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Inflation and squeezed quantum states

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Inflationary cosmology is analyzed from the point of view of squeezed quantum states. As noted by Grishchuk and Sidorov, the amplification of quantum fluctuations into macroscopic perturbations which occurs during cosmic inflation is a process of quantum squeezing. We carefully develop the squeezed state formalism and derive the equations that govern the evolution of a Gaussian initial state. We derive the power spectrum of density perturbations for a simple inflationary model and discuss its features. We conclude that the squeezed state formalism provides an interesting framework within which to study the amplification process, but, in disagreement with the claims of Grishchuk and Sidorov, that it does *not* provide us with any new physical results.

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I. INTRODUCTION

One of the impressive features of an inflationary cosmology is the prediction of a set of perturbations on the background Robertson-Walker metric. These perturbations are produced via the amplification of ground state quantum fluctuations during the inflationary period. This process has been widely studied and there is broad agreement regarding both methods and results [1]. The actual perturbations predicted depend on details of the inflationary period. A cosmology with a period of simple exponential inflation and with cold dark matter (CDM) forms the basis of the "standard CDM" model for the formation of galaxies and other structure in the Universe. This model has enjoyed great popularity, but it is also coming under increasing pressure from astronomical observations [2–5].

Recent work by Grishchuk and Sidorov [6, 7] has suggested that important quantum effects have been neglected in the standard approach. These authors claim that, because of quantum squeezing, inflation predicts features in the perturbations which have not been properly taken into account and which could result in striking observational consequences. In particular, they emphasize the phenomenon of desqueezing, which leads to approximate zeros in the power spectrum at calculable wavelengths.

We have systematically investigated the inflationary cosmology from the point of view of quantum squeezing, using Bardeen's gauge invariant variables [8]. We have found that indeed each mode of the perturbed field evolves as a squeezed state during the inflationary period but that the features discussed by Grishchuk and Sidorov in [6] and [7] are well known ones, which are essentially classical in nature. Although we note in Sec. VI an isolated error in the literature which may have prompted much of Grishchuk's criticism, we argue that the error can be (and usually is) avoided without appealing to the formalism of squeezed quantum states. We conclude that this perspective offers nothing more than an alternative set of words (and variables) with which to discuss the inflationary universe. We do however find the squeezed state formalism well suited to the problem [9] and it may prove useful in future work.

The structure of the paper is as follows. In Sec. II we look at a simple mechanical system, the inverted harmonic oscillator, and show how it exhibits squeezing behavior at both the classical and quantum levels. In Secs. III and IV we use the formalism of gauge invariant cosmological perturbations, as presented in [10], to construct the Hamiltonian operator. We then set up the time evolution operator and show that it can be factorized into a product of a squeeze operator and a rotation operator, which are characterized in terms of the squeeze factor $R_{\mathbf{k}}$, squeeze phase $\Phi_{\mathbf{k}}$ and the rotation angle $\Theta_{\mathbf{k}}$. $R_{\mathbf{k}}$ gives us a measure of the excitation of the state while $\Phi_{\mathbf{k}}$ gives us a measure of how the excitation is shared between canonical variables. We show how the evolution of the state can be characterized by a set of coupled first order ordinary differential equations for $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$. In Sec. V we study the behavior of this system of ordinary differential equations (ODE's), identifying different regimes according to the scale of the perturbations: on scales larger than the Hubble radius the squeeze phase freezes out and the squeeze factor grows; on scales smaller than the Hubble radius the squeeze parameters oscillate.

Having gained some insight into what to expect generically in such models we look at a simple inflationary model with baryonic matter coupled to photons (without dark matter) such that the evolution of perturbations can be well approximated by a single collective scalar field. We generate some typical power spectra, $|\delta_k|^2$, and see that they are Harrison-Zeldovich type on superhorizon scales ($|\delta_k|^2 \propto k$—no oscillations) and exhibit standard sound wave oscillations on subhorizon scales.

In Sec. VI we discuss the desqueezing effect emphasized by Grishchuk and Sidorov and argue that it is a familiar one properly taken into account in standard calculations. In Sec. VII we attempt to clarify the claim

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that these effects are of a distinctly quantum mechanical origin. We comment, using the language of squeezed states, on the classicality of the harmonic oscillator; we note that, for large squeezing, the squeezed state satisfies the WKB criterion for classicality. This is equivalent to the WKB classicality at late times in an inverted harmonic oscillator studied by Guth and Pi in Ref. [11]. The point of this section is to explain that the apparently very quantum mechanical squeezed state is in fact classical in the sense with which cosmologists are familiar. That the truly quantum mechanical features of these states which are probed, for example, in quantum optics might have cosmological implications is a fascinating claim but one which has no substance at present. In Sec. VIII we summarize briefly and conclude.

II. THE SINGLE INVERTED HARMONIC OSCILLATOR

The aim of this section is to familiarize the reader with the language of squeezed states. We apply the squeezed state formalism to a simple system—the inverted harmonic oscillator. We will show first how this system exhibits squeezing behavior at the classical level. We show how this behavior is due to the presence of one growing and one decaying solution and that essentially the same behavior carries over to the quantum mechanical system.

A. Classical

The inverted harmonic oscillator (with unit mass and spring constant) is described by the Hamiltonian

$$H = \frac{p^2}{2} - \frac{q^2}{2}. \tag{1}$$

A convenient choice of variables is

$$b_\pm \equiv \frac{1}{\sqrt{2}}(p \pm q). \tag{2}$$

The general solutions are

$$b_+(t) = b_+(0)e^t, \quad b_-(t) = b_-(0)e^{-t}. \tag{3}$$

The evolution of the inverted harmonic oscillator is illustrated in Fig. 1, which shows the trajectories in phase space of a few representative solutions. The phase space can be labeled equally well by p and q or b_+ and b_- (the rotated axes). As time goes on the value of b_+ gets exponentially large, while the value of b_- gets exponentially small. This is because all (but one) of the solutions eventually go "rolling down the hill." As this occurs, p and q each grow exponentially, while their difference exponentially approaches zero.

The trajectories in Fig. 1 describe squeezing in the sense that they get closer together in the b_- direction and further apart in the b_+ direction. For example, the circle in Fig. 1 evolves into the squeezed shape above it after a period of time. Any probability distribution in phase space will eventually become squeezed along the $p = q$ axis as the system evolves.

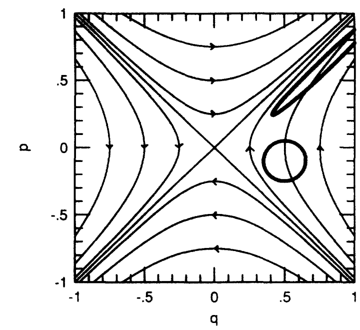

FIG. 1. Phase space trajectories for a classical upside-down harmonic oscillator. The presence of one growing and one decaying solution produces a "squeezing" effect even at the classical level. The circular region shown evolves with time into the squeezed shape above it.

B. Quantum

Now consider the quantum system described by Eq. (1). Using the usual $\hat{a}$ and $\hat{a}^\dagger$ defined for the right-side-up harmonic oscillator we find that

$$H = \frac{\hat{p}^2}{2} - \frac{\hat{q}^2}{2} \tag{4}$$

$$= i\frac{\hbar}{2}(\hat{a}^2 e^{2i\frac{\pi}{4}} - \text{H.c.}). \tag{5}$$

We have written the Hamiltonian in this way because this is the form directly comparable with the more general squeeze Hamiltonian which we will consider.

If the system starts in the vacuum state annihilated by $\hat{a}$ (which is just the Gaussian ground state of the right-side-up oscillator) it evolves into a "squeezed state" given by

$$|\Psi(t)\rangle = \mathcal{S}|0\rangle = e^{\frac{r}{2}(\hat{a}^2 e^{-2i\phi} - \text{H.c.})}|0\rangle. \tag{6}$$

The "squeeze operator" $\mathcal{S}$ is specified by two parameters: r, the "squeeze factor," and ϕ, the "squeeze phase." For a general squeeze operator r and ϕ can be complicated functions of time, but in this simple case they reduce to $r = t$ and $\phi = -\pi/4$.

We now discuss the squeezed state in connection with the Heisenberg uncertainty relationship. Using the relation

$$\mathcal{S}^\dagger a \mathcal{S} = \hat{a}\cosh r - -\hat{a}^\dagger e^{2i\phi}\sinh r, \tag{7}$$

it can be easily shown that

$$\hat{p}|\Psi(t)\rangle = \alpha(r,\phi)\hat{q}|\psi(t)\rangle, \tag{8}$$

where

$$\alpha(r,\phi) = i\frac{\cosh r + e^{2i\phi}\sinh r}{\cosh r - e^{2i\phi}\sinh r}. \tag{9}$$

It then follows that

$$\langle\Psi(t)|\hat{p}^2|\Psi(t)\rangle = |\alpha(r,\phi)|^2\langle\Psi(t)|\hat{q}^2|\Psi(t)\rangle \tag{10}$$

$$= \frac{\hbar}{2}(\cosh 2r + \sinh 2r \cos 2\phi), \tag{11}$$

and the uncertainty relationship is

$$[\langle\Psi(t)|\hat{q}^2|\Psi(t)\rangle\langle\Psi(t)|\hat{p}^2|\Psi t\rangle]^{\frac{1}{2}} = \frac{\hbar}{2}(1+\sin^2 2\phi\sinh^2 2r)^{\frac{1}{2}}. \quad (12)$$

Thus $\Delta q\Delta p \simeq \frac{1}{4}\hbar e^{2t}$ for $t \gg 1$. The initial minimum uncertainty Gaussian state which "sits at the top of the hill" spreads rapidly in q and p.

Consider however

$$\langle\Psi(t)|(\hat{p}\cos\phi-\hat{q}\sin\phi)^2|\Psi(t)\rangle = \frac{\hbar}{2}e^{2r} \quad (13)$$

and

$$\langle\Psi(t)|(\hat{p}\sin\phi+\hat{q}\cos\phi)^2|\Psi(t)\rangle = \frac{\hbar}{2}e^{-2r}. \quad (14)$$

For $\phi = -\pi/4$ these are just $(\Delta b_+)^2$ and $(\Delta b_-)^2$. Thus in the $p-q$ plane we say that the state is squeezed along an axis with slope $\tan\phi$. The fluctuations normal to this axis are exponentially small. This behavior mirrors that of phase space trajectories for the classical system (see Fig. 1) and likewise corresponds to the existence of one decaying and one growing solution.

