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Boulware state and the generalized second law of thermodynamics

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We show that the appropriate vacuum state for the interior of a box with reflecting walls being lowered adiabatically toward a Schwarzschild black hole is the Boulware state. This is concordant with the results of Unruh and Wald, who used a different approach to obtain the stress-energy inside the box, but does not agree with Li and Liu, who only consider the quantum state outside of the box.

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I. INTRODUCTION

One of the most remarkable developments in gravitational theory in the last century has been the discovery that fields quantized on a black hole background exhibit thermodynamical properties [1]. This discovery was presaged by the work of Bardeen, Carter, and Hawking [2] in which they pointed out an analogy between laws governing certain properties of black holes and the laws of ordinary thermodynamics. In particular, the analog of the second law of thermodynamics is Hawking's theorem that the surface area of a black hole is nondecreasing [3]: i.e.,

$$\frac{dA_{\rm BH}}{d\tau} \geq 0. \tag{1.1}$$

It was based on this analogy between (1.1) and the second law of thermodynamics that Bekenstein [4] conjectured a generalized second law (GSL) of thermodynamics: *The sum of the black hole entropy and the ordinary entropy in the black hole exterior never decreases.* More precisely, the GSL states that for any physical process

$$\delta S_{\rm matter} + \frac{1}{4}\delta A_{\rm BH} \geq 0 \tag{1.2}$$

(units $\hbar = c = G = k = 1$), where $S_{\rm matter}$ is the entropy of the matter outside the black hole. In (1.2), $\frac{1}{4}A_{\rm BH}$, one-quarter of the black hole's surface area, plays the role of the entropy of the black hole. This correspondence between the surface area and entropy of a black hole has become firmly established in the context of black hole thermodynamics, beginning with Hawking's discovery of the thermal radiation emitted by a black hole [1].

Bekenstein [5] further argued that an entropy bound on matter was required in order for the GSL to hold. His argument relied on the following gedankenexperiment. Imagine that a box of linear dimension R with reflecting walls is filled with ordinary matter of energy E_∞ and entropy S at a very large proper distance from a black hole. The box is then slowly (adiabatically) lowered toward the black hole of mass M. When the box is opened and the matter released into the black hole, the energy of the matter will have been reduced by the redshift factor $\chi = (1 - 2M/r)^{1/2}$ so that the black hole's energy is increased by

$$E = \chi E_\infty. \tag{1.3}$$

Since we can lower the box to approximately the distance R (the dimension of the box) from the event horizon before releasing the energy into the black hole, we can provide the black hole with as little as $E = [1-2M/(R+2M)]^{1/2}E_\infty$ energy. However, as Bekenstein demonstrated, this will lead to a change in the black hole entropy of

$$\delta S_{\rm BH} = \frac{1}{4}\delta A_{\rm BH} = 8\pi ME. \tag{1.4}$$

After the box is emptied, it can be slowly pulled back out to infinity. But observe that, if $R < S/(2\pi E_\infty)$, we will have $\delta S_{\rm BH} < S$ and the GSL will be violated. Therefore, Bekenstein concluded there was a bound on the entropy of matter with energy E that could be placed in a box of dimension R:

$$S/E \leq 2\pi R. \tag{1.5}$$

Unruh and Wald [6] (UW) have pointed out, however, that Bekenstein failed to consider black hole quantum effects in his analysis. In particular, they point out the effect of acceleration radiation on the box as it is being lowered. Since, in the reference frame of the almost stationary (hence accelerated) box, the black hole is surrounded by a bath of thermal radiation, there will be an upward pressure on the box. In fact, when this is taken into account, Unruh and Wald demonstrate that a box of negligible height will float when the energy contained in the box, E, is exactly the same as the energy of the acceleration radiation displaced by the box. In order to lower the box further, one will have to do work against this buoyancy force. Unruh and Wald go on to show that in order to minimize the entropy increase of the black hole, the box must be opened at the floating point. They further show that the matter released at this point will contribute at least enough energy to the black hole to increase its entropy by an amount

$$\delta S_{\rm BH} \geq S, \tag{1.6}$$

where S is the entropy of the matter in the box. Thus, they conclude, the GSL will hold independently of the validity of (1.5).

