A new method of generating exact inflationary solutions

Franz E. Schunck* and Eckehard W. Mielke†
Institute for Theoretical Physics, University of Cologne, D-50923 Köln, Germany
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The mechanism of the initial inflation of the universe is based on gravitationally coupled scalar fields ϕ. Various scenarios are distinguished by the choice of an *effective self-interaction potential* $U(\phi)$ which simulates a *temporarily* nonvanishing *cosmological term*. Using the Hubble expansion parameter H as a new "time" coordinate, we can formally derive the *general* Robertson-Walker metric for a *spatially flat* cosmos. Our new method provides a classification of allowed inflationary potentials and is broad enough to embody all known *exact* solutions involving one scalar field as special cases. Moreover, we present new inflationary and deflationary exact solutions and can easily predict the influence of the form of $U(\phi)$ on density perturbations.

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I. INTRODUCTION: EINSTEIN'S BIGGEST BLUNDER

The introduction of a *cosmological constant* Λ in the field equations of general relativity later on struck Einstein as "the biggest blunder of my life" [1,2]. Such an amendment was not even completely new, since von Seeliger [3] and Neumann [4], e.g., considered already in 1896 a corresponding term in the Poisson equation for the Newtonian potential in order to compensate the energy density of the 'æther.'

Nowadays, Einstein's dream of a completely geometrical description of fundamental physical interactions has evolved into supergravity [5] and superstring models [6] in a way which was unprecedented in his time. Nevertheless, the cosmological term is still a major problem of these new approaches, as can be inferred from the review of Weinberg [7].

The overall reason being that, in almost all quantized theories of particle interactions, the vacuum density $\rho_{\rm vac}$ gives rise to a huge *bare* cosmological constant $\Lambda_0 = \kappa\rho_{\rm vac}$, where κ is the gravitational constant. This can be traced back to the fact that the vacuum fluctuations "feel" all the complicated physics originating from Higgs fields, fermion condensates, etc., which enter into today's unified field theories. For much higher energies or, equivalently, to very short spacetime distances, the small scale behavior of the quantum world would determine the large scale structure of the universe.

On the other hand, it is known that the observed macroscopical energy density is extremely small. For the range of 45–100 km s^{-1} Mpc^{-1} of *today's* Hubble constant H_0, the critical density is estimated as $\rho_c = 0.5 - 2 \times 10^{-29}$ g/cm^3. From local as well as large scale astronomical measurements, the macroscopically observed cosmological constant Λ is estimated [8] to be less than 4×10^{-56} cm^{-2}. Since the vacuum energy may also be time dependent at the early stages of the universe, the exact fine tuning of the various vacuum contributions to a very small Λ in the low temperature regime of today appears to be one of the great mysteries about unification.

Higgs-type scalar fields become more and more important. They not only induce the masses for the elementary particles via the Higgs-Kibble mechanism, but they can also form stable boson stars [9] and kinks [10]. For spin-one particles, exact nonsingular solutions of the Einstein-SU(2)-Yang-Mills system are not yet known, but the new power series expansion technique of Ref. [11] can be regarded as a first attemption in this direction.

But the scalar fields, in disguise as the "inflaton" ϕ, can also dominate the early universe, the *epoch of inflation.* Before symmetry breaking, a self-interaction $U(\phi)$ of such gravitationally coupled scalar fields allows us to introduce a *variable* "cosmological term" without violating the Noether-Bianchi identities of Einstein's general relativity.

II. MODEL OF A UNIVERSE WITH INFLATION

From new astronomical observations (COBE) we know that the Universe expands and is rather homogeneous on the large scale and in the microwave background. However, the standard Friedmann model of the cosmos offers no solution to such issues as the singularity problem, the problem of flat space, the horizon problem, the homogeneity problem on great scales, the absence of magnetic monopoles [12], and the problem of large number of particles [13,14].

The idea of *inflation* (see Guth [15] and Linde [16]) attempts to solve several of these problems. Scalar fields (Higgs, axion) are expected to generate, shortly after the big bang, an exponential increase of the Universe. However, in these first attempts, there was no so-called *graceful exit* to the Friedmann cosmos, and the inflation-

*Electronic address: fs@thp.uni-koeln.de
†Electronic address: pke27@rz.uni-kiel.d400.de

ary phase did not end. This problem was solved in the *new inflationary universe.* In this model, the scalar field is ruled by a slightly different self-interaction potential which possesses a slow-roll part (a plateau) of the potential (acting as a vacuum energy), which dominates the universe at the beginning.

Later on, power-law models were constructed which possess no exponential but an $a(t) \sim t^n$ increase of the expansion factor of the universe [17,18]. The *intermediate inflation* is merely a combination of exponential and power-law increases [19]. Mathematically, inflation is described by a positive second derivative of the scale factor $a(t)$ of the universe. In general, this requires $\rho + 3p < 0$, where ρ is the density and p the pressure of the matter field.

In the models of the new and chaotic inflationary models, we have a fine-tuning problem, which consists of the combination of the largeness of the scale factor and an acceptable distribution for density perturbations [20]. Solutions for these problems are attempted in the scenarios of *extended inflation* [20–22].

In all of these models, the isotropy and homogeneity are prescribed. It was also shown that all initially expanding homogeneous models (the Bianchi and Sachs-Kantowski universes), which include a positive cosmological constant, approach asymptotically the de Sitter solution [23,24], which is isotropic. This is called the "cosmic no-hair" theorem. For models with scalar inflation, the question of damping a possibly initial anisotropy of the universe is not relevant, because the model merely has to a ensure a very small anisotropy in the universe after inflation [25].

In this paper, we present in Sec. V a general inflationary solution in terms of the Hubble parameter which comprises all previous exact solutions. This enables us, in Sec. VI, to classify the potential $U(\phi)$ for the scalar field according to the different onset of inflationary, deflationary, and Friedmann phases of the universe. Within this new description, some new exact solutions of the so-called *new* and *chaotic* type are found in Secs. VIII, IX, and X. The potentials found have a rather complicated form however, which so far have no motivation from field theory.

Recently *chaotic models* with *several* scalar fields ϕ_I including the inflaton have attracted much attention. Linde termed his model "hybrid inflation" [26]; cf. Copeland *et al.* [27]. In the vacuum dominated regime, the back reaction of the other scalar field on the inflaton is negligible and we can follow their evolution explicitly by using quantum field theory in curved spacetime, in the case of a de Sitter background; see [28], for example. In these hybrid models, all the other scalar fields vanish, if they sit in the false vaccum. For the remaining inflaton, we can then simply apply the general solution of Sec. V.