The state can in fact be represented as a phase space density, using the Wigner function [12], for which the contours are ellipses with one axis of length e^{2r} defined by the angle ϕ and the other axis of length e^{-2r} as in Fig. 1. The squeezed states which we will consider will have a time dependent ϕ so they can be pictured as ellipses rotating in the phase space.

Quantum squeezed states generate considerable interest in various areas of physics, e.g., nonlinear optics [13, 14], gravity waves [15, 16], gravity wave detectors, and quantum cosmology [17]. Their striking feature is that they exhibit dramatically the Heisenberg uncertainty relation, by allowing one variable to have arbitrarily small uncertainty. The conjugate variable has a compensating large uncertainty so the Heisenberg uncertainty relation is obeyed as an equality. In this sense squeezed states are very quantum mechanical. We will discuss the issue of classical versus quantum aspects in more detail in Sec. VII.

III. FORMALISM FOR COSMOLOGICAL PERTURBATIONS

The gauge invariant formalism of cosmological perturbations is well suited to the study of the evolution of vacuum fluctuations. As discussed in [10], the problem is reduced to the analysis of the evolution of a scalar field with a time-dependent mass.

If one looks solely at the scalar degrees of freedom of the metric perturbations,

$$\delta g_{\mu\nu} = a^2(\eta)\begin{pmatrix} 2\phi & -B_{|i} \\ --B_{|i} & 2(\psi\gamma_{ij}-E_{|ij}) \end{pmatrix}, \quad (15)$$

it is possible to combine the functions ϕ, ψ, E, B into two gauge invariant quantities (invariant under local coordinate transformations)

$$\Psi = \psi - \mathcal{H}(B-E'),$$
$$\Phi = \phi + (1/a)[(B-E')a]', \quad (16)$$

where $\mathcal{H} = a'/a$ is the conformal Hubble parameter, a denotes the scale factor, and the prime denotes the derivative with respect to conformal time. We can do the same thing with the matter fields; for example with a scalar field, $\varphi(\mathbf{x},\eta) = \varphi_0(\eta) + \delta\varphi(\mathbf{x},\eta)$, we can build a gauge invariant quantity

$$\delta\varphi^{(gi)} = \delta\varphi + \varphi_0'(B-E'). \quad (17)$$

These gauge invariant quantities can be combined into a single scalar field

$$v = a(\delta\varphi^{(gi)}_{\rm matt} + z\Psi), \quad (18)$$

where $\delta\varphi^{(gi)}_{\rm matt}$ denotes a generic matter field perturbation, z is given by

$$z = (a/c_s\mathcal{H})[\tfrac{2}{3}(\mathcal{H}^2-\mathcal{H}')]^{1/2}, \quad (19)$$

and $c_s = (\delta p_0/\delta\epsilon_0)^{1/2}$ denotes the speed of sound (in inflation the correct equations are obtained by setting $c_s \equiv 1$). The action for the perturbations can then be written as

$$S_{\rm pert} = \frac{1}{2}\int d^4x[(v')^2 - c_s^2(v_{,i})^2 + \frac{z''}{z}v^2], \quad (20)$$

which is the action for a free scalar field v with a time-dependent mass ($m^2 = -z''/z$) [10]. Up to a total derivative term this action is equivalent to the action

$$S'_{\rm pert} = \frac{1}{2}\int d^4x[(v')^2 - c_s^2(v_{;i})^2 - 2\frac{z'}{z}vv' + (\frac{z'}{z})^2v^2], \quad (21)$$

which we will find more convenient to work with. We can now proceed with the standard quantization. Constructing the Hamiltonian we get

$$H = \frac{1}{2}\int d^3x[\pi^2 + c_s^2(v_{,i})^2 + 2\frac{z'}{z}v\pi]. \quad (22)$$

Promoting the fields to operators and taking the Fourier decomposition so that

$$\hat{v} = \int\frac{d^3k}{(2\pi)^{3/2}}\hat{v}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\hat{\pi} = \int\frac{d^3k}{(2\pi)^{3/2}}\hat{\pi}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (23)$$

we get the two-mode Hamiltonian

$$\hat{\mathcal{H}}_{\mathbf{k}} = \hat{\pi}_{-\mathbf{k}}\hat{\pi}_{\mathbf{k}} + c_s^2k^2\hat{v}_{-\mathbf{k}}\hat{v}_{\mathbf{k}} + \frac{z'}{z}(\hat{\pi}_{-\mathbf{k}}\hat{v}_{\mathbf{k}} + \hat{v}_{-\mathbf{k}}\hat{\pi}_{\mathbf{k}}). \quad (24)$$

We want to work in the Schrödinger picture, in which the operators $\hat{v}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ are fixed at an initial time. We define modes with initial frequency equal to k which, suitably normalized, give

$$\hat{v}_{\mathbf{k}} = \frac{1}{\sqrt{2k}}(a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}})$$
$$\hat{\pi}_{\mathbf{k}} = -i\sqrt{\frac{k}{2}}(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}). \quad (25)$$

The two-mode Hamiltonian operator can be written in the simple form

$$\hat{\mathcal{H}}_{\mathbf{k}} = \hat{\mathcal{H}}_{\mathbf{k}}^{(0)} + \hat{\mathcal{H}}_{\mathbf{k}}^{(I)} = \Omega_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + a_{-\mathbf{k}}^{\dagger}a_{-\mathbf{k}} + 1) + i\lambda_{\mathbf{k}}(e^{-2i\varphi_{\mathbf{k}}}a_{\mathbf{k}}a_{-\mathbf{k}} - \mathrm{H.c.})\,, \tag{26}$$

where

$$\Omega_{\mathbf{k}} = \frac{k}{2}(1 + c_s^2),$$
$$\lambda_{\mathbf{k}} = \left[\left(\frac{k}{2}(1 - c_s^2)\right)^2 + \left(\frac{z'}{z}\right)^2\right]^{\frac{1}{2}},$$
$$\varphi_{\mathbf{k}} = -\frac{\pi}{2} + \frac{1}{2}\arctan\left(\frac{kz}{2z'}(1 - c_s^2)\right)\,. \tag{27}$$

Equations (26) and (27) describe the generic momentum-conserving quadratic Hamiltonian for a scalar field. It has a free evolution piece ${\mathcal{H}_{\mathbf{k}}}^{(0)}$ with a time-dependent frequency $\Omega_{\mathbf{k}}$, and a squeezing piece ${\mathcal{H}_{\mathbf{k}}}^{(I)}$ with a coupling strength $\lambda_{\mathbf{k}}(t)$. The evolution operator produced by this Hamiltonian can be factorized in the following way:

$$\mathcal{U}_{\mathcal{H}_{\mathbf{k}}}(\eta, \eta_0) = \mathcal{S}[R_{\mathbf{k}}, \Phi_{\mathbf{k}}]\mathcal{R}[\Theta_{\mathbf{k}}]\,, \tag{28}$$

where $\mathcal{R}$ is the two-mode rotation operator defined as

$$\mathcal{R}[\Theta_{\mathbf{k}}] = \exp[-i\Theta_{\mathbf{k}}(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + a_{-\mathbf{k}}^{\dagger}a_{-\mathbf{k}} + 1)] \tag{29}$$

and $\mathcal{S}$ is the two-mode squeeze operator defined as

$$\mathcal{S}[R_{\mathbf{k}}, \Phi_{\mathbf{k}}] = \exp\left[\frac{R_{\mathbf{k}}}{2}(e^{-2i\Phi_{\mathbf{k}}}a_{-\mathbf{k}}a_{\mathbf{k}} - \mathrm{H.c.})\right]\,. \tag{30}$$

This simple decomposition of the evolution operator is a general property of momentum preserving quadratic Hamiltonians [18]. The rotation operator alone gives ordinary oscillations (points in the phase space of a classical harmonic oscillator rotate about the origin). The squeeze operator alone produces squeezing as discussed in Sec. II. The complete solution to the problem we are considering reduces to finding $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$ as functions of time. (Note that $\Phi_{\mathbf{k}}$ is *not* the Bardeen variable which we shall write Φ^B.)

IV. EVOLUTION EQUATIONS

In this section we address the generation of cosmological perturbations by studying the evolution of the initial vacuum state with the Hamiltonian discussed in the preceding section.

A. The squeezed vacuum state

To begin with we have to define the initial conditions of our quantum field theory. We assume that all the modes of interest (i.e., the modes on subhorizon scales today) are well within the horizon at the initial time. In this case we have $k|\eta| \gg 1$, which (with $c_s = 1$) implies $\Omega_{\mathbf{k}} \simeq k \gg \lambda_{\mathbf{k}} \simeq (1/|\eta|)$, and Eq. (26) reduces to the free Hamiltonian ${\hat{\mathcal{H}}_{\mathbf{k}}}^{(0)}$. We then choose for the initial state the ground state of the free Hamiltonian, i.e., the Poincarè invariant vacuum state, which is defined by

$$a_{\mathbf{k}}|0\rangle_{\mathrm{in}} = 0\,, \qquad \forall \mathbf{k}\,.$$

The action of the rotation operator $\mathcal{R}$ produces an irrelevant phase

$$\mathcal{R}[\Theta_{\mathbf{k}}]|0\rangle_{\mathrm{in}} = e^{i\Theta_{\mathbf{k}}}|0\rangle_{\mathrm{in}}\,,$$

but, when acted upon by $\mathcal{S}[R_{\mathbf{k}}, \Phi_{\mathbf{k}}]$, the vacuum state transforms into a two-mode squeezed state [18]

$$|SS_{\mathbf{k}}\rangle = \mathcal{S}[R_{\mathbf{k}}, \Phi_{\mathbf{k}}]|0\rangle_{\mathrm{in}} = \sum_{n=0}^{\infty} \frac{1}{\cosh R_{\mathbf{k}}}(-e^{2i\Phi_{\mathbf{k}}}\tanh R_{\mathbf{k}})^n|n, \mathbf{k}; n, -\mathbf{k}\rangle\,, \tag{31}$$

where

$$|n, \mathbf{k}; n, -\mathbf{k}\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger})^n|0\rangle_{\mathrm{in}} \tag{32}$$

is the two-mode occupation number state. This part of the evolution operator is responsible for the amplification of the initial vacuum fluctuations; momentum-conserving pairs of quanta are created. The squeeze factor is related to the mean number of quanta, $n_{\mathbf{k}}$, in the squeezed vacuum state through the relation

$$n_{\mathbf{k}} = \langle SS|\hat{N}_{\mathbf{k}}|SS\rangle = \sinh^2 R_{\mathbf{k}}\,.$$