More recently, Li and Liu [7] have stated that the belief of Unruh and Wald that the Hawking radiation is thermal near the black hole is in error. In support of this statement they use the approximate stress-energy tensor for a massless scalar field surrounding a black hole found by Page [8]. Based on this stress energy they find that the pressure of the Hawking radiation near the black hole is not large enough to produce a substantial buoyancy effect, and they derive a bound on the entropy very similar to (1.5) of Bekenstein.

Page's approximate stress energy is for the Hartle-Hawking state associated with a conformally coupled massless scalar field in a Schwarzschild background. This is an adequate description of the state *outside* the box as seen by a freely falling observer. However, as demonstrated by UW, in the frame of a stationary (accelerated) observer the box is subject to a bath of acceleration radiation. Equivalently, UW show that in the frame of a freely falling (inertial) observer, this acceleration can be thought of as affecting the energy density *inside* the box. Li and Liu consider neither this change in the energy density inside the box nor the effect of acceleration radiation in their analysis of the buoyancy effects.

How can the acceleration of the reflecting walls of the box affect the energy inside? Let us first answer this question for a box of fixed proper length accelerating in flat space. It is well established that, when quantum effects are considered, a mirror experiencing nonuniform acceleration will radiate two fluxes of energy proportional to the change in acceleration, $dE/d\tau \propto da/d\tau$ [9]. One of these fluxes will be in the direction of the change in the acceleration of the mirror, and will have negative energy. The other, in the opposite direction, will have positive energy.

We first consider the situation from the point of view of an inertial observer watching an empty mirrored box accelerate from left to right. If the box increases its acceleration, two fluxes of energy will enter the box. The flux from the rear (left) wall will be negative and the flux from the front (right) wall will be positive. However, these fluxes will not be equal. As the box accelerates, it will undergo Lorentz contraction. The rear wall will therefore be forced to accelerate, and change its acceleration, at a higher rate than the front wall, and will thus emit a larger flux. As a result, the inertial observer sees a negative energy density developing inside the box.

Now, let us consider the situation from the point of view from an observer inside the box, accelerating with it. This observer does not notice a negative energy density developing inside the box. Indeed, this observer, who started in the empty Minkowski vacuum, still believes that the interior of the box is (apart from himself) empty. Thus, with respect to what he sees as the vacuum state, the exterior of the box is filled with a positive energy fluid. This fluid is none other than the acceleration radiation described by UW. It should be emphasized that the bath of acceleration radiation seen by the accelerating observer is an artifact of this observer measuring energy with respect to the vacuum of his non inertial (accelerated) frame. The true stress energy is properly described by the inertial observer, who sees a negative energy density inside the box.

Let us now return to the case of the rigid box being lowered toward a black hole. Both the top and bottom reflecting walls will undergo a change of acceleration when lowered. The positive energy flux from both mirrors will be toward the horizon, the negative energy fluxes away from the horizon. Thus, as we lower the box, positive energy will flow from the mirror at the top of the box into the box's interior, while at the same time, negative energy will flow from the bottom mirror into the box. But for a box of fixed proper length, the change in acceleration during lowering is larger at the bottom than at the top. Therefore, the flux from the bottom mirror will be larger, and there is a net negative energy flow into the box. The interior of a box which is initially empty will consequently acquire a negative energy density through the lowering process. This negative energy density is exactly what one would expect from the Boulware state.

Using a (1+1)-dimensional model, which we believe captures the essential features of the problem, we will show that energy of the contents of the box of negligible height is indeed correctly measured with respect to the (negative) energy of the Boulware state. Furthermore, it is easy to show that the measurement of the box's internal stress-energy with respect to the Boulware state is in full agreement with UW.

