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Mass and width determination of light-quark baryon resonances in QCD

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The spectrum of the proton and its excited states has been investigated in a finite-energy QCD sum rule approach using a Gauss-Weierstrass transform. The imaginary part of the polarization operator of the proton current has been saturated by the proton and two of its lowest-lying excited states. In the evaluation of the polarization operator using quark and gluon fields, perturbative corrections have also been incorporated. By taking various moments of the spectral function, a total of eight sum rules have been obtained. The masses of the two lowest-lying excited states of the proton, their widths, and the coupling constant, of the proton current with these states have been evaluated. In addition, contributions of all the excited states to the two-point function at zero momentum have also been determined. The values of the masses and widths obtained are in reasonable accord with their experimental values.

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I. INTRODUCTION

The study of the spectrum of hadron states is one of the most important problems of QCD. The QCD sum rule method, invented by Shifman, Vainshtein, and Zakharov (SVZ) [1], has been used to compute the masses and coupling constants of low-lying hadron states in a model-free way, resorting only to the vacuum condensates of quark and gluon fields as phenomenological parameters [2,3]. This was used to calculate the spectrum of ground state baryons with light quarks, first by Ioffe and his collaborators [4,5] and subsequently by others [6–8]. Attempts have also been made to calculate the masses of the first few excited states of baryons in the spectrum [5,9]. In the present work, we study the spectrum of the proton and its first two excited states within the combined framework of Gaussian sum rules and finite energy sum rules (FESR's) proposed in Ref. [10]. In the past, FESR's have been used mainly in the context of meson systems [10–12]. It will be interesting to apply them for studying excited states within baryon systems.

There are some obvious handicaps in dealing with baryon resonances as compared with meson resonances: The separation between ground state and excited states and that between first few excited states of baryons is not so clean as it is for mesons. Hence, the fit obtained may not be as good as it has been for mesons. In any event, FESR's should be more suitable for studying excited states as compared to the Borel (Laplace) transform version of QCD sum rules. As is well known, because of the exponential weight, the latter is more sensitive to the ground state whereas the former enhances the contribution of the high energy region, and hence of excited states, due to its polynomial kernel. Furthermore, FESR's can be employed to derive a number of sum rules for a given spectral function by calculating various moments. This is a relevant point in the present context, since one would require as many equations as are the unknown parameters to be determined. Another advantage of FESR's over Borel sum rules (BSR's) is that they do not mix different condensates of dimensions higher than a certain minimum dimension (depending upon the dimension of the current considered) when loop corrections are not considered in the Wilson coefficients of operator product expansion. Although we do take loop corrections in some of the Wilson coefficients derived in literature, the benefit still holds well, since the above property suppresses the contribution of higher dimensional condensates which are usually left. Lastly, the threshold of asymptotic freedom, s_0, is an adjustable parameter not fixed by BSR's. In FESR's, in principle s_0 can be fixed as an eigenvalue solution, or by implementing the principle that FESR's should be trusted only if results are stable in s_0 [13]. In a case where the contribution of some of the excited states, in addition to the ground state, has been parametrized separately from the continuum, it is not obvious whether the same value for s_0 should be chosen as is normally chosen in cases where only the ground state contribution is parametrized separately. By relying on the principle of Pich and de Rafael [13], we avoid this ambiguity and, moreover, save one equation from being consumed in solving for s_0.

Our construction of the phenomenological model for the two-point function is such that it explicitly includes first two excited states of the proton (having the same spin and valence quarks as the proton), in addition to the ground state (proton) itself. This introduces six additional unknowns in the phenomenological model: the mass, the width, and the coupling constant of the current with the state for each of these two states. In Refs. [5,9], different equations have been obtained by working at different Lorentz structures. For the current that we use, we have only two Lorentz structures; nevertheless we exploit the fact that different Hermite moments of the Gauss-Weierstrass transform of theoretical two-point function yield different moments of the phenomenological spectral function [10,11] and thus we get different equations.

We have also considered the functions

$$E_i(p^2)=[F_i(p^2)-F_i(0)]/p^2 ,$$

where F_1 and F_2 are the two independent functions appearing in the two-point function. From the zeroth Hermite moment of $E_i(p^2)$, we have evaluated $F_i(0)$. This gives us an estimate of the contribution of all the excited states to these functions at zero momentum. As far as the author is aware, the values of widths and coupling constants of the current with the excited states and of $F_1(0)$ and $F_2(0)$ have been calculated for the first time in the present work.

The method is general enough and can be easily extended to other members of the octet family. In the next section, we briefly describe the method. In Sec. III, we describe our phenomenological model and improve the results with renomalization group equations. In Sec. IV, we do an extensive stability analysis of our results and give the final results and conclusions. Finally, in the Appendix, we give some mathematical details.