B. Evolution equations

The problem is to determine the functions $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$. The time evolution operator is given by the time-ordered exponential

$$\mathcal{U}(\eta, \eta_0) = \mathcal{T}\exp\left(-i\int_{\eta_o}^{\eta} d\eta' \mathcal{H}_{\mathbf{k}}(\eta')\right) = \mathcal{T}\exp\left[\int_{\eta_o}^{\eta} d\eta' \lambda_{\mathbf{k}}(\eta')(e^{-2i\varphi_{\mathbf{k}} + 2i\int_{\eta'}^{\eta} \Omega_{\mathbf{k}}(\eta'')d\eta''}a_{\mathbf{k}}a_{-\mathbf{k}} - \mathrm{H.c.})\right]\exp\left[-i\int_{\eta_0}^{\eta} d\eta' \Omega_{\mathbf{k}}(\eta')(a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + a_{-\mathbf{k}}^{\dagger}a_{-\mathbf{k}} + 1)\right]\,. \tag{33}$$

We divide the evolution into infinitesimal time intervals[1] ϵ. The composite property of the evolution operator implies

[1]Note that in contrast with the evolution operator which describes parametric amplification in [18], the time-ordering problem is nontrivial.

$$\mathcal{U}(\eta+\epsilon,\eta_o) = \mathcal{U}(\eta+\epsilon,\eta)\mathcal{U}(\eta,\eta_0)\,. \tag{34}$$

We can recast this in terms of the squeeze operator $\mathcal{S}$ and the rotation operator $\mathcal{R}$ in the form

$$\mathcal{S}[R_{\mathbf{k}},\Phi_{\mathbf{k}}]\,\mathcal{R}[\Theta_{\mathbf{k}}] = \mathcal{S}[\delta R_{\mathbf{k}},\delta\phi_{\mathbf{k}}]\,\mathcal{R}[\delta\theta_{\mathbf{k}}]\,\mathcal{S}[R^0_{\mathbf{k}},\phi^0_{\mathbf{k}}]\,\mathcal{R}[\theta^0_{\mathbf{k}}]\,. \tag{35}$$

Taking account of Eq. (33) we infer that for small ϵ: $\delta R_{\mathbf{k}} \simeq \lambda_{\mathbf{k}}(\eta)\epsilon$, $\delta\theta_{\mathbf{k}} \simeq \Omega_{\mathbf{k}}(\eta)\epsilon$, and $\delta\phi_{\mathbf{k}} \simeq \varphi_{\mathbf{k}}$. Using the computation properties of the squeeze and rotation operators, the right-hand side of (35) can be written as

$$\text{right-hand side (RHS)} = \mathcal{S}[\delta R_{\mathbf{k}},\delta\phi_{\mathbf{k}}]\,\mathcal{S}[R^0_{\mathbf{k}},\phi^0_{\mathbf{k}}-\delta\theta_{\mathbf{k}}]\,\mathcal{R}[\theta^0_{\mathbf{k}}+\delta\theta_{\mathbf{k}}]\,. \tag{36}$$

In order to express the product of the two squeeze operators in terms of a single squeeze operator we use the standard composition property, as given in [18]

$$\mathcal{S}[\delta R_{\mathbf{k}},\delta\phi_{\mathbf{k}}]\,\mathcal{S}[R^0_{\mathbf{k}},\phi^0_{\mathbf{k}}-\delta\theta_{\mathbf{k}}] = \mathcal{S}[R_{\mathbf{k}},\Phi_{\mathbf{k}}]\,\mathcal{R}[\bar\theta_{\mathbf{k}}]\,,$$

where

$$\begin{aligned} e^{i\bar\theta_{\mathbf{k}}}\cosh R_{\mathbf{k}} &= \cosh R^0_{\mathbf{k}}\cosh\delta R_{\mathbf{k}} + e^{2i(\phi^0_{\mathbf{k}}-\delta\phi_{\mathbf{k}}-\delta\theta_{\mathbf{k}})}\sinh R^0_{\mathbf{k}}\sinh\delta R_{\mathbf{k}}\\ e^{i(2(\Phi_{\mathbf{k}}-\phi^0_{\mathbf{k}}+\delta\theta_{\mathbf{k}})+\bar\theta_{\mathbf{k}})}\sinh R_{\mathbf{k}} &= \sinh R^0_{\mathbf{k}}\cosh\delta R_{\mathbf{k}} + e^{-2i(\phi^0_{\mathbf{k}}-\delta\phi_{\mathbf{k}}-\delta\theta_{\mathbf{k}})}\sinh\delta R_{\mathbf{k}}\cosh R^0_{\mathbf{k}}\,. \end{aligned} \tag{37}$$

For sufficiently small ϵ we can expand the left-hand side (LHS) in $\delta R_{\mathbf{k}}$ and $\delta\theta_{\mathbf{k}}$ to obtain the recursion relations

$$\begin{aligned} R_{\mathbf{k}}(\eta+\epsilon) &= R_{\mathbf{k}}(\eta) + \lambda_{\mathbf{k}}(\eta)\epsilon\cos 2[\varphi_{\mathbf{k}}(\eta) - -\Phi_{\mathbf{k}}(\eta)],\\ \Phi_{\mathbf{k}}(\eta+\epsilon) &= \Phi_{\mathbf{k}}(\eta) - \Omega_{\mathbf{k}}(\eta)\epsilon + \lambda_{\mathbf{k}}(\eta)\frac{\epsilon}{2}[\tanh R_{\mathbf{k}}(\eta) + \coth R_{\mathbf{k}}(\eta)]\sin 2[\varphi_{\mathbf{k}}(\eta) - -\Phi_{\mathbf{k}}(\eta)],\\ \Theta_{\mathbf{k}}(\eta+\epsilon) &= \Theta_{\mathbf{k}}(\eta) + \Omega_{\mathbf{k}}(\eta)\epsilon - \lambda_{\mathbf{k}}(\eta)\epsilon\tanh R_{\mathbf{k}}(\eta)\sin 2[\varphi_{\mathbf{k}}(\eta) - \Phi_{\mathbf{k}}(\eta)]\,, \end{aligned} \tag{38}$$

where $\Theta_{\mathbf{k}} = \theta^0_{\mathbf{k}} + \delta\theta_{\mathbf{k}} + \bar\theta$. The differential form of these equations is

$$\begin{aligned} R'_{\mathbf{k}} &= \lambda_{\mathbf{k}}\cos 2(\varphi_{\mathbf{k}} - \Phi_{\mathbf{k}}),\\ \Phi'_{\mathbf{k}} &= -\Omega_{\mathbf{k}} + \frac{\lambda_{\mathbf{k}}}{2}(\tanh R_{\mathbf{k}} + \coth R_{\mathbf{k}})\sin 2(\varphi_{\mathbf{k}} - \Phi_{\mathbf{k}}),\\ \Theta'_{\mathbf{k}} &= \Omega_{\mathbf{k}} - \lambda_{\mathbf{k}}\tanh R_{\mathbf{k}}\sin 2(\varphi_{\mathbf{k}} - \Phi_{\mathbf{k}})\,. \end{aligned} \tag{39}$$

These are the equations of motion of our system.

The analogous equations for gravitational waves have been derived in Ref. [19]. These can be obtained from (39) by specifying $\lambda_{\mathbf{k}} = a'/a$, $\Omega_{\mathbf{k}} = k$, and $\phi_{\mathbf{k}} = -\pi/2$, which is obtained by the formal substitution $c_s^2 = 1$ in Eq. (27).

An alternative derivation of these equations is given in Appendix A, where we use the fact that the mode functions in the Heisenberg picture can be expressed in terms of the Schrödinger picture variables $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$. We then show that the Hamilton equations for the mode functions reduce to those in Eq. (39).

V. APPLICATION TO A SIMPLE INFLATIONARY MODEL

Our aim is to study the growth of cosmological perturbations in the squeezed state formalism for a simple inflationary model. This section is mostly concerned with studying the solutions to Eq. (39). In general, when $\lambda_{\mathbf{k}}$, $\Omega_{\mathbf{k}}$, and $\varphi_{\mathbf{k}}$ are some complicated functions of time, it is not possible to solve Eq. (39) analytically. However, before we proceed to a discussion of the numerical solution, we can get some insight into the dynamics of the system using analytical techniques.

A. Analytic approach

We assume that $\lambda_{\mathbf{k}}$, $\Omega_{\mathbf{k}}$, and $\varphi_{\mathbf{k}}$ are slowly varying functions of time, i.e., for $k\eta < 1$ we have $\Delta\Omega_{\mathbf{k}}/\Omega_{\mathbf{k}}$, $\Delta\lambda_{\mathbf{k}}/\lambda_{\mathbf{k}}$, $\Delta\varphi_{\mathbf{k}}/\varphi_{\mathbf{k}} \ll 1$.

In the *strong coupling* or *squeeze-dominated* regime ($\lambda_{\mathbf{k}} > \Omega_{\mathbf{k}}$), the squeeze angle $\Phi_{\mathbf{k}}$ and the rotation angle $\Theta_{\mathbf{k}}$ approach a stable fixed point (freeze-out). The squeeze factor grows monotonically with time, which reflects the fact that in the course of evolution the growing mode becomes more and more dominant over the decaying mode. In the *weak coupling* regime ($\lambda_{\mathbf{k}} < \Omega_{\mathbf{k}}$), the solution is oscillatory, with the squeeze factor remaining essentially constant and the dominant features are the oscillations of the squeezed state, which are revealed physically as the pressure oscillations in the hydrodynamic fluid.