III. FRIEDMANN SPACETIME

For a rather general class of inflationary models the Lagrange density reads

$$\mathcal{L} = \frac{1}{2\kappa}\sqrt{|g|}\{R + \kappa[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - 2U(\phi)]\}\ , \quad (3.1)$$

where ϕ is the scalar field and $U(\phi)$ the self-interaction potential. We use natural units with $c = \hbar = 1$. A constant potential $U_0 = \Lambda/\kappa$ would simulate the cosmological constant Λ. Scalar coupled Jordan-Brans-Dicke type [29] models can be reduced to (3.1) via the Wagoner-Bekenstein-Starobinsky transformation [30–33]. We are looking for solutions of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}\ , \quad (3.2)$$

which are of the Robertson-Walker type

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

$$k = 0, \pm 1\ , \quad (3.3)$$

where $a(t)$ is the expansion factor with the dimension *length.* An open, flat, or closed universe is characterized by $k = -1, 0, 1$, respectively. This means that we will investigate *homogeneous and isotropic* spacetimes. The scalar field depends only on the time t, i.e., $\phi = \phi(t)$. Then, the only nonvanishing components of the energy-momentum tensor read

$$\rho = T_0{}^0 = \frac{1}{2}\dot{\phi}^2 + U\ , \quad (3.4)$$

$$p = -T_1{}^1 = -T_2{}^2 = -T_3{}^3 = \frac{1}{2}\dot{\phi}^2 - U\ . \quad (3.5)$$

IV. REPARAMETRIZED SELF-INTERACTION

Let us assume that $a(t) \neq 0$; furthermore, we express our result in terms of the Hubble expansion rate

$$H := \frac{\dot{a}(t)}{a(t)}\ . \quad (4.1)$$

Only the diagonal components of the Einstein equation are nonvanishing. The (0,0) component is

$$3\left(H^2 + \frac{k}{a^2}\right) = \kappa\rho\ . \quad (4.2)$$

It describes the conservation of the energy. The (1,1), (2,2), and (3,3) components are given by

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\kappa p\ . \quad (4.3)$$

The resulting Klein-Gordon equation is

$$\ddot{\phi} = -3H\dot{\phi} - U'(\phi)\ , \quad (4.4)$$

which, after multiplication by $\dot{\phi}$, can be transformed into

$$\frac{1}{2}[(\dot{\phi})^2]^{\cdot} = -3H(\dot{\phi})^2 - \dot{U}\ . \quad (4.5)$$

From (4.2) and (4.3) we obtain by linear combination

$$\dot{H} = \frac{k}{a^2} - \frac{\kappa}{2}(\rho + p) = \frac{k}{a^2} - \frac{\kappa}{2}\dot{\phi}^2 \tag{4.6}$$

and

$$\dot{H} + 3H^2 + \frac{2k}{a^2} = \frac{\kappa}{2}(\rho - p) = \kappa U . \tag{4.7}$$

Observe that (4.7) is, in view of (4.5) and (4.6), a *first integral* of (4.4) for all values of the normalized extrinsic curvature scalar k. Alternatively, if we eliminate the k/a^2 terms in Eqs. (4.2) and (4.3), we obtain the Raychaudhuri equation

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) . \tag{4.8}$$

There are several options to calculate solutions for the given system of differential equations (4.2) and (4.3) or (4.6) and (4.7), respectively. The first possibility is to assume a reasonable functional dependence of the scale factor $a(t)$ and then to calculate simply the Hubble expansion rate $H(t)$. However, even for $k = 0$ the equation (4.6) is not easily integrable in closed form.

Secondly, one could imagine a potential $U(\phi)$ which possesses the physically desirable features, and consider (4.6) and (4.7), which, for $k = 0$, form the autonomous nonlinear system

$$\dot{H} = \kappa U(\phi) - 3H^2 , \tag{4.9}$$

$$\dot{\phi} = \pm\sqrt{\frac{2}{\kappa}}\sqrt{3H^2 - \kappa U(\phi)} . \tag{4.10}$$

In the phase space [34], the equilibrium states of this system are given by the constraint $\{\dot{H}, \dot{\phi}\} = 0$. This constraint is fulfilled by $\kappa U(\phi) = 3H^2$, where the Hubble expansion rate is constant, i.e., $H_0 = \sqrt{\Lambda/3}$. For $\dot{\phi} = 0$, we obtain a de Sitter-type inflation with $a(t) = \exp(\sqrt{\Lambda/3}\,t)$.

For $\kappa U(\phi) \neq 3H^2$, we find $\{\dot{H}, \dot{\phi}\} \neq 0$, which implies that the solution $\phi = \phi(t)$ and $H = H(t)$ are *invertible.* Then we can write the potential in (4.9) and (4.10) in the *reparametrized* form

$$U(\phi) = U(\phi(t)) = U(\phi(t(H))) = \widetilde{U}(H) . \tag{4.11}$$

Another question is whether it is possible to construct $H = H(t)$ from the *inverse* function $t = t(H)$ in closed form. Only in this case, the Hubble expansion parameter and the scalar field can be expressed explicitly as a function of time, and the self-interaction potential $U(\phi)$ can be recovered from $\widetilde{U}(H)$.

V. GENERAL METRIC OF A SPATIALLY FLAT INFLATIONARY UNIVERSE

In view of (4.2), (4.3), and (4.11), for $k = 0$, the density and the pressure can be reexpressed as

$$\rho = \frac{3}{\kappa}H^2 , \tag{5.1}$$

$$p = -\rho - \frac{2\dot{H}}{\kappa} \tag{5.2}$$

$$= \rho - 2\widetilde{U} . \tag{5.3}$$

Hence, the density ρ is always a positive function, whereas the pressure p is indefinite and changes sign at $\kappa\widetilde{U} = 3H^2/2$.

For $\kappa\widetilde{U} \neq 3H^2$, we find from (4.9) and (4.11) the formal solution for the coordinate time

$$t = t(H) = \int \frac{dH}{\kappa\widetilde{U} - 3H^2} . \tag{5.4}$$

In formal expressions involving indefinite integrals, we omit the constant of integration. The scale factor in the metric follows from the definition (4.1) of the Hubble expansion rate as $a(t) = a_0 \exp(\int H dt)$ where a_0 is a constant with dimension *length.* Inserting (5.4), we thus can determine the general solution as

$$a = a(H) = a_0 \exp\left(\int \frac{HdH}{\kappa\widetilde{U} - 3H^2}\right) . \tag{5.5}$$

This implies for $k = 0$ that the reparametrized Robertson-Walker metric for inflation reads

$$ds^2 = \frac{dH^2}{\left(\kappa\widetilde{U} - 3H^2\right)^2} - {a_0}^2 \exp\left(2\int \frac{HdH}{\kappa\widetilde{U} - 3H^2}\right) \times \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] . \tag{5.6}$$

Note that the Hubble expansion rate H has become the (inverse) time coordinate. This resembles the reparametrization of Hughston (cf. [2], p. 731) for the Friedmann solution, in which $a(t)$ serves as the new time parameter. In view of (5.4), the general solution of (4.10) for the scalar field can be calculated in terms of the Hubble parameter H as

$$\phi = \phi(H) = \mp\sqrt{\frac{2}{\kappa}}\int \frac{dH}{\sqrt{3H^2 - \kappa\widetilde{U}}} . \tag{5.7}$$

Our general formula (5.7) resembles the Wagoner-Starobinsky transformation from the conformal Brans-Dicke frame to an Einstein frame, cf. [32]. In metric-affine gauge theories of gravity [35] this transformation has a rather natural origin from generalized conformal changes of the metric.