II. OUTLINE OF THE METHOD

We start with the most general form of the quark current which has all the quantum numbers of a proton and has the lowest possible dimension:

$$\eta(x)=2\varepsilon^{abc}\{[u^a(x)C\gamma_5 d^b(x)]u^c(x) + t[u^a(x)Cd^b(x)]\gamma_5 u^c(x)\} . \tag{1}$$

The choice $t=-1$ corresponds to the current used and advocated by Ioffe [4,14] whereas $t=-0.2$ gives the current advocated by Chung *et al.* [7]. We shall do the calculation with the general combination (1).

Next, consider the two-point function

$$\pi(p)=i\int d^4x\, e^{ipx}\langle 0|T\{\eta(x),\bar{\eta}(0)\}|0\rangle . \tag{2}$$

The general tensor structure of $\pi(p)$ is

$$\pi(p)=\not{p}F_1(p^2)+F_2(p^2) . \tag{3}$$

For $F_1(p^2)$ and $F_2(p^2)$, we use the expressions derived in Ref. [7], where radiative corrections of Wilson coefficients appearing in the operator product expansion are also derived. However, for the Wilson coefficients of leading order terms in the operator product expansion (OPE), we use the more recent results of Ref. [15] (μ is the renormalization point):

$$-F_1(p^2)=[A+B\ln(-p^2/\mu^2)]p^4\ln(-p^2/\mu^2)+A_4\ln(-p^2/\mu^2)\langle(\alpha_s/\pi)G^2\rangle + [A_6+B_6\ln(-p^2/\mu^2)](1/p^2)\langle\bar{q}q\rangle^2+A_8(1/p^4)\langle\bar{q}q\rangle\langle g_s\bar{q}\sigma Gq\rangle , \tag{4a}$$

$$-F_2(p^2)=C_3p^2\ln(-p^2/\mu^2)\langle\bar{q}q\rangle+[C_5+D_5\ln(-p^2/\mu^2)]\ln(-p^2/\mu^2)\langle g_s\bar{q}\sigma Gq\rangle+C_7(1/p^2)\langle\bar{q}q\rangle\langle(\alpha_s/\pi)G^2\rangle . \tag{4b}$$

where

$$A=\frac{1}{32(2\pi)^4}(5+2t+5t^2)\left[1+\frac{71}{12}\frac{\alpha_s}{\pi}\right] ,$$

$$B=\frac{-1}{64(2\pi)^4}(5+2t+5t^2)\frac{\alpha_s}{\pi} ,$$

$$A_4=\frac{1}{64(2\pi)^2}(5+2t+5t^2) ,$$

$$A_6=\frac{1}{6}\left[7\left[1-\frac{43}{42}\frac{\alpha_s}{\pi}\right]-2t\left[1-\frac{1}{6}\frac{\alpha_s}{\pi}\right] -5t^2\left[1-\frac{29}{30}\frac{\alpha_s}{\pi}\right]\right] ,$$

$$B_6=\frac{1}{6}\left[\frac{62}{3}-\frac{4}{3}t-\frac{46}{3}t^2\right]\frac{\alpha_s}{\pi} ,$$

$$A_8=\frac{1}{18}[58-8t-50t^2] ,$$

$$C_3=\frac{-1}{4(2\pi)^2}\left[7\left[1+\frac{15}{14}\frac{\alpha_s}{\pi}\right]-2t\left[1+\frac{3}{2}\frac{\alpha_s}{\pi}\right] -5t^2\left[1+\frac{7}{10}\frac{\alpha_s}{\pi}\right]\right] ,$$

$$C_5=\frac{3}{4(2\pi)^2}\left[1+\frac{79}{18}\frac{\alpha_s}{\pi}-t^2\left[1+\frac{103}{18}\frac{\alpha_s}{\pi}\right]\right] ,$$

$$D_5=\frac{3}{4(2\pi)^2}\left[\frac{-7}{12}\right](1-t^2)\frac{\alpha_s}{\pi} ,$$

$$C_7=\frac{-1}{288}[31-2t-29t^2] .$$

To derive FESR's from the above equations, different methods discussed in the literature [16] may be used. We find proceeding via a Gauss-Weierstrass (GW) transform in the manner of Bertlmann, Launer, and de Rafael [10] particularly useful. It facilitates the treatment of log terms appearing in the perturbative expansion of Wilson coefficients. We outline below briefly the steps involved in arriving at the GW transform.