We wish to stress, however, that this does not necessarily lend credence to UW's conclusion that the buoyancy is sufficient to preserve the GSL on its own. The necessity of a bound of the form (1.5) is, in our opinion, still an open question. Indeed, Bekenstein has recently pointed out that the floating point for such a box will be at a proper distance above the event horizon similar to the height of the box [10]. In this case, our assumption that the height of the box is negligible is not valid near the floating point, and thus we will not be able to draw a conclusion about the GSL from it. Therefore, we will not consider here the floating condition, nor the effect of the buoyancy on the validity of the GSL.

Rather, we wish to simply point out that the effect of the accelerating mirrors on the quantum fields cannot be ignored in analyzing the buoyancy, and that their contribution can be understood as imposing a Boulware state background for the matter inside the box. We also wish to emphasize that we obtain the Boulware vacuum energy from considering only the acceleration of the reflecting walls of the box in a *flat* background. This, we feel, is a remarkable result which may be exploited more fully in the future.

II. (1+1)-DIMENSIONAL BLACK HOLE VACUUM STATES

Let us begin by considering a (1+1)-dimensional black hole with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)},$$
$$f(r_0) = 0, \quad f'(r_0) = 2\kappa, \tag{2.1}$$

where r_0 is the horizon radius, f' denotes df/dr, and κ is a constant. The redshift factor for metric (2.1) is $\chi = \sqrt{f}$ and $dz := dr/\sqrt{f}$ defines z which measures the proper distance from the horizon.

Let us introduce null coordinates u and v defined by

$$dv := dt + \frac{dr}{f(r)} = -n_a dx^a,$$
$$du := dt - \frac{dr}{f(r)} = -l_a dx^a, \tag{2.2}$$

where indices $a, b, c, \ldots$ range over $0, 1$. The expectation value for the stress-energy tensor of a massless field on the background (2.1) can be written in the form

$$T^{ab} = \frac{1}{2} T^c_c g^{ab} + E(l^a l^b + n^a n^b) + F l^a l^b, \tag{2.3}$$

where the unspecified functions $T^c_c(r), E(r)$, and $F(r)$ correspond to vacuum polarization, an isotropic radiation field, and a net outward flux, respectively.

For a massless scalar field, the function T^c_c is given by the "trace anomaly":

$$T^c_c = \frac{1}{24\pi} R = -\frac{1}{24\pi} f'', \tag{2.4}$$

where R is the curvature scalar for metric (2.1) and f' denotes the derivative of f with respect to r. One can obtain the remaining components from the conservation law, $T^{ab}{}_{|a} = 0$, where the vertical bar denotes covariant differentiation. In terms of E and F the conservation law takes the form

$$F'(r) = 0,$$
$$E'(r) = -\frac{1}{4} f(r) \frac{d}{dr} T^c_c(r). \tag{2.5}$$

The specific vacuum state with respect to which the expectation value of the stress-energy tensor is taken is given by the boundary conditions which are imposed on (2.5).

We will be interested in two types of vacuum stress energy here. The Boulware state appears empty, apart from the vacuum polarization represented by (2.4), to stationary observers. This is expressed by the boundary condition $T^{ab} \to 0$ as $r \to \infty$. This condition and the conservation equations (2.5) imply

$$E \equiv E_B = \frac{1}{48\pi}\left(\frac{1}{2} f f'' - \frac{1}{4} f'^2\right),$$
$$F \equiv F_B = 0. \tag{2.6}$$

When (2.4) and (2.6) are substituted into (2.3) the stress energy takes the form of a stationary fluid with energy density and pressure:

$$\rho_B = \frac{1}{24\pi}\left(f'' - \frac{f'^2}{4f}\right), \tag{2.7}$$

$$P_B = -\frac{1}{24\pi}\frac{f'^2}{4f}, \tag{2.8}$$

respectively.