If we introduce the conformal time T via $dt = a(t)\,dT$, the Robertson-Walker metric (3.3) (for $k = 0$) acquires the manifest conformally flat form

$$ds^2 = a^2(t)\left[dT^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] . \tag{5.8}$$

For our general solution, the conformal time can be expressed by the relation

$$dT = \frac{dH}{\kappa\widetilde{U} - 3H^2}\exp\left(\int \frac{HdH}{3H^2 - \kappa\widetilde{U}}\right) . \tag{5.9}$$

Our general solution holds for $k = 0$ and for $\widetilde{U} \neq$

$3H^2/\kappa$. Since this singular case leads to the de Sitter inflation, we try in the explicit models the ansatz

$$\widetilde{U}(H) = \frac{3}{\kappa}H^2 + \frac{g(H)}{\kappa} \tag{5.10}$$

for the potential, where $g(H)$ is a nonzero function for the *graceful exit.*

VI. ALLOWED INFLATIONARY POTENTIALS

Inflation and deflation necessarily occur for all potentials which satisfy the matter condition $\rho + 3p < 0$, i.e., $\ddot{a}(t) > 0$. In order to discriminate inflationary from deflationary models, one has to take into account also the rate of change of the scale factor $a(t)$ or the sign of the Hubble expansion rate, respectively. For *inflation*, we require

$$\ddot{a} > 0\,, \quad \dot{a} > 0 \quad \Longleftrightarrow \quad H > 0\,, \tag{6.1}$$

whereas for *deflation* we require

$$\ddot{a} > 0\,, \quad \dot{a} < 0 \quad \Longleftrightarrow \quad H < 0\,. \tag{6.2}$$

For a classification of the potentials, we follow Ref. [2] (p. 773) and call

$$q(t) := -\frac{\ddot{a}a}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right) \tag{6.3}$$

$$= 2 - \kappa\frac{\widetilde{U}}{H^2} \tag{6.4}$$

the *deceleration parameter.* Because $\dot{a}^2$ and a are positive, an accelerating cosmos ($\ddot{a} > 0$) is described by negative q values. Thus, acceleration can only occur for a potential satisfying

$$\kappa\widetilde{U} > 2H^2\,. \tag{6.5}$$

According to (5.3) the pressure is then necessarily negative and *drives* the inflation. Another constraint is found by looking at the general solution for the scalar field (5.7). The scalar field remains real only if the potential fulfills $\kappa\widetilde{U} < 3H^2$. [Otherwise, we would have a scalar "ghost" in the Lagrangian (3.1).] From Fig. 1, we can read off the different regions of the potential. All values within the parabola, i.e., $\kappa\widetilde{U} > 3H^2$, are forbidden. All points on the curve $\kappa\widetilde{U} = 3H^2$ are singular for our system (4.9) and (4.10) and describe the de Sitter solution. The origin is the flat and empty Minkowski spacetime. Solutions within the domain

$$2H^2 < \kappa\widetilde{U} < 3H^2 \quad \text{and} \quad H > 0\,, \tag{6.6}$$

bounded by parabolas, describe universes with *inflation.* Solutions within the domain

$$2H^2 < \kappa\widetilde{U} < 3H^2 \quad \text{and} \quad H < 0 \tag{6.7}$$

describe universes with *deflation.* If these solutions leave this area through $\kappa\widetilde{U} = 2H^2$, they make contact with a Friedmann cosmos for which $\ddot{a} < 0$.

According to [28], the discrimination between inflation and deflation depends on the choice of the conformal frame. For the scalar matter, we find from (4.8) that $\rho + 3p = 2(\dot{\phi}^2 - \widetilde{U})$. For potential-dominated eras this term is negative and hence inflation can occur. The condition for inflation $\ddot{a} > 0$ is then equivalently to $\dot{H} > -H^2$.

For $k = 0$ and scalar matter, we can infer from (4.6) that always $\dot{H} < 0$, i.e., $-H^2 < \dot{H} < 0$. For other types of matter it is possible that $\dot{H} > 0$, cf., e.g., the spin driven inflation [36]. Such physical models are also called *superinflationary*, in contrast to the *subinflationary* ones [37,38,17] considered here.

Several models are now conceivable which have combinations of in- and deflationary potentials. One can construct models where inflation never ends, those with a combined inflation-Friedmann cosmos, or some where the universe enters the deflationary regime. In the following, we will recover known models from our general formalism and present some new ones, too.

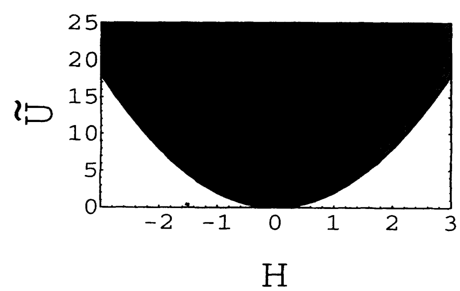

FIG. 1. Classification of inflationary potentials $\widetilde{U}(H)$. $\widetilde{U}$ is measured in units of $(1/\kappa^2)$ and H in units of $(1/\kappa^{1/2})$.

VII. POWER-LAW AND INTERMEDIATE INFLATION

The ansatz

$$g(H) = -A\,H^n\,, \tag{7.1}$$

where n is real and A a positive constant, leads to several known and new solutions. The integration constants C_1, C_2, C_3 are, of course, different in every model. As it turns out, $n = 0, 1, 2$ are special cases which we consider first.

For $n = 0$, we find the following solution:

$$H(t) = -(At + C_1)\,, \qquad a(t) = a_0 \exp\left(-\frac{1}{2A}(At + C_1)^2 + C_2\right)\,, \tag{7.2}$$

and

$$\phi(t) = \pm\sqrt{\frac{2A}{\kappa}}(At + C_1 - C_3)\,. \tag{7.3}$$

The chronology in this model is the following: at first, there is an inflationary phase for which the maximal size of the universe is $a_0 \exp[-C_1/(2A) + C_2]$. This is very extended for $C_1 \ll 0$ or $C_2 \gg 0$. The transition from inflation to the standard Friedmann cosmos occurs at the point $H = +\sqrt{A}$. There exists also a transition from the standard Friedmann cosmos to deflation which occurs at $H = -\sqrt{A}$. This all can also be recognized by looking at the classification diagram (Fig. 1). The self-interaction potential is only *quadratic* as it is normally investigated in the chaotic scenario:

$$U(\phi) = \frac{1}{\kappa}\left[3\left(\sqrt{\frac{\kappa}{2A}}\phi + C_3\right)^2 - A\right] . \tag{7.4}$$

For $n = 1$ we have

$$H(t) = C_1 \exp(-At) ,$$
$$a(t) = a_0 \exp\left(-\frac{C_1}{A}\exp(-At) + \frac{C_2}{A}\right) , \tag{7.5}$$

and

$$\phi(t) = \pm\sqrt{\frac{8}{A\kappa}}\left[\sqrt{C_1}\exp\left(\frac{-At}{2} - C_3\right)\right] , \tag{7.6}$$

so that $C_1 > 0$. The universe in this model starts with an inflationary phase up to the point $H = A$, where it crosses the boundary $\kappa\widetilde{U} = 2H^2$ and evolves towards a conventional Friedmann cosmos. The universe reaches the size $a_0 \exp(C_2/A)$ after infinitely long time. The potential

$$U(\phi) = \frac{A}{8}e^{2C_3}\phi^2\left(\frac{3\kappa A}{8}e^{2C_3}\phi^2 - A\right) \tag{7.7}$$

has here a linear combination of ϕ^2 and ϕ^4 terms which are familiar from the Higgs potential of spontaneous symmetry breaking. For a pure ϕ^4 potential, an exact and an approximate solution is found in Refs. [39] and [40].

For $n = 2$ we find

$$H(t) = \frac{1}{At + C_1} , \quad a(t) = a_0[C_2(At + C_1)]^{1/A} , \tag{7.8}$$

and

$$\phi(t) = \pm\sqrt{\frac{2}{A\kappa}}\ln\left(\frac{1}{C_3(At + C_1)}\right) . \tag{7.9}$$

The self-interaction is the exponential potential

$$U(\phi) = \frac{3 - A}{\kappa}{C_3}^2 \exp(\pm\sqrt{2\kappa A}\,\phi) . \tag{7.10}$$

This case describes power-law inflation $t^{1/A}$ if $0 < A < 1$, which means that $2H^2 < \kappa U < 3H^2$ and $H > 0$. For $A = 3/2$ the pressure (5.3) of the scalar field vanishes and we get $a(t) \simeq t^{2/3}$ as in the matter-dominated Friedmann cosmos [41]. One recognizes that for $A = 3$ the scalar field possesses a vanishing potential (see the Appendix).