1. *Strong coupling: freeze-out*

For $\lambda_{\mathbf{k}} > \Omega_{\mathbf{k}}$ there is a *fixed point* (in $\Phi_{\mathbf{k}}$ and $\Theta_{\mathbf{k}}$) to the equations of motion

$$\Phi^{*'}_{\mathbf{k}} = \Theta^{*'}_{\mathbf{k}} = 0,\;\; R'_{\mathbf{k}} = \lambda_{\mathbf{k}}\cos 2(\varphi_{\mathbf{k}} - \Phi^*_{\mathbf{k}}) \tag{40}$$

with

$$\begin{aligned} \sin 2(\varphi_{\mathbf{k}} - \Phi^*_{\mathbf{k}}) &= \frac{2\Omega_{\mathbf{k}}}{\lambda_{\mathbf{k}}}(\tanh R_{\mathbf{k}} + \coth R_{\mathbf{k}})^{-1}\\ &\underset{|R_{\mathbf{k}}|\gg 1}{\sim} \operatorname{sgn}(R_{\mathbf{k}})\left(\frac{\Omega_{\mathbf{k}}}{\lambda_{\mathbf{k}}}\right)\,. \end{aligned} \tag{41}$$

Using this condition, we can now integrate Eq. (40) for the squeeze factor to obtain

$$R_{\mathbf{k}} \simeq \int (\lambda_{\mathbf{k}}^2 - \Omega_{\mathbf{k}}^2)^{1/2} d\eta \,, \tag{42}$$

so that $R_{\mathbf{k}}$ grows monotonically. Most of the squeezing occurs in the strong coupling regime.

2. Weak coupling: oscillations

For $\lambda_{\mathbf{k}} \ll \Omega_{\mathbf{k}}$ and taking $\lambda_{\mathbf{k}}$, $\Omega_{\mathbf{k}}$, and $\varphi_{\mathbf{k}}$ constant, we get the solution

$$\tan(\Phi_{\mathbf{k}} - \varphi_{\mathbf{k}}) = \cos\alpha_{\mathbf{k}} \tan[-\Omega_{\mathbf{k}}(\eta - \eta_0) + \alpha_{\mathbf{k}}] - \tan\alpha_{\mathbf{k}}$$

in which $\sin\alpha_{\mathbf{k}} = \lambda_{\mathbf{k}}/\Omega_{\mathbf{k}}$. In the case where $\alpha_{\mathbf{k}} \ll 1$, this solution reduces to the form

$$\begin{aligned} \Phi_{\mathbf{k}} &= \varphi_{\mathbf{k}} - \Omega_{\mathbf{k}}(\eta - \eta_0), \\ R_{\mathbf{k}} &= R_{\mathbf{k}}^0 + \frac{\lambda_{\mathbf{k}}}{2\Omega_{\mathbf{k}}} \sin 2\Omega_{\mathbf{k}}(\eta - -\eta_0) \\ &= R_{\mathbf{k}}^0 + \frac{\lambda_{\mathbf{k}}}{2\Omega_{\mathbf{k}}} \sin 2(\varphi_{\mathbf{k}} - \Phi_{\mathbf{k}}) \,. \end{aligned} \tag{43}$$

We can consider this oscillatory solution as a reasonable approximation for modes well within the horizon in both the inflationary era, when $\lambda_{\mathbf{k}}/\Omega_{\mathbf{k}} \simeq 1/k|\eta| \ll 1$, and the radiation-dominated era, when $\lambda_{\mathbf{k}}/\Omega_{\mathbf{k}} \simeq 1/2$. For these modes $R_{\mathbf{k}}$ is constant on average; i.e., there is no net squeezing and perturbations do not grow.

For modes that cross the horizon during the matter era, where $\Omega_{\mathbf{k}} \simeq \lambda_{\mathbf{k}}$, we cannot apply this simple analysis.

3. An exact solution: the Bunch-Davies vacuum

In the exponentially expanding de Sitter stage, when $\Omega_{\mathbf{k}} = k$, $\lambda_{\mathbf{k}} = 1/|\eta|$, and $\varphi_{\mathbf{k}} = -\pi/2$, there is an exact solution to the equations of motion (39):

$$\begin{aligned} R_{\mathbf{k}} &= \operatorname{arcsinh}\frac{1}{2k\eta}, \\ \Phi_{\mathbf{k}} &= -\frac{\pi}{4} - \arctan\frac{1}{2k\eta}, \\ \Theta_{\mathbf{k}} &= k\eta + \arctan\frac{1}{2k\eta} \,. \end{aligned} \tag{44}$$

This solution corresponds to the Bunch-Davies vacuum [20], which is an attractor. If the initial state (for the modes within the horizon) is not already highly squeezed, one finds that as the modes get driven to superhorizon scales they evolve toward the Bunch-Davies vacuum. In the language of squeezed state parameters this corresponds to the freeze-out of $\Phi_{\mathbf{k}}$ and $\Theta_{\mathbf{k}}$; we see this behavior in the limit $k|\eta| \ll 1$ of Eq. (44).

B. Squeezing in a simple inflationary model

Having established that most of growth occurs on superhorizon scales, we now use a simple model to estimate the amount of squeezing in the perturbation field. We have found that all of the relevant squeezing occurs on superhorizon scales, i.e., when $k|\eta| < 1$. In Fig. 2 this corresponds to the interval $[\eta_{1x}, \eta_{2x}]$, where $\eta_{1x} = -1/k$ and $\eta_{2x} \simeq 1/k$ $(2/k)$ in the radiation (matter) era.

The relevant squeezing occurs for the couplings for which $\lambda_{\mathbf{k}} \gg \Omega_{\mathbf{k}}$ when $\lambda_{\mathbf{k}} \simeq z'/z$. We can then integrate Eq. (42) to obtain

$$R_{\mathbf{k}} = \int_{\eta_{1x}}^{\eta_{2x}} d\ln z \,. \tag{45}$$

During the inflationary era (superscript i), z can be approximated by $z(\eta) \simeq (2/3)^{1/2} a/l_P$ (where $l_P = (8\pi G/3)^{1/2}$ is the Planck length). The amount of squeez-

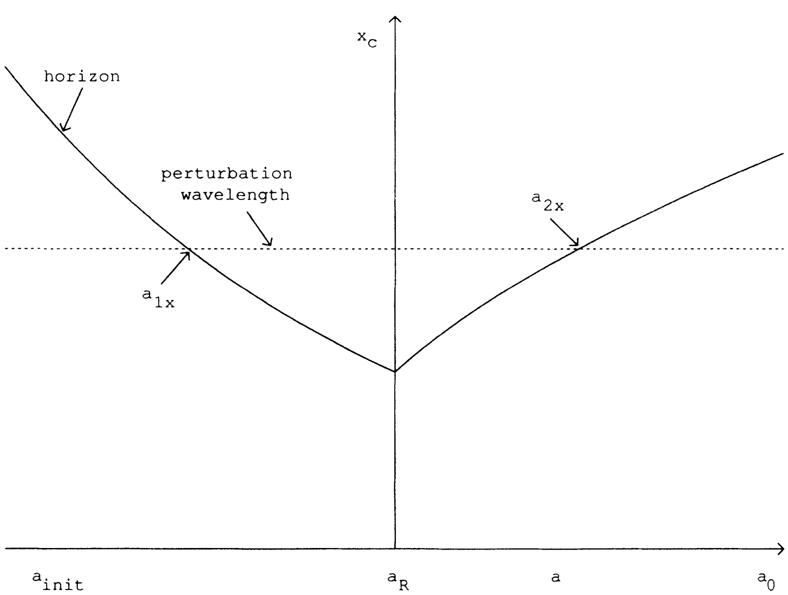

FIG. 2. Evolution of scales in an inflationary universe model. x_c denotes the comoving scale and a_R the end of the inflationary stage. The perturbation is on superhorizon scales in the interval $[a_{1x}, a_{2x}]$.

ing is given by $\Delta R^i_{\mathbf{k}} \simeq \ln(a_R/a_{1x}) \simeq \ln(1/|\eta_R|k)$. In the hydrodynamical era, if a mode crosses the horizon during the radiation era, then $z(\eta) \simeq 2^{1/2}a$ and $\Delta R^{\rm rad}_{\mathbf{k}} \simeq \ln(a_{2x}/a_R) \simeq \ln(1/|\eta_R|k)$. If the mode crosses the horizon in the matter era for which $z(\eta) \simeq a^{3/2}2^{1/2}$ we have $\Delta R^{\rm rad}_{\mathbf{k}} \simeq \ln(\eta_{\rm eq}/\eta_R)$, and $\Delta R^{\rm matt}_{\mathbf{k}} \simeq (3/2)\ln(a_{X2}/a_{\rm eq}) \simeq 3\ln(1/k\eta_{\rm eq})$. This last term is a poor approximation in the matter era; in fact $\lambda_{\mathbf{k}} \simeq \Omega_{\mathbf{k}}$ and the squeezing angle $\Phi_{\mathbf{k}}$ is not completely frozen resulting in a slower growth of $R_{\mathbf{k}}$. In the case of gravitational waves (45) gives $\Delta R_{\mathbf{k}} \simeq \ln(a_{2x}/a_{1x})$, which is in agreement with the result first obtained by Grishchuk and Sidorov in [16].

C. Numerical analysis

We shall now study numerically the evolution of perturbations in our simple model. It is important to point out that during the hydrodynamic era we are looking only at the collective field of baryonic matter and radiation (we are ignoring cold dark matter, or any other field which cannot be accurately described by a single collective scalar field). In addition we ignore decoupling of matter and radiation. We do not expect to get results which agree completely with the highly refined calculations which already exist in the literature [21]. However we do expect approximate agreement if we look solely at the baryonic and radiation sector of these simulations; in particular in the radiation era and on superhorizon scales.

The evolution is given by the recursion relations (38) and we shall assume the following time dependence for the scale factor:

$$a_i = -\frac{1}{H\eta} \quad (-\infty < \eta < -\eta_R) \ \text{ inflationary era},$$

$$a = \frac{1}{4}\left(\frac{\eta+\theta}{\eta_*}\right)^2 + \left(\frac{\eta+\theta}{\eta_*}\right) \quad (\eta_R < \eta < \infty)$$

$$\text{hydrodynamical era}, \qquad (46)$$

where $H = \mathcal{H}/a = a'/a^2$ is the Hubble constant during inflation, θ and η_* are chosen such that $a(\eta_R) = a_i(\eta_R)$ and $a'(\eta_R) = a_i'(\eta_R)$, where conformal time η_R denotes the end of inflation (we assume instantaneous reheat).