The Hartle-Hawking state is the one which is appropriate for an eternal black hole inside a cavity with reflecting walls, in thermal equilibrium with its own radiation. It appears empty (modulo vacuum polarization) to free-falling observers at the horizon. This corresponds to the boundary condition that the stress energy be regular on both the past and future event horizons. By imposing this boundary condition on Eqs. (2.5) we find that E and F take the form

$$E \equiv E_{\rm HH} = \frac{1}{48\pi}\left(\frac{1}{2} f f'' + \kappa^2 - \frac{1}{4} f'^2\right),$$
$$F \equiv F_{\rm HH} = 0. \tag{2.9}$$

Thus, the expectation value of the stress-energy in the Hartle-Hawking state also takes the form of a stationary fluid with energy density and pressure:

$$\rho_{\rm HH} = \frac{1}{24\pi}\left(f'' - \frac{4\kappa^2 - f'^2}{4f}\right), \tag{2.10}$$

$$P_{\rm HH} = -\frac{1}{24\pi}\frac{4\kappa^2 - f'^2}{4f}, \tag{2.11}$$

respectively. Notice that as $r \to \infty$ we have $P_{\rm HH} \approx \rho_{\rm HH} \approx \kappa^2/24\pi$. This is the thermodynamical equation of state for black-body radiation at temperature

$$T \equiv T_{\rm BH} = \kappa/2\pi. \tag{2.12}$$

$T_{\rm BH}$ is taken to be the temperature of the black hole.

III. ACCELERATION RADIATION AND THE BOULWARE STATE

Fulling and Davies [9] have found the regularized in-in vacuum expectation value of the stress-energy tensor for a massless scalar field in (1+1)-dimensional Minkowski space bounded by a mirror executing arbitrary motion. They find

$$\begin{aligned}\langle \text{in}|T_{tt}|\text{in}\rangle &= \langle \text{in}|T_{xx}|\text{in}\rangle \\ &= -\langle \text{in}|T_{tx}|\text{in}\rangle \\ &= -\frac{3\ddot{z}^2\,\dot{z} + \dddot{z} - \dddot{z}\,\dot{z}^2}{12\pi(1-\dot{z}^2)^2\,(1-\dot{z})^2},\end{aligned} \tag{3.1}$$

where x and t are the usual Minkowski coordinates, $x = z(t)$ is the trajectory of the mirror, and the right-hand side is evaluated at the retarded time t_R defined implicitly by $t_R - z(t_R) = t - x$. From this tensor it is straightforward to show that the energy flux from the mirror is

$$\frac{dE}{d\tau} \equiv T_{ab} v^a \frac{a^b}{a} = -\frac{1}{12\pi}\frac{da}{d\tau}, \tag{3.2}$$

where τ, v^a, and a^b are the proper time, velocity vector and acceleration vector of the mirror, and $a = \sqrt{g_{ab} a^a a^b}$ is the magnitude of a^a.

For the spacetime (2.1), a is given at a proper distance z from the horizon by

$$a = \frac{1}{\chi}\frac{d\chi}{dz} = \frac{f'}{2\sqrt{f}}. \tag{3.3}$$

Thus, the magnitude of the energy flux from a mirror is

$$\frac{dE}{d\tau} = -\frac{1}{24\pi}\left(f'' - \frac{f'^2}{2f}\right)\frac{dz}{d\tau}. \tag{3.4}$$

Now, let us consider a rigid box with reflecting walls being lowered adiabatically toward the black hole. We will assume the top and bottom walls are rigidly separated by a proper length ℓ which is much less than the radius of the black hole. We will consider the energy flux at an arbitrary fixed surface (a point, in this case) labeled by z_i ($z_B < z_i < z_T$), in the interior of the box. Consider the flux from, say, the top mirror of the box. In terms of the proper time τ_i for stationary observer at z_i the flux at the top will have the form

$$\frac{dE}{d\tau_i}(z_T) = -\frac{1}{24\pi}\left(f'' - \frac{f'^2}{2f}\right)_T \frac{dz_T}{d\tau_i}, \tag{3.5}$$

where the subscripts i and T denote quantities at z_i and the top of the box, respectively. At the surface z_i, the flux will be blue-shifted:

$$\frac{dE}{d\tau_i}(z_i) = -\frac{1}{24\pi}\frac{\sqrt{f(z_T)}}{\sqrt{f(z_i)}}\left(f'' - \frac{f'^2}{2f}\right)_T \frac{dz_T}{d\tau_i}. \tag{3.6}$$