For $n \neq 0, 1, 2$ the constant A has the dimension *length*$^{n-2}$. For the Hubble expansion rate, we get

$$H = [A(n-1)(t + C_1)]^{1/(1-n)} , \tag{7.11}$$

whereas the scale factor reads

$$a(t) = a_0 \exp\left[(A(n-1))^{1/(1-n)}\frac{1-n}{2-n} \times (t + C_1)^{(2-n)/(1-n)}\right] . \tag{7.12}$$

The scalar field is then given by

$$\phi(t) + C_3 = \sqrt{\frac{2}{A\kappa}}\frac{2}{2-n}[A(n-1)(t + C_1)]^{(2-n)/(2(1-n))} . \tag{7.13}$$

The corresponding potential reads

$$U(\phi) = \frac{1}{\kappa}\left[\sqrt{\frac{\kappa A}{8}}(2-n)\left(\phi + C_3\right)^{2/(2-n)}\right]\left[3\frac{\kappa A}{8}(2-n)^2\left(\phi + C_3\right)^2 - A\left(\frac{\kappa A}{8}\right)^{n/2}(2-n)^n\left(\phi + C_3\right)^n\right] . \tag{7.14}$$

Hence, we recover the models [19] of intermediate inflation $\exp(t^{(2-n)/(1-n)})$ with $0 < (2-n)/(1-n) < 1$, which is equivalent to $1 < n < 2$. For $(2-n)/(1-n) = 2/3$, the flat Harrison-Zel'dovich spectrum is recovered. Translated into our model this holds for $n = 4$, i.e.,

$$\kappa\widetilde{U} = 3H^2 - AH^4 . \tag{7.15}$$

VIII. NEW POTENTIAL IN THE FRAMEWORK OF CHAOTIC INFLATION

Another model can be obtained from the ansatz

$$g(H) = \pm\frac{4}{C}\sqrt{ACH - H^2}\,H , \tag{8.1}$$

where A, C are constants. For this model we find the scale factor

$$a(t) = a_0 \exp[C \arctan(At + B) + F] . \tag{8.2}$$

See Fig. 2. For

$$a(t) = a_0 \exp[C \operatorname{arccot}(-At) + F] , \tag{8.3}$$

we find the same model, but the scale factor reaches for $t \to \infty$ another limit ($A > 0$): $a_0 \exp(C\pi)$.

The Hubble expansion rate is

$$H(t) = \frac{AC}{1 + (At + B)^2} , \tag{8.4}$$

which vanishes for $t \to \infty$. The solution of the scalar field (Fig. 3) was determined via the computer algebra system MACSYMA as

$$\begin{aligned}\phi(t) = \pm\sqrt{\frac{2C}{\kappa}}\Bigg\{ &\Big[\arctan\Big(\sqrt{2(At+B)}+1\Big) \\ &+ \arctan\Big(\sqrt{2(At+B)}-1\Big)\Big] \\ &-\frac{1}{2}\Big[\ln\Big(At+B+\sqrt{2(At+B)}+1\Big) \\ &-\ln\Big(At+B-\sqrt{2(At+B)}+1\Big)\Big]\Bigg\} + C_1 . \end{aligned} \tag{8.5}$$

In the following, we will only consider this model with positive scalar field. The potential U depending on the time t reads

$$U(t) = \frac{A^2 C}{\kappa} \frac{3C - 2At - 2B}{[1 + (At + B)^2]^2} . \tag{8.6}$$

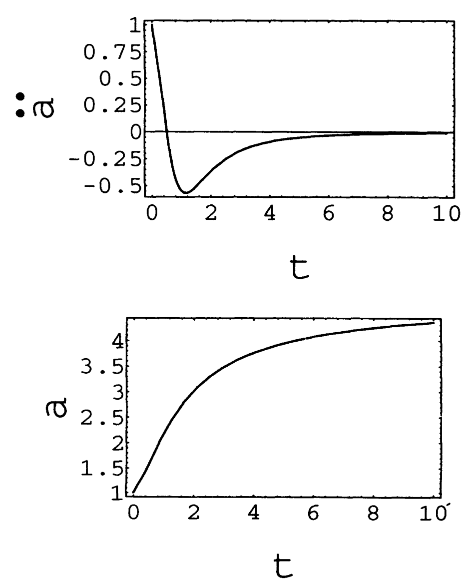

FIG. 2. Below: The scale function $a(t)$ in units of (a_0) and time t in units of $(1/A)$ ($C = 1$). Above: Its second time derivative.

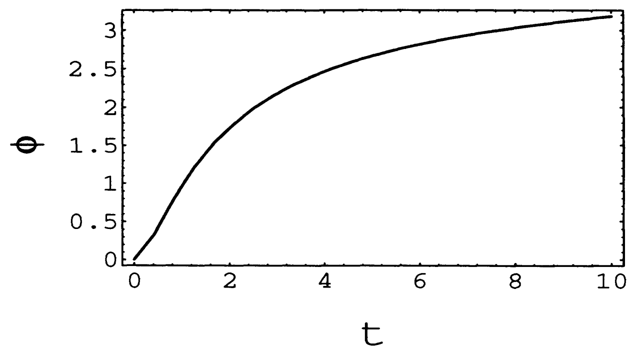

FIG. 3. The scalar field $\phi(t)$ in units of $(2C/\kappa)^{1/2}$ and time t in units of $(1/A)$ ($D = 0$).

Because of the complicated functional dependence of $\phi(t)$, we have not found a closed form of the inverse function, but have numerically determined $U(\phi)$; see Fig. 4. This potential has *no* plateau at the origin $\phi = 0$. But it possesses a minimum with a negative value of U. After the minimum a limiting point follows, for which the potential vanishes: The scalar field needs infinitely long in order to reach this point.

Equation (8.5) gives the constraints $C > 0$ and $At + B > 0$ on the integration constants. Real solutions occur only for initial times $t_i \geq -B/A$. The condition [42] for an inflationary phase is in general given by (6.6). For $A > 0$ we have $\dot{a} > 0$ and find

$$\ddot{a}(t) = \frac{A^2 C \exp[C \arctan(At + B)]}{(1 + B^2 + 2ABt + A^2 t^2)^2}(C - 2At - 2B) . \tag{8.7}$$

In each model, depending on the constants A, B, C, the scale factor starts with the value $a(-B/A) = 1$, the velocity $\dot{a}(-B/A) = AC$, and the acceleration $\ddot{a}(-B/A) = A^2C^2$. Then the universe inflates exponentially up to the time $t_f = (C - 2B)/(2A) \Leftrightarrow \ddot{a} = 0$ (f means final). Then, a positive pressure of the scalar field prevents a further expansion of the universe. Hence, the duration of inflation, is $t_f - t_i = C/(2A)$. Only the constant C determines the strength of inflation, whereas both constants A and C determine the duration of the inflation.

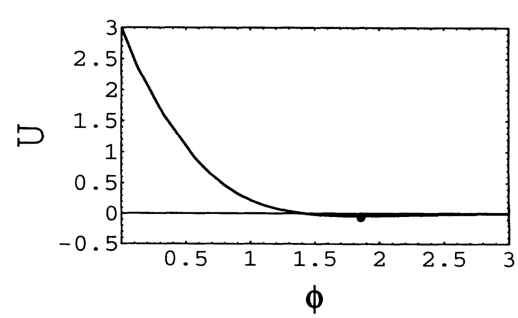

FIG. 4. The numerically determined potential $U(\phi)$ in units of (A^2C/κ) and scalar field ϕ in units of $(2C/\kappa)^{1/2}$, where we have set $C = 1$.