We normalize a such that $a_{\rm eq} = 1$ and we set $\eta_* = 1$. As in [10] we assume $z = a\varphi_0'/\mathcal{H}$ in the inflationary era. During the early radiation era, when most of the matter particles are relativistic, we take $c_s^2 = 1/3$. We assume that there is a time, $\eta = \eta_{\rm rel}$, when matter particles become nonrelativistic. For $\eta > \eta_{\rm rel}$ we have $c_s^2 = (\delta p_0/\delta\epsilon_0)_S = \frac{1}{3}(1 + 3a/4)^{-1}$. (We checked that the choice of $\eta_{\rm rel}$ does not influence squeezing of the state.) The wave function is continuous at $\eta = \eta_R$, which means that the functions $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$ are continuous at η_R. The overall amplitude of the perturbations in the hydrodynamical era will be dependent on the amount of squeezing in the inflationary era, which in turn depends on reheat temperature specified by η_R.

1. Evolution of the squeeze parameters

Figure 3 is a plot of the evolution of the squeeze factor $R_{\mathbf{k}}$ as a function of the scale factor a. Most of the growth in the squeeze factor occurs on the superhorizon scales between the marks *1x* and *2x*. When $k\eta \ll 1$, the analytic result discussed subsequent to Eq. (45) is an excellent approximation: $R_{\mathbf{k}} \simeq \ln a/a_{1x}$.

What about subhorizon scales ($k|\eta| > 1$)? In inflation (when $\lambda_{\mathbf{k}} \ll \Omega_{\mathbf{k}}$) $R_{\mathbf{k}} \simeq 0$, while in the radiation era $R_{\mathbf{k}}$ oscillates [see Eq. (43)]. For the modes which enter the horizon in the matter era (case $k\eta_{\rm eq} = 0.1$ on Fig. 2), the squeeze factor $R_{\mathbf{k}}$ continues growing as $\Delta R_{\mathbf{k}} \simeq C_{\mathbf{k}}\ln a$, where $C_{\mathbf{k}} \simeq 1.3$ for $k\eta_{\rm eq} = 0.1$ and $C_{\mathbf{k}} \to 1.5$ for $k\eta_{\rm eq} \ll 1$. This means that, as a consequence of large coupling in the matter era ($\Omega_{\mathbf{k}} < \lambda_{\mathbf{k}}$), the squeeze angle remains frozen ($\Phi_{\mathbf{k}} \simeq$ const) and the squeezing continues. Physically, this is related to the classical process of gravitational collapse.

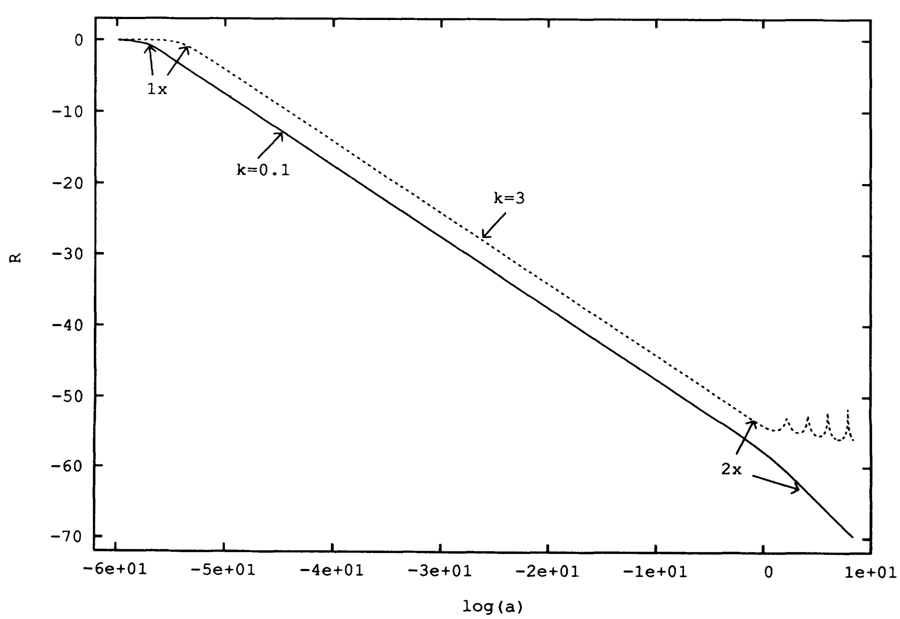

FIG. 3. Evolution of the squeeze factor R as a function of the scale factor $\log a$ in an inflationary universe model for two scales: $k\eta_{\rm eq} = 0.1$ and $k\eta_{\rm eq} = 3$. Most of the growth occurs on superhorizon scales (period between the marks $1x$ and $2x$).

There is a critical wave vector $k_{\rm crit}\eta \simeq 2$ (which corresponds to the scale $\lambda_{\rm phys} \simeq \pi\lambda_{\rm eq} \simeq 40(\Omega_0 h^2)^{-1}$ Mpc; $\lambda_{\rm eq} \simeq 13(\Omega_0 h^2)^{-1}$ Mpc is scale today corresponding to the horizon size at equal matter and radiation, Ω_0 is the fraction of the critical density today, and h is the present Hubble parameter in units of 100 km/s Mpc). For $k > k_{\rm crit}$ the state oscillates and $R_{\mathbf{k}}$ does not grow, and for $k < k_{\rm crit}$ the state is frozen and $R_{\mathbf{k}}$ grows.

Figure 4 shows the evolution of the squeeze factor with respect to the squeeze phase. On subhorizon scales in inflation $\Phi_{\mathbf{k}}$ grows, while $R_{\mathbf{k}} \simeq 0$. At the horizon crossing, the squeeze angle freezes out: $\Phi^*_{\mathbf{k}} = n\pi$ ($n \in \mathbf{Z}$) becomes an attractor and, as we can read of from Fig. 4: $\Phi^*_{\mathbf{k}} = 0$ (-2π) if $k = 1$ (10), which is in agreement with Eq. (40). For the subcritical case $k\eta_{\rm eq} = 1$, after the mode crosses the horizon at $2x$, the angle $\Phi \simeq \Phi^*$ remains frozen and $R_{\mathbf{k}}$ continues growing. On the other hand, for $k\eta_{\rm eq} = 10 > k_{\rm crit}\eta_{\rm eq}$, after time η_{2x} the mode starts oscillating with $\Delta R_{\mathbf{k}} \simeq 1$. The amplitude of oscillations in $R_{\mathbf{k}}$ is slightly bigger than predicted by the simple formula (43); the reason being that the condition for validity of Eq. (43) $(\lambda_{\mathbf{k}}/\Omega_{\mathbf{k}} \ll 1)$ is not strictly satisfied (here we have $\lambda_{\mathbf{k}}/\Omega_{\mathbf{k}} \to 1/2$). During one oscillation $\Delta\Phi_{\mathbf{k}} = \pi$ and $R_{\mathbf{k}}$ remains constant on average. Looking back at Eqs. (30) and (43) we observe that if $\Phi_{\mathbf{k}}$ grows (oscillations), the squeeze operator produces and destroys, on average, equal number of particle pairs, i.e., there is no net squeezing.

2. *Evolution of physical quantities*

We are interested in looking at physical quantities in the hydrodynamical era, typically the Bardeen variable Φ^B (which corresponds to the Newtonian potential inside the horizon) and the energy density perturbations $\delta\epsilon/\epsilon$. In the standard notation

$$\frac{\delta\epsilon}{\epsilon} = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\delta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{47}$$

and

$$|\delta_{\mathbf{k}}|^2 = k\langle\Phi^B_{-\mathbf{k}}\Phi^B_{\mathbf{k}}\rangle, \tag{48}$$

$$\Phi^B_{\mathbf{k}} = -\sqrt{\frac{3}{2}}\, l_P \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}_{\mathbf{k}} c_s^2}\frac{1}{z}\pi_{\mathbf{k}}\,, \tag{49}$$

where $l_P = (8\pi G/3)^{-\frac{1}{2}}$ is the Planck length.

To make contact with the existing work on power spectra from inflation, we plot in Fig. 5 the growth of the power spectrum $|\delta_{\mathbf{k}}|^2$ defined in Eqs. (47) and (48) against the scale factor for the modes: $k\eta_{\rm eq} = 0.1$ and $k\eta_{\rm eq} = 3$. On superhorizon scales, during the radiation era ($\ln a < 0$), the power grows as $|\delta_{\mathbf{k}}|^2 \propto a^4 \propto \eta^4$, which agrees with the estimates based on Eq. (45). In the matter era, for the modes $k\eta_{\rm eq} < 2$, the power grows as $|\delta_{\mathbf{k}}|^2 \propto a^2 \propto \eta^4$, while for $k\eta_{\rm eq} > 2$, the state starts oscillating and the growth becomes very slow.

Figure 6 shows the power spectrum at two different time slices: $\eta = 0.1\eta_{\rm eq}$ and $\eta = 0.5\eta_{\rm eq}$. The spectrum is scale invariant, $|\delta_{\mathbf{k}}|^2 \sim k$, on superhorizon scales and $|\delta_{\mathbf{k}}|^2 \sim k^{-1}$ on subhorizon scales. The turning point, caused by the oscillations of the squeezed state (see Fig. 5), is at $k\eta_{\rm eq} \simeq 4$ for both time slices. The first dip in the power spectrum is at $k\eta_{\rm eq} \simeq 8 - 9$, which corresponds to the wavelength $\lambda \simeq (0.8 - 0.9)\lambda_{\rm eq} \simeq 11(\Omega_0 h^2)^{-1}$ Mpc. These dips correspond to the acoustic oscillations in the fluid.