The net flux of energy at the surface z_i will be (3.6) plus the energy flux from the bottom of the box, redshifted to the appropriate value:

$$\frac{dE_{\rm net}}{d\tau}(z_i) = \frac{1}{24\pi}\left\{\frac{\sqrt{f(z_T)}}{\sqrt{f(z_i)}}\left(f'' - \frac{f'^2}{2f}\right)_T - \frac{\sqrt{f(z_B)}}{\sqrt{f(z_i)}}\left(f'' - \frac{f'^2}{2f}\right)_B\right\}\frac{dz}{d\tau_i}, \tag{3.7}$$

where the subscript B denotes quantities at the bottom of the box. In obtaining (3.7) we have taken advantage of the fact that since the proper length of the box, ℓ, is assumed constant, $z_B = z_T + \ell$, and therefore $dz_B = dz_T = dz$.

Let us now assume the length of the box is small compared to other relevant length scales in the problem. This will not be true close to the horizon and will therefore prevent us from considering the floating point of the box. Nonetheless, when the box is far from the horizon and ℓ is small,

$$\frac{dF}{dz} \simeq \frac{F(z+\ell) - F(z)}{\ell}, \tag{3.8}$$

for any function $F(z)$. Therefore, we rewrite (3.7):

$$\frac{dE_{\rm net}}{d\tau}(z_i) \simeq \frac{\ell}{24\pi\sqrt{f}}\left\{\frac{d}{dz}\left[\sqrt{f}\left(f'' - \frac{f'^2}{2f}\right)\right]\right\}\frac{dz}{d\tau_i}. \tag{3.9}$$

The total energy is the sum of the energies entering the box as it is lowered from infinity to z:

$$\begin{aligned} E_{\rm net}(z) &\simeq \frac{\ell}{24\pi}\int_{\tau_i(z_i=\infty)}^{\tau_i(z_i=z)} \frac{1}{\sqrt{f}} \times\left\{\frac{d}{dz}\left[\sqrt{f}\left(f'' - \frac{f'^2}{2f}\right)\right]\right\}\frac{dz}{d\tau_i}d\tau_i \\ &\simeq \frac{\ell}{24\pi}\left(f'' - \frac{f'^2}{4f}\right)_z . \end{aligned} \tag{3.10}$$

But this is just the energy of the Boulware state (2.7). Thus, the energy of the matter content of the box is properly measured with respect to the Boulware vacuum energy.

Alternatively, let us examine the forces involved in lowering the box toward the black hole. By taking the Boulware state to be the vacuum state inside the box, and the Hartle-Hawking state to be the state outside the box, one can recover the results of UW. To prove this, let us consider the contribution of radiation pressure on the top and bottom of the box to the force needed to lower it. The pressure on each of the reflecting walls is the difference between the pressure of the Hartle-Hawking fluid on the outside of the box and the pressure of the Boulware fluid on the inside:

$$P_{\rm net} = P_{\rm HH} - P_B = \frac{\kappa^2}{24\pi f}. \tag{3.11}$$

Thus the net contribution to the force needed to lower the box is

$$F_{\rm net} = \frac{\kappa^2}{24\pi}\left(\frac{1}{f_T} - \frac{1}{f_B}\right) = \frac{\pi}{6}(T_T^2 - T_B^2), \tag{3.12}$$

where $T_T = T_{\rm BH}/\chi_T$ and $T_B = T_{\rm BH}/\chi_B$ are the redshifted values of the black hole temperature at the top and bottom of the box, respectively. This is precisely the expression obtained by UW for the contribution of the acceleration radiation to the force needed to lower the box.