The constant B has no geometrical meaning. At the end of inflation the scale factor is constant. After infinitely long time, in practice, very soon after the inflation phase, a Minkowski spacetime emerges.

It is also possible to take into account a de Sitter-type expansion, i.e., the Hubble expansion rate is becoming constant after a short starting phase determined by the new model. We may connect the two models by

$$a(t) = a_0 \exp[C \arctan(At + B) + Dt + E] \ . \qquad (8.8)$$

Then, we find

$$H = \frac{AC}{1 + (At + B)^2} + D \longrightarrow D \ . \qquad (8.9)$$

We get the same solution for the scalar field, whereas the time-dependent potential is changed to

$$U(t) = \frac{3}{\kappa}\left(\frac{AC}{1 + (At + B)^2} + D\right)^2 - \frac{2A^2}{\kappa}\frac{(At + B)C}{[1 + (At + B)^2]^2} \ , \qquad (8.10)$$

so that in the limit $t \to \infty$ we have: $U \to 3D^2/\kappa$. The disadvantage of this model is that the de Sitter inflation very soon plays the decisive role and inflation never ends.

IX. EXACT SOLUTION OF NEW INFLATION

For the graceful exit function, we consider a polynomial in H up to second order, i.e.,

$$g(H) = \frac{1}{G}H^2 + \left(D - \frac{2A}{G}\right)H + \frac{A^2}{G} - AD \ , \qquad (9.1)$$

where A, D, G are constants. Again, it is possible to calculate the model completely. For the Hubble expansion rate, we find

$$H(t) = A - \frac{DG \exp(Dt + F)}{1 + \exp(Dt + F)} \ . \qquad (9.2)$$

In the limit of infinitely long time, H is becoming the constant $A - DG$. For this universe the scale factor reads (Fig. 5)

$$a(t) = a_0 \frac{\exp(At + K)}{[1 + \exp(Dt + F)]^G} \ . \qquad (9.3)$$

From (5.7) we get the scalar field (Fig. 6)

$$\phi(t) = \pm\sqrt{\frac{8G}{\kappa}} \arctan\left[\exp\left(\frac{Dt + F}{2}\right)\right] + C \ . \qquad (9.4)$$

The constants F, K, C are further integration constants. We read off the reality condition $G > 0$.

The potential is given by

$$U(\phi) = \frac{1}{\kappa\left[1 + \tan^2\left(\pm\sqrt{\kappa}(\phi - C)/\sqrt{8G}\right)\right]^2}\left[3A^2 + \left(6A^2 - 6ADG - D^2G\right)\tan^2\left(\pm\sqrt{\frac{\kappa}{8G}}(\phi - C)\right) + 3\left(A - DG\right)^2 \tan^4\left(\pm\sqrt{\frac{\kappa}{8G}}(\phi - C)\right)\right] \ , \qquad (9.5)$$

cf. Fig. 7.

The constants A and D have the dimension $1/length$. For large times, the scale factor (9.3) reaches the value

$$a(t) \simeq a_0 \exp[(A - DG)t] \ . \qquad (9.6)$$

One recognizes that for $A \neq DG$ the scale factor is either exponential in- or decreasing depending on the sign of $A - DG$. Hence, in these cases we find models with either in- or deflationary behavior. Only for $A = DG$, we have a limiting value. In this case, the constant a_0 determines the limiting value a_∞. From the potential (9.5), we can distinguish three types of different physical behavior of our model universe.

Hence, in this model, a fine-tuning problem arises. Only if three constants fulfill the exact condition $A = DG$ is the graceful exit secured. In all other cases, the inflation can first stop, then occurs again or the resulting universe will recollapse to its original state.

For $A = DG$, the potential $U(\phi)$ is vanishing with increasing scalar field. These models are the exact solutions of the new inflationary theory.

For $A < DG$, the potential $U(\phi)$ reaches a positive value with increasing scalar field. Therefore a *deflation* of the universe is born later. The chronology of this model is inflation, Friedmann cosmos, deflation. The universe contracts again after a maximal radius.

For $A > DG$, the inflationary phase first ends, then it is renewed and never ends. In order to be precise, for increasing deviations of A from DG, the limiting value of the potential becomes more and more negative, and finally the minimum of the potential vanishes. Hence the chronology is the following inflation, Friedmann cosmos, inflation. The duration of the Friedmann phase decreases with increasing A and fixed product DG. The slow-roll approximation occurs again.

Because all models start with an inflation phase, we can first calculate the end of inflation, which we define by $\ddot{a}(t_f) = 0$. It occurs at the time t_f which is implicitly given by

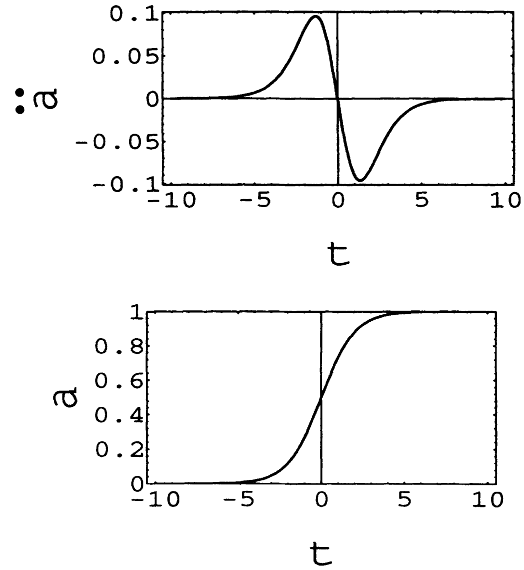

FIG. 5. Below: The scale factor $a(t)$ in units of (a_0) and time t in units of $(1/A)$ for the graceful exit solution with $A = DG$ and $F = K = 0$. Above: The second time derivative of the scale factor $a(t)$ in the same units determines the different parts of the universe model. For $\ddot{a} > 0$ the universe is inflationary.

$$e^{Dt_f+F} = \frac{1}{2(A-DG)^2}\Big[-2A^2 + 2ADG + D^2G \quad (9.7)$$

$$\pm\sqrt{D^2G(D^2G + 4ADG - 4A^2)}\Big] . \quad (9.8)$$

For the case $A = DG$ we find

$$t_f = -\frac{F}{D} + \frac{1}{D}\ln G . \quad (9.9)$$

There exist several constraints on the constants in order to fulfill today's astrophysical estimates [43]. We suppose that inflation ends at $t_f := \tau = 10^{-35}$ s. At the Planck time $t_i = 10^{-43}$ s the initial extension of the universe was the Planck length $\ell_0 \simeq 10^{-33}$ cm where the inflation has started. Nowadays, the Hubble expansion rate is $H(t_f) = 3.24 \times 10^{-18}\ \mathrm{s}^{-1}$.