VI. COMPARISON WITH PREVIOUS WORK

The features of the power spectrum just discussed are those expected. We obtained the correct growth on superhorizon scales and found acoustic oscillations in the modes which reenter the horizon in the radiation dominated era, as described, for example, Bardeen *et al.* in

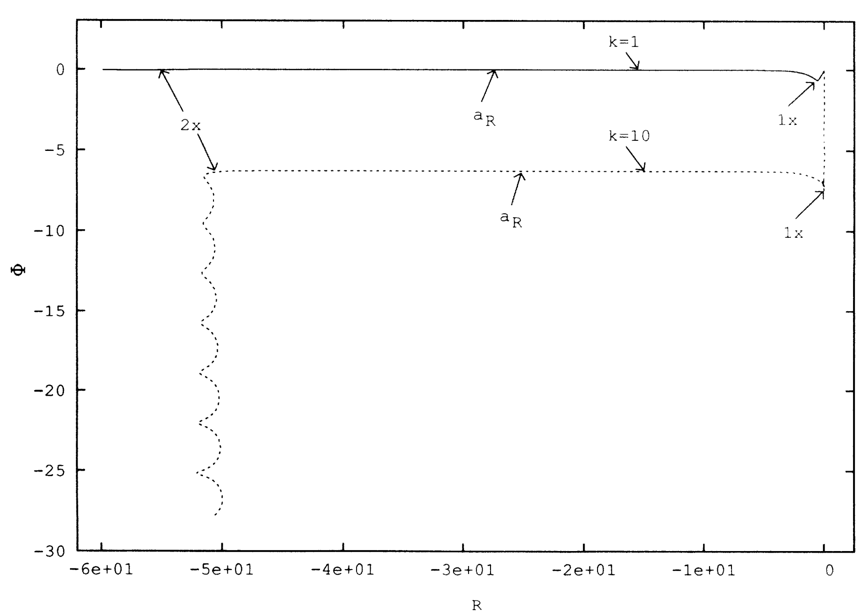

FIG. 4. The squeeze phase vs squeeze factor ($\Phi - R$) diagram for two scales: $k\eta_{\rm eq} = 1$ and $k\eta_{\rm eq} = 10$. The squeeze angle freezes out on superhorizon scales. On subhorizon scales it exhibits two types of behavior: for scales below critical ($k_{\rm crit}\eta_{\rm eq} \sim 2$), the $\Phi - R$ curve exhibits oscillatory behavior, and for scales above critical the phase remains frozen.

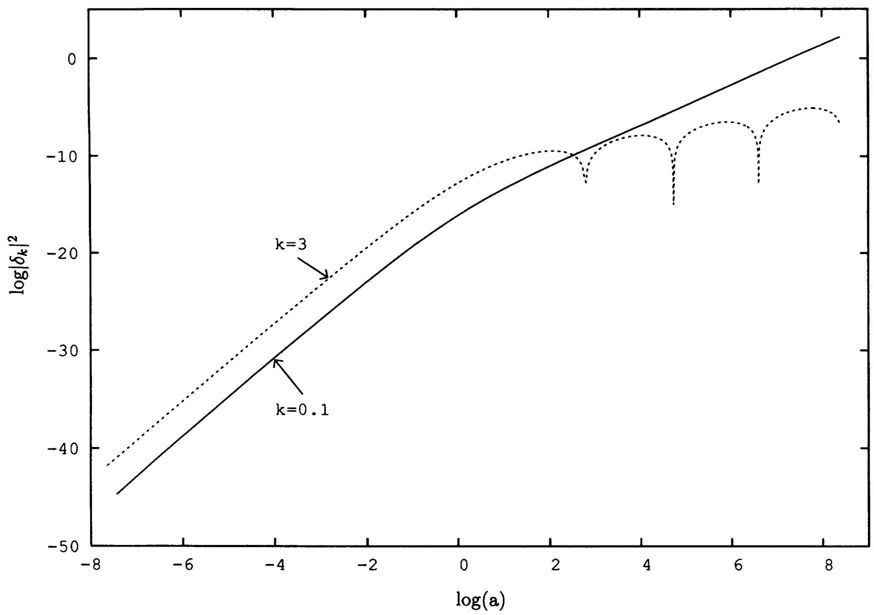

FIG. 5. The growth of the power spectrum $|\delta_{\mathbf{k}}|^2$ against the scale factor a for $k\eta_{\rm eq} = 0.1$ and $k\eta_{\rm eq} = 3$. In both cases we observe the same power law growth on superhorizon scales. In the subcritical case ($k\eta_{\rm eq} = 0.1$) the growth continues (with somewhat slower rate), while in the supercritical case ($k\eta_{\rm eq} = 3$) the power exhibits oscillations after the horizon crossing.

Ref. [1] and Efstathiou in Ref. [21]. We have simply illustrated that these phenomena can be described in a different way in the squeezed state framework.

For a more direct comparison with the work of Grishchuk and Sidorov in Ref. [7], in particular their discussion of "desqueezing," we treat analytically a model in which matter and radiation instantaneously decouple. We work with the action of Eq. (20) as in [7]. We take

$$a = -\frac{1}{H\eta}, \quad \eta < \eta_2 < 0 \quad \text{(inflation)}, \tag{50}$$

$$a = \frac{\eta}{\eta_1}, \quad -\eta_2 < \eta < \eta_1 \quad \text{(radiation)}, \tag{51}$$

$$a = \frac{\eta^2}{(2\eta_1)^2}, \quad \eta > 2\eta_1 \quad \text{(matter)}. \tag{52}$$

The most convenient way to solve explicitly for the squeeze factor R is to solve the equation for v derived from the action and then to use the transformations relating the two sets of variables derived in Appendix A. The solutions for v in the three eras are

$$v_i = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}, \tag{53}$$

$$v_r = \sqrt{\frac{\sqrt{3}}{2k}} e^{-\frac{ik\eta}{\sqrt{3}}}, \tag{54}$$

$$v_m = \sqrt{\frac{\eta_1}{3}} \left[\left(\frac{\eta}{2\eta_1}\right)^2 + i\frac{2\eta_1}{\eta}\right], \tag{55}$$

and $\pi = v'$. The normalizations are chosen so that $v'v^* - v'^*v = -i$ in each case. Matching v and v' (and therefore R and Φ) continuously at each of the two boundaries we obtain the following expressions for the squeeze factor to leading order in $k\eta_2$:

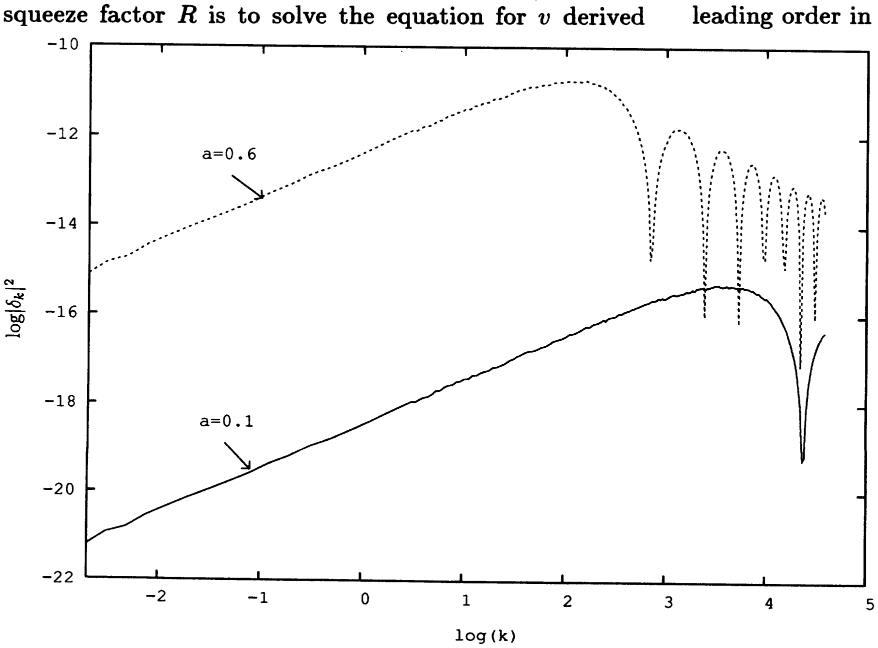

FIG. 6. The snap-shot of the power spectrum ($\log|\delta_{\mathbf{k}}|^2 - \log k$ plot) for two times: $a = 0.1a_{\rm eq}$ and $a = 0.6a_{\rm eq}$. The spectrum is scale invariant on superhorizon scales: $|\delta_{\mathbf{k}}|^2 \sim k$ and after the turning point at $k\eta_{\rm eq} \simeq 4$ it shows oscillatory behavior and decays as $|\delta_{\mathbf{k}}|^2 \sim k^{-1}$.

$$\sinh^2 R_{\mathbf{k}} = \frac{1}{4(k\eta)^4}\ , \quad \eta < \eta_2 \tag{56}$$

$$\sinh^2 R_{\mathbf{k}} = \frac{1}{4(k\eta_2)^4}\left(2 - \cos\frac{2k\eta}{\sqrt{3}}\right)\ , \quad -\eta_2 < \eta < \eta_1 \tag{57}$$

$$\sinh^2 R_{\mathbf{k}} = \frac{1}{4(k\eta_2)^4}\Bigg\{ (\alpha+\beta)\left[\left(\frac{\eta}{2\eta_1}\right)^4 + \frac{1}{(k\eta_1)^2}\left(\frac{\eta}{2\eta_1}\right)^2\right] + \gamma\left(\frac{\eta}{2\eta_1} - -\frac{1}{(k\eta_1)^2}\frac{\eta_1}{\eta}\right) + (\alpha-\beta)\left[\left(\frac{2\eta_1}{\eta}\right)^2 + \frac{1}{4(k\eta_1)^2}\left(\frac{2\eta_1}{\eta}\right)^4\right]\Bigg\},\quad \eta > 2\eta_1 \tag{58}$$

where

$$\alpha = \frac{1}{12}\left\{\left(5 + \frac{8(k\eta_1)^2}{3}\right) - \left(5 - \frac{8(k\eta_1)^2}{3}\right)\cos\frac{2k\eta_1}{\sqrt{3}} - \frac{4k\eta_1}{\sqrt{3}}\sin\frac{2k\eta_1}{\sqrt{3}}\right\}, \tag{59}$$

$$\beta = \frac{1}{4}\left\{-1 + \cos\frac{2k\eta_1}{\sqrt{3}} + \frac{4k\eta_1}{\sqrt{3}}\sin\frac{2k\eta_1}{\sqrt{3}}\right\}, \tag{60}$$

$$\gamma = -\frac{2}{3}\left\{\left(-1 + \frac{2(k\eta_1)^2}{3}\right) + \left(1 + \frac{2(k\eta_1)^2}{3}\right)\cos\frac{2k\eta_1}{\sqrt{3}} - \frac{k\eta_1}{\sqrt{3}}\sin\frac{2k\eta_1}{\sqrt{3}}\right\}. \tag{61}$$

