The constraints now depend crucially on the duration of this dust phase but the upper limits on the rms amplitude are always strengthened.

In order to constrain the spectral index n of the density fluctuations, we have normalized the rms amplitude on the COBE DMR quadrupole scale and applied the

above results to the special case in which the spectral index is constant over all the scales of interest. This example has recently been investigated within the context of the hybrid inflationary scenario. The limit on n depends crucially on the reheat temperature T_{RH}. If there is no early dust phase, the relic constraint provides the strongest upper limit for $T_{\text{RH}} > 10^{14}$ GeV and this corresponds to PBH's with masses smaller than 10^6 g. The limit is $n \approx 1.42$ for $T_{\text{RH}} \approx 10^{16}$ GeV and $n \approx 1.49$ for $T_{\text{RH}} \approx 10^{14}$ GeV. The constraint is tightened if there was an early dust phase, and it is found that the range $1.3 < n < 1.4$ has a number of potentially interesting astrophysical consequences. In particular, relics can provide the critical density for $T_{\text{RH}} > 10^{9.5}$ GeV and 10^{15}-g PBH's can contribute appreciably to the observed γ-ray and cosmic-ray spectra at 100 MeV for $T_{\text{RH}} > 10^6$ GeV [59]. It is interesting that the current best fits to the second-year COBE DMR data appear to favor this range for the spectral index, although it must be emphasized that these fits are also consistent with a flat $(n = 1)$ spectrum. It is also intriguing that the burst and transient source experiment (BATSE) detector on the Compton Gamma Ray Observatory (CGRO) [60] has observed γ-ray bursts that appear to exhibit the characteristics associated with PBH explosions at the present epoch [61]. It remains to be seen, however, whether this range of values for the spectral index can be reconciled with current data from large-scale structure.

A number of assumptions have been made in this work. For simplicity we have assumed that the transition from the coherent oscillation phase to the radiation phase occurs instantaneously, although this is not necessarily the case [34]. Moreover, during the oscillation phase, the scalar field only behaves as a dust fluid on average over one cycle. The formation of PBH's in such a case may be different from that assumed in Ref. [33], and presumably this would affect the observational limits. Moreover, in deriving the constraints on the spectral index, we assumed that n is constant over all relevant scales.

In conclusion, therefore, the above constraints are useful for a number of reasons. Firstly they extend to scales many orders of magnitude smaller than those accessible to large-scale structure observations. This implies that errors in the COBE DMR detection due, for example, to cosmic variance are not important when one normalizes the spectrum on the scales associated with large angle CMB experiments. Likewise, any contribution to the large-scale CMB anisotropy from a primordial spectrum of gravitational waves is generally also negligible. Moreover, both the COBE DMR and the PBH limits are independent of the form of the dark matter in the Universe and the constraints on the spectral index are therefore independent of any transfer functions and biasing parameters. Consequently, although they are somewhat weaker than those derived from current large-scale structure observations alone, they have the advantage that they are relatively insensitive to any specific choice of cosmological model.

APPENDIX

In the case of hybrid inflation, the constancy of the spectral index is a natural result [22,23]. In this scenario one of the scalar fields is initially at zero and the potential takes the asymptotic form

$$V(0, \phi) = V_0 + \tfrac{1}{2}m^2\phi^2 , \qquad (A1)$$

where $V_0 = M^4/4\lambda^2$ and M is a free parameter. Such models assume $\lambda m^2 \ll M^2 g^2$ and $3\lambda m^2 M_{\rm Pl}^2 \ll 2\pi M^4$, where λ and g are coupling constants typically of order unity [22]. The inflationary phase ends when ϕ reaches $\phi_e = M/g$ and triggers the rapid rolling of the second field. The spectral index and number of e folds are given by [16]

$$n = 1 + \frac{M_{\rm Pl}^2}{4\pi}\frac{V''}{V} - \frac{3M_{\rm Pl}^2}{8\pi}\left(\frac{V'}{V}\right)^2 , \qquad (A2)$$

$$N = \frac{8\pi}{M_{\rm Pl}^2}\int_{\phi_e}^{\phi} \frac{V}{V'}d\phi , \qquad (A3)$$

where the quantities on the right-hand side of Eq. (A2) are evaluated when the scale first crosses the Hubble radius during inflation. If one requires $n > 1$, the false vacuum term V_0 in the potential must dominate the $m^2\phi^2$ contribution [19]. This is the case near the end of inflation, so the changes in n and N are given by

$$\Delta n \simeq -\frac{3M_{\rm Pl}^2}{4\pi}\frac{m^4\phi}{V_0^2}\Delta\phi \qquad (A4)$$

and

$$\Delta N \simeq \frac{8\pi}{M_{\rm Pl}^2}\frac{V_0}{m^2\phi}\Delta\phi , \qquad (A5)$$

respectively. Eliminating $\Delta\phi$ and substituting for V_0 with $\phi = \phi_e$ gives

$$\Delta n \simeq -\Delta N \frac{8\pi\lambda^2}{3}\left[\frac{mM_{\rm Pl}}{M^2}\left(\frac{3\lambda}{2\pi}\right)^{1/2}\right]^4 \frac{\lambda m^2}{M^2 g^2} \qquad (A6)$$

and for $\Delta N \simeq 60$ this implies that $\Delta n \ll 1$ over the range of interest. We conclude, therefore, that n is effectively constant over the last 60 e foldings of inflation. This may be understood physically by considering the form of the effective potential (A1) as ϕ rolls towards the minimum. As ϕ decreases, the form of Eq. (A1) approaches the secant potential that leads to an exactly constant spectral index.

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