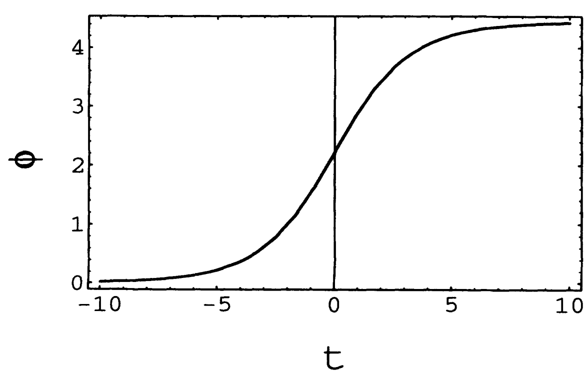

FIG. 6. The scalar field $\phi(t)$ in units of $(8G/\kappa)^{1/2}$ and t in units of $(1/A)$ where we have set $C = F = 0$. It monotonically increases to a nonvanishing limit.

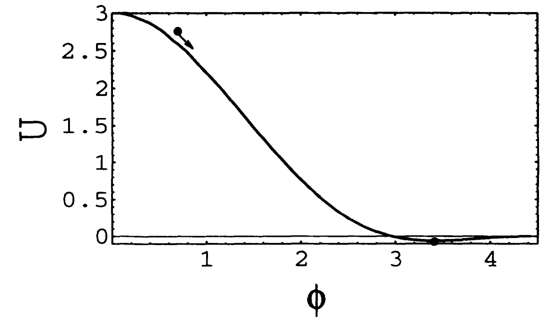

FIG. 7. The potential $U(\phi)$ in units of (A^2/κ) and the scalar field ϕ in units of $(8G/\kappa)^{1/2}$. We have set $G = 1$ and $C_1 = 0$. It has all features which are demanded by the theory of the new inflation, but with one exception: it does not possess an increasing potential wall after the inflationary phase. Instead we have there a limiting point which prevents the scalar field from reaching lower points on the potential. The limiting point simulates the potential wall for large ϕ values.

Let us concentrate on the graceful exit case, i.e., $A = DG$. There we set $G = 1$. Because the inflation is arising till 10^{-35} s, we set the constants $A = D = 10^{36}$/s. In order to get at least our size of the present universe, Eq. (9.6) requires $a(t_f) \simeq a_0 = 10^{28}$ cm. We find from (9.9)

$$F = \ln G - D\tau = 130.5 . \quad (9.10)$$

Using the Planck length at t_i gives the condition $K = 4.54 \times 10^{-5}$.

At the origin $\phi = 0$, the potential (9.5) possesses the following power series expansion (the inflationary part)

$$U(\phi) \simeq \frac{3\,A^2}{\kappa} + \left(\frac{-3\,A\,D}{4} - \frac{D^2}{8}\right)\phi^2 + \left(\frac{3\,D^2\,\kappa}{64} + \frac{A\,D\,\kappa}{32\,G} + \frac{D^2\,\kappa}{48\,G}\right)\phi^4 + O(\phi)^5 . \quad (9.11)$$

It is also interesting to calculate the minimum of the potential U. This minimum occurs at the time $t_{\min}$, where

$$e^{Dt_{\min}+F} = \frac{6A + D}{6DG + D - 6A} . \quad (9.12)$$

Observe that no minimum exists if the right-hand side is negative or zero. For the constraint $A = DG$, we can find a minimum in every case. For this special choice of constants, we find

$$t_{\min} = \frac{1}{D}\left[\ln(6G+1) - F\right] , \quad (9.13)$$

and a negative value of the potential at the minimum

$$U(t_{\min})_{A=DG} = -\frac{D^2G}{4\kappa(1+3G)} . \quad (9.14)$$

After the initial submission of the present paper, further exact inflationary solutions were published [44]. The first solution is a special case of (9.1) for $DG = 2A$, i.e.,

$$g(H) = -2H^2/\hat{A}^2 + 2\hat{A}^2\lambda^2 \tag{9.15}$$

(in the notation of [44]). Then the solution reads

$$\phi(t) = \hat{A}\ln[\tanh(\lambda t)] , \tag{9.16}$$

$$H(t) = \hat{A}^2\lambda\coth(2\lambda t) , \tag{9.17}$$

$$a(t) = a_0[\sinh(2\lambda t)]^{\hat{A}^2/2} , \tag{9.18}$$

$$U(\phi) = \hat{A}^2\lambda^2\left[(3\hat{A}^2 - 2)\cosh^2\left(\frac{\phi}{\hat{A}}\right) + 2\right] . \tag{9.19}$$

The second solution of [44] corresponds, in our notation, to the following ansatz:

$$g(H) = 3\hat{A}^{-10/3}\lambda^{-2/3}6^{2/3}H^{8/3} + \hat{A}^{-2}(3\hat{A}^2 - 9)H^{6/3} - \frac{3}{2}6^{1/3}\hat{A}^{-2/3}\lambda^{2/3}(\hat{A}^2 + 1)H^{4/3} + \frac{6^{2/3}}{12}\hat{A}^{2/3}\lambda^{4/3}H^{2/3} . \tag{9.20}$$

In the notation of [44] the solution is then

$$\phi(t) = \hat{A}\,\mathrm{csch}(\lambda t) , \tag{9.21}$$

$$H(t) = \frac{\hat{A}^2\lambda}{6}\coth^3(\lambda t) , \tag{9.22}$$

$$a(t) = a_0[\sinh(2\lambda t)]^{\hat{A}^2/2}\exp\left[-\frac{\hat{A}^2}{12}\coth^2(\lambda t)\right] , \tag{9.23}$$

$$U(\phi) = \frac{\lambda^2}{12\hat{A}^2}\phi^2(\phi^2 + \hat{A}^2)\left(\frac{\phi^4}{\hat{A}^4} + 2\phi^2 + \hat{A}^2 - 6\right) . \tag{9.24}$$

X. DEFLATIONARY UNIVERSES

One could suppose that the Hubble expansion rate increased in the beginning of the Universe and then became constant. Having this idea in mind, we try the ansatz

$$g(H) = \frac{AC}{1 + \tan^2(H/C)} = AC\cos^2(H/C) , \tag{10.1}$$

which yields

$$H = C\arctan(At + B) . \tag{10.2}$$

Then the scale factor is

$$a(t) = a_0\frac{\exp\left[(Ct + BC/A)\arctan(At + B) + F\right]}{[1 + (At + B)^2]^{C/(2A)}} , \tag{10.3}$$

and the scalar field reads

$$\phi(t) = \sqrt{-\frac{2C}{A\kappa}}\mathrm{arsinh}(At + B) - D . \tag{10.4}$$

The potential is given by (Fig. 8)

$$U(\phi) = \frac{1}{\kappa}\left\{3C^2\arctan^2\left[\sinh\left(\sqrt{-\frac{A\kappa}{2C}}(\phi + D)\right)\right]\right. \tag{10.5}$$

$$\left. + \frac{AC}{1 + \sinh^2\left(\sqrt{-\frac{A\kappa}{2C}}(\phi + D)\right)}\right\} . \tag{10.6}$$

In order to have a real scalar field solution, we have to require $C/A < 0$. A more detailed investigation of the behavior of the scale factor shows that for negative $At + B$ we find an inflationary phase before $a(t)$ reaches a maximum. For positive $At + B$, a deflationary phase starts.

A further solution is found if one supposes that the Hubble parameter is not becoming constant but is increasing logarithmically

$$H = C\ln(At + B) , \tag{10.7}$$

which follows from the ansatz

$$g(H) = AC\exp(-H/C) . \tag{10.8}$$

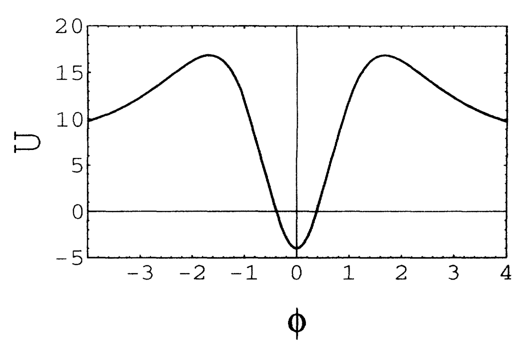

FIG. 8. The potential $U(\phi)$ in units of (C^2/κ) and the scalar field ϕ in units of $[-2C/(A\kappa)]^{1/2}$ for $g(H) = AC\cos^2(H/C)$. We have put $A = C = 1$ and $D = 0$.