From the second expression in (58) we see that the squeeze factor is modulated only slightly in the radiation era in agreement with what we found earlier. For the matter era however one can show that the coefficients $\alpha+\beta$ and γ of the terms which grow with η vanish when the condition

$$\frac{k\eta_1}{\sqrt{3}} + \arctan\frac{2k\eta_1}{\sqrt{3}} = n\pi \qquad (n \quad \text{integer}) \tag{62}$$

is satisfied. This leads to a significant amount of "desqueezing" of these modes as the squeeze factor for these modes is given approximately by

$$\sinh^2 R_{\mathbf{k}} = \sinh^2 R^0_{\mathbf{k}}\left(\frac{2\eta_1}{\eta}\right)^2 \frac{1+\frac{1}{(k\eta)^2}}{1+\frac{1}{(2k\eta_1)^2}}\ , \tag{63}$$

where $R^0_{\mathbf{k}} = R_{\mathbf{k}}(2\eta_1)$ is the squeeze factor at the decoupling. In terms of the scale factor,

$$R_{\mathbf{k}} \simeq R^0_{\mathbf{k}} - \frac{1}{2}\ln\left(\frac{a}{a_{\rm dec}}\right) \qquad \text{for } k\eta \gg 1. \tag{64}$$

The existence of this "desqueezing" is again a familiar phenomenon expressed in a different set of words. When one matches the oscillating solutions of the radiation era onto the growing and decaying solutions of the matter era one finds that certain modes match completely onto the decaying solution. In fact this is the simplest way to derive the condition (62) above. These modes lose power and we have approximate zeros in the power spectrum. These oscillations in the power spectrum are known as Sakharov oscillations [22]. In order to obtain the position of the zeroes, we solve Eq. (62) and obtain $k\eta_{\rm rec} = 2k\eta_1 = \{6.36, 16.7, 27.4, 38.2, 49.1, 59.9, 70.8, \ldots\}$, which correspond to today's scales: $\lambda = \{89, 34, 20.7, 14.8, 11.5, 9.5, 8.0, \ldots\}\ h^{-1}$ Mpc (h is the Hubble constant today in units 100 km/s Mpc). The occurrence of these oscillations depends crucially on the matching at the inflation-radiation transition. In order to match purely onto the decaying solution (in the matter era), one must have standing wave solutions in the radiation era and this in turn depends on having the correct input from the inflationary epoch. It is indeed the squeezing of all of the physical momentum out of the superhorizon modes during inflation that produces the standing waves at the end of inflation, which one requires to produce this effect.

Grishchuk and Sidorov suppose this crucial ingredient to be missing in standard treatments of the growth of perturbations. They claim that incorrect assumptions about the perturbations produced by inflation are often made, which lead to traveling wave solutions in the radiation dominated era and the resultant absence of these Sakharov oscillations in the power spectrum. For example, Grishchuk states in [6] that "the unavoidable property of squeezing manifests itself in the fact that the phases of primordial density perturbations are fixed and correlated, *in contrast with the usually made assumption* that the phases are distributed randomly and evenly. In other words, the primordial density perturbations, similarly to the case of gravitational waves, must form a set of standing waves with definite phases" (our italics). In fact these two points are not in conflict. Indeed there *are* "standing waves with definite phases," but there are other phases which are distributed randomly and evenly. One must be careful about which phases one is talking about. Each standing wave has a "phase of oscillation" which distinguishes among solutions which are at different points in their period of oscillation. This is the phase which is fixed (relative to the time of horizon crossing) in inflationary cosmologies.

However, inflation does *not* predict the location of the nodes in the standing wave. There is another "spatial" phase which distinguishes among standing waves which differ by a translation in space. Since the wave function assigns equal probability to solutions that differ only by a translation, one can choose a random spatial phase. This amounts to making a particular random choice of $\delta(x)$ from among the many possible ones.

We are aware of one place in the literature where an error is made regarding which phases are random. In a passage in [23] [preceding the paragraph containing Eq. (7)], Peebles argues that the temporal phase of the standing waves may be taken to be random. This statement is incorrect and, to the extent that Grishchuk's criticisms refer to it, we are in agreement with him.[2] However, this is an isolated error and is not of significance in either the work of this author or others who produce detailed predictions based on specific models (see, e.g., Ref. [21]).

Typically the correct standing wave solutions are used without making reference to the squeezed state terminology. That this is so can most simply be seen by the fact that the usual Bunch-Davies vacuum matched onto the oscillating radiation era solutions gives precisely the standing waves noted by Grishchuk and Sidorov. In [21], for example, the matching is described in terms of growing and decaying modes in the radiation era, but amounts to the choice of standing waves and indeed both the acoustic oscillations and Sakharov oscillations which result are seen in these simulations. The reason why so little attention is paid to these features is that they occur only in the baryonic component of matter and are almost completely swamped in dark matter dominated models. It is an interesting possibility that this difference might be exploited to distinguish between baryonic and dark matter dominated models. Attempts have in fact been made to look for these Sakharov oscillations but the results are inconclusive [24].

The other important claim of Grishchuk and Sidorov is that these features can be said to be of a distinctly quantum mechanical origin. Speaking of desqueezing, they state in [7] that "we relate this *quantum effect* to the effect of the so-called Sakharov oscillations known in the classical theory of matter-density perturbations." In [6] Grishchuk opines that "it is quite possible that the very specific properties of the large scale density perturbations *related to their quantum mechanical origin* can be revealed in the appropriate observations" (our italics). We will attempt to clarify this question in the next section.

VII. THE CLASSICALITY OF SQUEEZED STATES

A squeezed state seems to be an especially quantum mechanical state. It is not well localized in p and q and therefore cannot be represented by a point in classical phase space. It may instead be viewed as a *coherent superposition* of many localized wave packets. It is very unlike the archetype classical state, the coherent state, being very squeezed in one variable. It is this feature which generates so much interest in these states in quantum optics and other areas of physics and leads to their characterization as very "nonclassical" [14].

In general, there is not universal agreement on a precisely defined boundary between quantum and classical. We will now show, however, that the squeezed states are *very* classical in the WKB sense. We then discuss the relationship between the WKB classicality and the various treatments of perturbations from inflation.

A. WKB classicality of squeezed states

Consider the q representation of the squeezed state in the static inverted harmonic oscillator which we considered earlier:

$$\psi(q) = N e^{-(B+iC)\frac{q^2}{2\hbar}}, \tag{65}$$

where

$$N = \left(\frac{B}{\hbar\pi}\right)^{\frac{1}{4}}, \quad B = \frac{1}{\cosh 2r}, \quad C = \tanh 2r. \tag{66}$$

We will show that for large squeezing this wave function is *very* classical in the WKB sense and becomes increasingly so with time. The wave function can be written

$$\psi(q) = \rho(q) e^{iS(q)}. \tag{67}$$

If $S(q)$ varies much more rapidly with q than $\rho(q)$ the state is a WKB state for which

$$\hat{p}|\psi\rangle \simeq [\hbar\partial_q S(q)]|\psi\rangle. \tag{68}$$

To the extent that this holds the state assigns momentum and position simultaneously according to

$$p(q) = \hbar\partial_q S(q). \tag{69}$$

While $p(q)$ need not be localized, it does represent a distribution in classical phase space which evolves classically in the WKB limit.

For the evolved state given by Eq. (65) we have

$$\rho(q) = N e^{-B\frac{q^2}{2\hbar}}, \tag{70}$$

$$S(q) = -C\frac{q^2}{2\hbar}. \tag{71}$$

The WKB condition is met when the quantity $\rho(\partial_q S(q)/\partial_q \rho(q))$ is large. From Eq. (65) we find

$$\left|\rho\frac{\partial_q S(q)}{\partial_q \rho(q)}\right| = \frac{C}{B} = \sinh 2r. \tag{72}$$

Therefore as the initial state evolves and becomes more squeezed, it also becomes more classical in the WKB sense.

Equivalently this can be seen from Eqs. (8)–(11) since they imply

$$|\langle\Psi(t)|\hat{q}\hat{p}|\Psi(t)\rangle| = \frac{\hbar}{2}(1+\sinh^2 2r)^{\frac{1}{2}} \simeq \frac{\hbar}{4}e^{2t}. \tag{73}$$

This just expresses more directly the effective irrelevance of the noncommutativity of the position and momentum operators on the state for large squeezing. It is precisely these properties of the inverted harmonic oscillator which were used by Guth and Pi in [11] to illustrate how a quantum mechanical state can be treated in certain cases as an ensemble of classical states.

[2]We are grateful to Jim Peebles for a discussion of this point.

B. Implications of WKB classicality

The WKB classicality means that the squeezed state can be approximated in its evolution as a classical phase space distribution, as long as one only measures classical quantities. When a particle in a spread out WKB state interacts with another system which responds to (or "measures") the value of p or q, one can predict the outcome using only the probability distribution in classical phase space.

In fact, it is well established that when such a measurement takes place correlations are set up which cause the quantum coherence to be lost (see for example [25]). From that point on the particle is in a density matrix rather than a pure state, and the possibility of observing the effects of quantum coherence is even more remote. (See [26] for a discussion of quantum coherence in the context of WKB classicality.)

In quantum optics, where squeezed states of the electromagnetic field can be produced, detections are often represented as measurements in the number eigenstates of the field modes. Because these are not classical variables one *cannot* predict these results using a classical probability distribution. Interesting quantum coherence effects have been calculated, which can be thought of as interference among the many coherently superposed classical trajectories comprising the squeezed state. However, because of practical limitations Zhu and Caves [27] argue that such effects have not been experimentally observed.

The crucial question then is: when matter interacts with a density field in a squeezed quantum state, does it respond to (or measure) the classical field values or something else? We have given this question some thought, and find it hard to see anything other than very classical processes in these interactions. The matter, after all, evolves according to the values of things such as the Newtonian potential, which is local in the field variables. Furthermore as the Universe evolves and the matter responds to the perturbations, correlations will be set up which destroy the initial coherence as discussed above.

If one wishes to show that the initial quantum coherence of the squeezed state is of physical importance, one must demonstrate interactions which measure something other than classical quantities *before* the ordinary interactions destroy the quantum coherence. It would be very interesting if this could be done, but we do not see how.

The particular features of the power spectrum discussed by Grishchuk and Sidorov are *not* the result of quantum coherence. They are features which appear in individual classical solutions (e.g., properties of each trajectory in classical phase space) and do not represent quantum interference among different classical solutions. The physical origin of the fluctuations (the vacuum fluctuations) is quantum mechanical but their known physical effects are indistinguishable from fluctuations from a classical stochastic field.