IV. CONCLUSION

We have examined once more the long debated *gedankenexperiment* of lowering a box containing matter fields from infinity to a finite proper distance from a black hole. We have shown that the effect of the acceleration radiation on the energy density inside the box is exactly the same as obtained by considering the vacuum state of the interior of the box to be the Boulware state, and concluded that the acceleration of the box has induced a Boulware state inside it.

That the in vacuum for the box's interior is the Boulware state is to be expected on general grounds. The interior vacuum is initially the Boulware state (the vacuum for a stationary observer at infinity) and is invariant under the adiabatic (quasistatic) process of lowering. This is the real quantum state of the boxes interior. It will be natural, however, for different observers to calibrate energies in different ways. Table I summarizes the observations made by both inertial (freely falling) observers and accelerated (static) observers.

It is clear from Table I why the analysis of the buoyancy force by Li and Liu [7] is in error. They assume that the exterior state is the Hartle-Hawking state. This is the natural state as observed by an inertial observer and corresponds to column (i) from Table I. However, for such an observer the energy density of the radiation inside the box is decreasing as the box is lowered by an amount equal to the Boulware energy density [row (c) of column (i)]. By failing to take this into account, Li and Liu have implicitly assumed the point of view of a static

TABLE I. Energy densities outside and inside a box lowered adiabatically to a proper distance z from the event horizon of a n dimensional black hole as observed by (i) a local free-falling observer and (ii) a local static observer. Note that the energy densities in column (ii) are those which have a real gravitational effect while those in (ii) are recalibrated so that the interior of the box is taken as the ground state. $a = \frac{f'}{2\sqrt{f}}$ is the acceleration at z and only the dominant contribution as $z \to 0$ has been included.

	(i) Energy density as observed by free falling (inertial) observer	(ii) Energy density as observed by static (accelerated) observer
(a) Outside box (Hartle-Hawking state)	$\approx \rho_{\mathrm{HH}}$ bounded as $z \to 0$	$\approx \rho_{\mathrm{acceleration\ rad}}$ $\approx (a/2\pi)^n$
(b) Inside empty box[a] (Boulware state)	ρ_{Boulware} $\approx -(a/2\pi)^n$	Bounded as $z \to 0$
(c) Inside box filled at $z = \infty$ with radiation[b] ($T_{\mathrm{rad}} >> T_{\mathrm{HH}}$)	$\approx \rho_{\mathrm{rad}} - (a/2\pi)^n$	$\approx \rho_{\mathrm{rad}}$

[a]Box is empty at $z = \infty$ when lowering process begins.
[b]Box is filled with high density radiation at $z = \infty$ before being lowered.

observer [column (ii)] for the measurement of the interior energy density. By measuring the interior and exterior energy densities with respect to different states, Li and Liu obtain the wrong result for the buoyancy force on the box.

Thus, the Li-Liu criticism that the buoyancy forces are negligible based soley on Page's stress-energy tensor is unfounded. We have, furthermore, shown that using the Boulware state for the interior of the box leads to agreement with UW as to the form of the buoyancy force for a box of negligible height. As pointed out by Bekenstein [10], the assumption of negligible height breaks down near the floating point, however.

It may be surprising that the acceleration of an empty box in flat spacetime can be used to obtain the energy density of the Boulware state. However, this is not difficult to understand. As we mentioned, it is expected that the state inside an adiabatically lowered box is the Boulware state. However, a small enough box sees the gravitational field of the black hole as being essentially homogeneous. By the equivalence principle, such a box is unable to determine whether it is accelerating in a gravitational field or in Rindler space [11]. We have demonstrated how this allows us to obtain the Boulware energy density simply by considering the energy balance between two accelerating mirrors in (1+1)-dimensional Rindler space. We are presently investigating possibility of carrying out this procedure in (3+1)-dimensions.

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