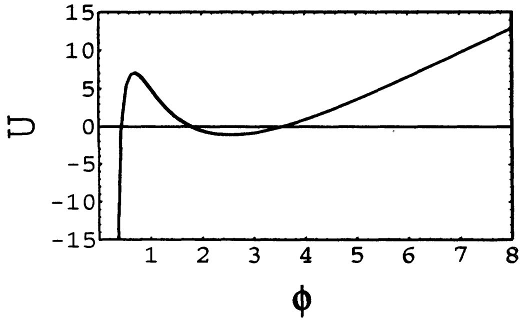

FIG. 9. The potential $U(\phi)$ in units of (C^2/κ) and the scalar field ϕ in units of $[-8C/(A\kappa)]^{1/2}$ for $g(H) = AC\exp(-H/C)$. We have put $A = C = 1$ and $D = 0$.

Then we find the exact solution (Fig. 9)

$$a(t) = a_0(At+B)^{(Ct+BC/A)}e^{-(Ct+BC/A)+F} , \tag{10.9}$$

$$\phi(t) = \sqrt{-\frac{8C}{A\kappa}}\sqrt{At+B} - D , \tag{10.10}$$

$$U(\phi) = \frac{1}{\kappa}\left\{3C^2\left[\ln\left(-\frac{A\kappa}{8C}(\phi+D)^2\right)\right]^2 - \frac{8C^2}{\kappa}\frac{1}{(\phi+D)^2}\right\} . \tag{10.11}$$

For a real scalar field, the constraints $C/A < 0$ and $\sqrt{At+B} > 0$ have to be fulfilled. The inflation starts at the time $t_i = -B/A$ with a short increasing (because of the exponential function in the scale factor), but then decreases very rapidly to zero (because it becomes a function of the type $1/t^t$). Hence, the solution has a purely deflationary character. However, the discrimination between inflation and deflation is also depending on the conformal frame [45].

XI. DENSITY PERTURBATIONS

For a long time it was thought that the spectrum of density perturbations was described by the scale-invariant Harrison-Zel'dovich form [46–48]. But new observations by COBE [49] show the possibility of small deviations. The spectra of scalar and transverse-traceless tensor perturbations [50,51] are given by

$$P_{\mathcal{R}}^{\frac{1}{2}}(\hat{k}) = \left(\frac{H^2}{4\pi\mid H'\mid}\right)\Bigg|_{aH=\hat{k}} , \tag{11.1}$$

$$P_g^{\frac{1}{2}}(\hat{k}) = \left(\frac{H}{2\pi}\right)\Bigg|_{aH=\hat{k}} , \tag{11.2}$$

where $\mathcal{R}$ denotes the perturbation in the spatial curvature, $H' = dH/d\phi$, and $\hat{k}$ the wave number. The expressions on the right-hand side have to be evaluated at that comoving scale $\hat{k}/a$ which is leaving the horizon during the inflationary phase. The results are only valid in first order slow-roll approximation [51]. The scalar and the gravitational spectral indices in first order approximation read

$$n_s := 1 + \frac{d\ln P_{\mathcal{R}}}{d\ln\hat{k}} = 1 - 4\epsilon + 2\eta , \tag{11.3}$$

$$n_g := \frac{d\ln P_g}{d\ln\hat{k}} = -2\epsilon , \tag{11.4}$$

where the two slow-roll parameters ϵ and η are defined [44] as follows:

$$\epsilon := 3\frac{\dot{\phi}^2}{2U+\dot{\phi}^2} = \frac{2}{\kappa}\left(\frac{H'}{H}\right)^2 , \tag{11.5}$$

$$\eta := -3\frac{\ddot{\phi}}{3H\dot{\phi}} = \frac{2}{\kappa}\frac{H''}{H} . \tag{11.6}$$

In general, they are scale dependent and have to be evaluated at the horizon. The parameter ϵ describes the relation between the kinetic and the total energy, whereas η is a measure for the relation between the "acceleration" of the scalar field and its "curvature-dependent velocity." In the slow-roll approximation ϵ and η are small quantities. Actually, the phase of acceleration (the slow-roll approximation, $\ddot{a} > 0$) is now equivalently to the condition $\epsilon < 1$. The *flat spectrum* of the Harrison-Zel'dovich form is obtained for $n_s = 1$.

By using (4.10), the first slow-roll parameter is given by

$$\epsilon = -\frac{g}{H^2} . \tag{11.7}$$

After inserting (4.4) and (4.6), the second parameter can be expressed as

$$\eta = 3 - \frac{\kappa}{2H}\frac{d\tilde{U}}{dH} = -\frac{1}{2H}\frac{dg}{dH} . \tag{11.8}$$

Thus, the condition $n_s = 1$ for a flat spectrum of the Harrison-Zel'dovich type converts into the relation

$$H\frac{dg}{dH} = 4g \tag{11.9}$$

for the graceful exit function g. This Euler-type relation for homogeneous functions is solved by $g = CH^4$, which corresponds to our earlier Eq. (7.15); cf. [42].

In the era of inflation, the scale of the universe has to explode at least by a factor $e^{60} \simeq 10^{30}$. The number of e foldings between scalar field values ϕ_1 and ϕ_2 is given by

$$\ln a(\phi_1,\phi_2) = -\frac{\kappa}{2}\int_{\phi_1}^{\phi_2}\frac{H(\tilde{\phi})}{H'(\tilde{\phi})}d\tilde{\phi} . \tag{11.10}$$

In the new two models of chaotic and new inflation, the slow-roll phase appears for the regime of small ϕ values. Here we investigate only the new inflationary model because of its explicit ϕ dependence. Only for $A = DG$, we find an inflationary model for our universe which works rather well. Hence, we can calculate (for small ϕ values up to order ϕ^2)

$$n_s \simeq 1 - \frac{1}{\kappa G} - \frac{3\phi^2}{8\kappa G^2} , \tag{11.11}$$

$$n_g \simeq -\frac{\phi^2}{4\,\kappa G^2} \,. \tag{11.12}$$

Observe that the constant $1/G$ as well as ϕ occurs in second order. Only if $G \gg 1$ we find a nearly scale-invariant spectrum. For very large G we have $g = 0$, i.e., the de Sitter solution. For $\kappa G = 1$ the scalar spectral index is very small and proportional to ϕ^2.

Of further interest is the relative contribution of the tensor and scalar modes to the microwave background signal. The lth multipole of the spherical harmonic expansion [51] of the anisotropy of the temperature is, on this scale, given by

$$R_l(\epsilon) := \frac{\Sigma_l^2(\text{tensor})}{\Sigma_l^2(\text{scalar})} \simeq 12.4\,\epsilon \simeq \frac{12.4\,\phi^2}{8\,\kappa G^2} \,. \tag{11.13}$$

The scalar field ϕ has to be evaluated at that time scale where the corresponding lth multipole leaves the horizon. This result is very similar to the one found for the intermediate inflation (see Eq. (21) of Ref. [42]).