Regarding the phases of modes which oscillate inside the horizon, these are predicted regardless of whether there is quantum coherence. The prediction is based on the fact that the modes in question have spent a long time outside the horizon, where there is one growing and one decaying solution to the equations of motion. The growing component becomes completely dominant for the modes which are amplified during inflation. This growing solution has a uniquely determined oscillatory behavior when it enters the horizon, and thus the phase of the oscillations is predicted. The original work on this subject has correctly accounted for these predictions [1].

The quantum squeezing is also a consequence of the presence of one growing and one decaying solution, but that does not mean that observing the phases of the oscillatory behavior amounts to a test of quantum coherence. An incoherent superposition (such as would result from the establishment of correlations with particles and photons mentioned above) would provide the same results, as long as each mode was dominated by the growing solution. The particular features of the power spectrum discussed by Grishchuk and Sidorov are only quantum mechanical in origin in the mundane sense in which all perturbations in inflation are. The physical origin of the fluctuations is quantum mechanical but they are in their known physical effects indistinguishable from fluctuations from a classical stochastic field.

VIII. CONCLUSION

We developed the squeezed state formalism to study the growth of cosmological perturbations. The formalism is then applied to a simple inflationary model with baryonic matter. We discussed how the standard features, such as acoustic oscillations and Sakharov oscillations, are characterized in the squeezed state formalism. At late times density perturbations are semiclassical and, for all practical purposes, can be well represented by a classical probability distribution function.

Confusion can be avoided if one keeps in mind that there are three very different phases which enter into the discussion. *Firstly*, there is the complex phase of the wave function. To the extent that the system being studied is WKB classical, this phase is simply absorbed in the construction of the classical phase space distribution. The classicality of the subsequent measurements determine whether there are any interesting quantum cohernece effects. *Secondly*, there is the phase of oscillation of standing waves in the density field. These are very precisely fixed in inflationary cosmologies and this can lead to predictable *Sakharov oscillations* at late times. Note that this second phase is a *classical* phase. *Thirdly*, there are classical phases for each Fourier mode which correspond to translations in physical space. Since the inflationary universe assigns equal probability to density fields which differ only by a translation, these spatial phases are random within the linear approximation.

The use of squeezed states in a cosmological setting was first advocated and implemented by Grishchuk and Sidorov to calculate the power spectrum of primordial gravitational waves [16]. The treatment is entirely analogous to that of cosmological perturbations; it is possible to reduce the problem to, again, quantizing a scalar field with a time dependent mass ($z = a$). The power spectrum of this scalar field exhibits oscillations on certain scales. It is possible, as Grishchuk claims, to predict the position of the dips in the power spectrum. However, this feature is also present in the standard Heisenberg formalism as treated by Abbott and Harari [28]. The squeezed state formalism gives us an intuitive way of looking at the generation and evolution of cosmological perturbations. However, the formalism we have developed is not restricted to cosmological applications; the equations of motion are quite generic of systems with quadratic Hamiltonians that can be put in the form of Eq. (26).

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APPENDIX A: RELATING THE HEISENBERG AND SCHRÖDINGER PICTURES

In this appendix we show how to parametrize the Schrödinger picture variables in terms of the squeezed state parameters $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$. We then demonstrate that the classical equations of motion for the mode functions reduce to the evolution equations for $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$ derived in Sec. III [Eq. (39)]. This shows how the Schrödinger picture problem can be reduced to solving the classical equations of motion with an appropriate reparametrization.

The Heisenberg picture operators $\hat{v}(\mathbf{x},\eta)$ and $\hat{\pi}(\mathbf{x},\eta)$ can be written as

$$\begin{aligned}\hat{v}(\mathbf{x},\eta) &= \mathcal{U}^\dagger(\eta,\eta_0)\hat{v}(\mathbf{x},\eta_0)\mathcal{U}(\eta,\eta_0)\\ &= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}(u^*_{\mathbf{k}}(\eta)a_{\mathbf{k}} + u_{-\mathbf{k}}(\eta)a^\dagger_{-\mathbf{k}}),\\ \hat{\pi}(\mathbf{x},\eta) &= \mathcal{U}^\dagger(\eta,\eta_0)\hat{\pi}(\mathbf{x},\eta_0)\mathcal{U}(\eta,\eta_0)\\ &= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}(w^*_{\mathbf{k}}(\eta)a_{\mathbf{k}} + w_{-\mathbf{k}}(\eta)a^\dagger_{-\mathbf{k}}) . \end{aligned} \quad \text{(A1)}$$

It is now easy to show, using the Heisenberg equations of motion for $\hat{v}$ and $\hat{\pi}$, that the mode functions $u_{\mathbf{k}}(\eta)$, $w_{\mathbf{k}}(\eta)$ satisfy the Hamilton equations

$$\begin{aligned} u'_{\mathbf{k}} &= w_{\mathbf{k}} + \frac{z'}{z}u_{\mathbf{k}},\\ w'_{\mathbf{k}} &= -c_s^2k^2u_{\mathbf{k}} - \frac{z'}{z}w_{\mathbf{k}} . \end{aligned} \quad \text{(A2)}$$

These are the configuration and momentum variables of the classical field theory given by the action in Eq. (21). With the initial choice of $u_{\mathbf{k}}(\eta_0) = (2k)^{-1/2}$ and $w_{\mathbf{k}}(\eta_0) = i(k/2)^{1/2}$, corresponding to an initial (right moving) traveling wave, the solutions to Eq. (A2) are uniquely defined for all times. At $\eta = \eta_0$ we obtain the Schrödinger picture operators (25). At some later time η we have

$$\begin{aligned} \hat{v}_{\mathbf{k}}(\eta) &= \frac{1}{\sqrt{2k}}[a_{\mathbf{k}}(\eta) + a^\dagger_{-\mathbf{k}}(\eta)],\\ \hat{\pi}_{\mathbf{k}}(\eta) &= -i\sqrt{\frac{k}{2}}[a_{\mathbf{k}}(\eta) - a^\dagger_{-\mathbf{k}}(\eta)] , \end{aligned} \quad \text{(A3)}$$

where $a_{\mathbf{k}}(\eta)$ and $a^\dagger_{-\mathbf{k}}(\eta)$ are the Heisenberg picture annihilation and creation operators defined by

$$\begin{aligned} a_{\mathbf{k}}(\eta) &\equiv \mathcal{U}^\dagger(\eta,\eta_0)a_{\mathbf{k}}\mathcal{U}(\eta,\eta_0)\\ &= \mathcal{R}^\dagger(\Theta_{\mathbf{k}})\mathcal{S}^\dagger(R_{\mathbf{k}},\Phi_{\mathbf{k}})a_{\mathbf{k}}\mathcal{S}(R_{\mathbf{k}},\Phi_{\mathbf{k}})\mathcal{R}(\Theta_{\mathbf{k}})\\ &= \cosh R_{\mathbf{k}}e^{-i\Theta_{\mathbf{k}}}a_{\mathbf{k}} - \sinh R_{\mathbf{k}}e^{i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}a^\dagger_{-\mathbf{k}}. \end{aligned} \quad \text{(A4)}$$

From Eq. (A1) we then get

$$\begin{aligned} \hat{v}_{\mathbf{k}}(\eta) &= \frac{1}{\sqrt{2k}}\left[a_{\mathbf{k}}\left(\cosh R_{\mathbf{k}}e^{-i\Theta_{\mathbf{k}}} - \sinh R_{\mathbf{k}}e^{-i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}\right) + a^\dagger_{-\mathbf{k}}\left(\cosh R_{\mathbf{k}}e^{i\Theta_{\mathbf{k}}} - \sinh R_{\mathbf{k}}e^{i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}\right)\right],\\ \hat{\pi}_{\mathbf{k}}(\eta) &= -i\sqrt{\frac{k}{2}}\left[a_{\mathbf{k}}\left(\cosh R_{\mathbf{k}}e^{-i\Theta_{\mathbf{k}}} + \sinh R_{\mathbf{k}}e^{-i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}\right) - a^\dagger_{-\mathbf{k}}\left(\cosh R_{\mathbf{k}}e^{i\Theta_{\mathbf{k}}} + \sinh R_{\mathbf{k}}e^{i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}\right)\right] . \end{aligned} \quad \text{(A5)}$$

Comparing Eq. (A1) with Eq. (A5) we can identify the mode functions to be

$$\begin{aligned} u_{\mathbf{k}}(\eta) &= \frac{1}{\sqrt{2k}}\left(\cosh R_{\mathbf{k}}e^{i\Theta_{\mathbf{k}}} - \sinh R_{\mathbf{k}}e^{i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}\right) ,\\ w_{\mathbf{k}}(\eta) &= i\sqrt{\frac{k}{2}}\left(\cosh R_{\mathbf{k}}e^{i\Theta_{\mathbf{k}}} + \sinh R_{\mathbf{k}}e^{i(\Theta_{\mathbf{k}}+2\Phi_{\mathbf{k}})}\right) \end{aligned} \quad \text{(A6)}$$

and these define the transformation that we seek between the Schrödinger picture variables and the Heisenberg picture mode functions. It is now a matter of algebra to show that Hamilton's equations for the mode functions (A2) give the equations of motion for $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$ (39).

APPENDIX B: INVARIANCE OF THE EQUATIONS OF MOTION FOR R_k, Φ_k, AND Θ_k

Here we show that for the two actions (20) and (21) differing by the total derivative term $[(z'/z)v^2]'$, the equations of motion for $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$ have invariant form. For the action (21) $\lambda_{\mathbf{k}}$, $\Omega_{\mathbf{k}}$, and $\varphi_{\mathbf{k}}$ are defined in Eq. (27). On the other hand for the action (20) we have

$$\begin{aligned} \lambda_{\mathbf{k}} &= \frac{k}{2}(1-c_s^2) + \frac{z''}{2kz}, \\ \Omega_{\mathbf{k}} &= \frac{k}{2}(1+c_s^2) - \frac{z''}{2kz}, \\ \varphi_{\mathbf{k}} &= -\frac{\pi}{4}\,. \end{aligned} \tag{B1}$$

Even though canonically related Hamiltonians give different evolution for $R_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\Theta_{\mathbf{k}}$, the physically measurable quantities are invariant. We have not investigated how generic is the invariance of the equations of motion (39). We leave this as an exercise to an inquisitive reader.

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