The relation between the relative amplitude and the scalar spectral index is given by

$$n_s \simeq 1 - \frac{1}{\kappa G} - \frac{3\,R_l}{12.4} \,. \tag{11.14}$$

Again, we can read off that, for $G \gg 1$, we have a scale-independent spectrum.

It is quite useful to compare this result with those calculated from other inflationary models. For the intermediate inflation,

$$n_s \simeq 1 + \frac{n-4}{12.4} R_l \tag{11.15}$$

was found [42], whereas for the power-law inflation,

$$n_s \simeq 1 - \frac{R_l}{6.2} \tag{11.16}$$

holds. In the last case, both n_s and R_l are scale independent.

We can now compare these ϕ-dependent results with those emerging from the H dependence. For $A = DG$ we find

$$\epsilon = -\frac{1}{G}\left(1 - \frac{DG}{H}\right) \,. \tag{11.17}$$

The inflation occurs for $|\ H - DG\ | \ll 1$. The scalar spectral index reads

$$n_s = 1 + \frac{2}{G} - \frac{3D}{H} \,. \tag{11.18}$$

Hence, in the era of inflation,

$$n_s \simeq 1 - \frac{1}{G} \tag{11.19}$$

holds. The relation between R_l and n_s reads

$$n_s = 1 - \frac{4\,R_l}{12.4} - \frac{2\kappa}{G} + \frac{D\kappa}{H} \,, \tag{11.20}$$

which in the inflationary era reduces to (11.14).

ACKNOWLEDGMENTS

We would like to thank Peter Baekler, Friedrich W. Hehl, and Yuval Ne'eman for helpful comments. We are grateful to Yuri Obukhov for some instructive remarks on the critical points of our system of nonlinear first order equations and to Vitaly N. Melnikov for drawing our attention to Ref. [30]. Research support for F.E.S. was provided by the Deutsche Forschungsgemeinschaft, Project No. He 528/14 − 1, whereas E.W.M. thanks D. Stauffer for his leave of absence from Cologne, paid by the Canada Council.

APPENDIX A: SOLUTIONS FOR CONSTANT SELF-INTERACTION POTENTIAL

In this appendix, we derive all solutions for a constant self-interaction potential $U(\phi) = \Lambda/\kappa$ for closed, open, and flat universes. Then, a first integral of the Klein-Gordon equation (4.4) is

$$\dot{\phi} = \frac{C_1}{\sqrt{\kappa} a^3} \,. \tag{A1}$$

Equation (4.2) yields

$$\dot{a}^2 + k = \frac{1}{3}\Lambda a^2 + \frac{{C_1}^2}{6a^4} \,, \tag{A2}$$

where we have replaced H by the scale factor $a(t)$.

The general integral of Eq. (A2) reads

$$t = \pm\sqrt{6}\int \frac{a^2\,da}{\sqrt{2\Lambda a^6 - 6ka^4 + {C_1}^2}} \,, \tag{A3}$$

for which C_2 is a second integration constant; cf. [2], p. 731 for the reparametrization of the time coordinate.

For $U = 0$, we obtain $H = 1/(3t)$ as the solution of (4.9).

The integral (A3) belongs to the *hyperelliptic integrals* ([52], part 251, 6). It can be transformed by means of $y := a^2$ into an *elliptic integral* ([52], part 244),

$$t = \pm\frac{\sqrt{6}}{2}\int \frac{y\,dy}{\sqrt{2\Lambda y^4 - 6ky^3 + {C_1}^2 y}} \,. \tag{A4}$$

The solutions depend on the special form of the zeros of the quartic equation

$$2\Lambda y^4 - 6ky^3 + {C_1}^2 y = 0 \,. \tag{A5}$$

Case $k = \Lambda = 0$. The solution reads

$$a(t) = \left(\pm C_1\sqrt{\frac{3}{2}}t + C_2\right)^{1/3} \tag{A6}$$

and

$$\phi(t) = \pm\sqrt{\frac{2}{3\kappa}}\ln\left(\pm C_1\sqrt{\frac{3}{2}}t + C_2\right) + C_3 \,. \tag{A7}$$

Case $k=0$ and $\Lambda \neq 0$. For $\Lambda > 0$, the solution is

$$a(t) = \left(\frac{C_1{}^2}{2\Lambda}\right)^{1/6} \sinh^{1/3}[\pm\sqrt{3\Lambda}(t+C_2)] \,, \qquad \text{(A8)}$$

$$\phi(t) = \pm\sqrt{\frac{2}{3\kappa}} \ln\tanh\left(\pm\frac{\sqrt{3\Lambda}}{2}(t+C_2)\right) + C_3 \,. \qquad \text{(A9)}$$

For $\Lambda < 0$, the hyperbolic functions are converted into the trigonometric ones:

$$a(t) = \left(-\frac{C_1{}^2}{2\Lambda}\right)^{1/6} \sin^{1/3}[\pm\sqrt{-3\Lambda}(t+C_2)] \,, \qquad \text{(A10)}$$

$$\phi(t) = \pm\sqrt{\frac{2}{3\kappa}} \ln\tan\left(\pm\frac{\sqrt{-3\Lambda}}{2}(t+C_2)\right) + C_3 \,. \qquad \text{(A11)}$$

Case $\Lambda = 0$ and $k = \pm 1$. We have

$$t = \pm \int \frac{a^2\, da}{\sqrt{C_1{}^2/6 - \varepsilon a^4}} \,, \qquad \text{(A12)}$$

where $\varepsilon = 1$, for $k=+1$, and $\varepsilon = i$, the imaginary unit, for $k=-1$. This integral is again of elliptic type. In terms of the elliptic integrals

$$F(\varphi, \tilde{k}) = \int_0^{\varphi} \frac{d\psi}{\sqrt{1-\tilde{k}^2 \sin^2\psi}} \qquad \text{(A13)}$$

and

$$E(\varphi, \tilde{k}) = \int_0^{\varphi} \sqrt{1-\tilde{k}^2 \sin^2\psi}\, d\psi \qquad \text{(A14)}$$

of the first and second kind, respectively, we find the solution

$$t + C_2 = \pm\frac{1}{\sqrt{2}}\frac{1}{\sqrt{\varepsilon^3}}\left(\frac{C_1{}^2}{6}\right)^{1/4}\left[2E\left(\varphi, \frac{1}{\sqrt{2}}\right) - F\left(\varphi, \frac{1}{\sqrt{2}}\right)\right] \,, \qquad \text{(A15)}$$

where $a = -(1/\sqrt{\varepsilon})\left(C_1{}^2/6\right)^{1/4}\cos\varphi$.

For $k=+1$, the solution is also given by the integral tables of Gröbner and Hofreiter ([52], part 244, 8b15c), whereas for $k=-1$, we find the solution in ([52], p. 91, part 244, 8c8) (with $s := r_1 = s_1 = s_2 = -r_2 = \sqrt{C_1}/(24)^{1/4}$).

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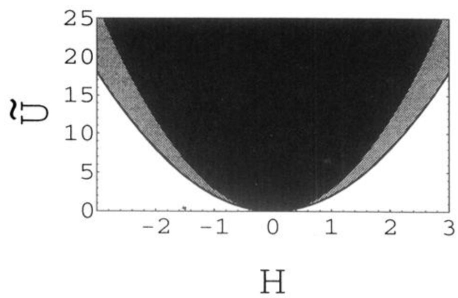

FIG. 1. Classification of inflationary potentials $\widetilde{U}(H)$. $\widetilde{U}$ is measured in units of $(1/\kappa^2)$ and H in units of $(1/\kappa^{1/2})$.