To the dispersion relation

$$F_i(q^2)=\int_0^\infty ds\frac{1}{s-q^2-i\varepsilon}\frac{1}{\pi}\,\mathrm{Im}F_i(s) + \text{subtraction terms} \tag{5}$$

one applies the Borel transform operator ($Q^2=-q^2>0$)

$$\hat{B}\equiv \lim_{\substack{N\to\infty\\ Q^2\to\infty\\ N/Q^2=\sigma}} \frac{(-1)^N}{(N-1)!}(Q^2)^N\frac{d^N}{(dQ^2)^N} . \tag{6}$$

This gives Borel transform $M_i(\sigma)$:

$$M_i(\sigma)=\int_0^\infty ds\, e^{-s\sigma}\frac{1}{\pi}\,\mathrm{Im}F_i(s) \tag{7}$$

and subtraction terms on the right-hand side (RHS) get washed out. The Borel transforms needed for our purpose are listed in the Appendix. It is clear from a δ-function type of parametrization of spectral functions $\mathrm{Im}F_i(s)$, where widths have been taken to be zero for simplicity, that the excited state contributions, in this sum rule, will get exponentially suppressed compared to the ground state contribution. In order to calculate the GW transform, one applies the operator

$$\hat{L} \equiv \lim_{\substack{N\to\infty \\ \sigma^2\to\infty \\ N/\sigma^2=\tau}} \frac{(-1)^N}{(N-1)!}(\sigma^2)^N \frac{d^N}{(d\sigma^2)^N} \tag{8}$$

to the expression for $(1/\sigma)M_i(\sigma)e^{-\hat{s}\sigma}$, thus constructing $G_i(-\hat{s},\tau)$:

$$\frac{1}{2\tau}\hat{L}\left[\frac{1}{\sigma}M_i(\sigma)e^{-\hat{s}\sigma}\right]=\frac{1}{\sqrt{4\pi\tau}}\int_0^\infty ds\,\exp\left[\frac{-(s+\hat{s})^2}{4\tau}\right]\frac{1}{\pi}\,\mathrm{Im}F_i(s)\equiv G_i(-\hat{s},\tau)\ . \tag{9}$$

From $G_i(-\hat{s},\tau)$, $G_i(\hat{s},\tau)$ may be obtained by analytic continuation. The derivation of the GW transform for power terms is straightforward, as given in Ref. [10]. However, working out the GW transform for log terms is a tedious task. We have relegated the mathematical details to the Appendix. We write below our result of the calculation:

$$\begin{aligned} G_1(s,\tau)=\sqrt{2/\pi\tau}\exp(-s^2/8\tau)\Big[&-2\{A+B[3+2\psi(1)-2\ln\mu^2]\}D_{-3}(-s/\sqrt{2\tau}) \\ &-A_4\frac{1}{2\tau}D_{-1}(-s/\sqrt{2\tau})\langle\alpha_s G^2/\pi\rangle-\{A_6+B_6[\psi(1)-\ln\mu^2]\}(2\tau)^{-3/2} \\ &\times D_0(s/\sqrt{2\tau})\langle\bar{q}q\rangle^2 \\ &-A_8(2\tau)^{-2}D_1(s/\sqrt{2\tau})\langle\bar{q}q\rangle\langle g_s\bar{q}\sigma Gq\rangle\Big] \\ &+2BI(-s,\tau;2,-2)+B_6\tfrac{1}{2}I(-s,\tau;-1,-2)\langle\bar{q}q\rangle^2\ , \end{aligned} \tag{10a}$$

$$\begin{aligned} G_2(s,\tau)=-\sqrt{\tau/\pi}\exp(-s^2/8\tau)\Big[&C_3D_{-2}(-s/\sqrt{2\tau})\langle\bar{q}q\rangle+\{C_5-2D_5[\ln\mu^2-\psi(1)]\}(2\tau)^{-1/2}D_{-1}(-s/\sqrt{2\tau})\langle g_s\bar{q}\sigma Gq\rangle \\ &+C_7\frac{1}{2\tau}D_0(s/\sqrt{2\tau})\langle\bar{q}q\rangle\langle\alpha_sG^2/\pi\rangle\Big]+D_5I(-s,\tau;0,-2)\langle g_s\bar{q}\sigma Gq\rangle\ . \end{aligned} \tag{10b}$$

The notation followed in Eqs. (10) is that of Ref. [10]. It is clear that the short-distance expansion of $G_i(s,\tau)$ appears as a double series in powers of $1/\sqrt{2\tau}$, the nonperturbative contributions, and in powers of the running coupling constant, the perturbative contributions. Next, we construct even and odd functions:

$$U_i^{(\pm)}(s,\tau)=G_i(s,\tau)\pm G_i(-s,\tau)\ . \tag{11}$$

We compute zeroth and second Hermite moments of $U_i^{(+)}$ and the first Hermite moment of $U_i^{(-)}$. Introducing a cutoff s_0, which is identified with the onset of the QCD continuum, and taking the limit $\tau\to 0$ we get the desired sum rules:

$$\int_0^{s_0}ds\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\left[A+B\left[2\ln\frac{s_0}{\mu^2}-\frac{2}{3}\right]\right]\frac{s_0^3}{3}+A_4\langle\alpha_sG^2/\pi\rangle s_0+\left[A_6+B_6\ln\frac{s_0}{\mu^2}\right]\langle\bar{q}q\rangle^2\ , \tag{12a}$$

$$\int_0^{s_0}ds\,s\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\left[A+B\left[2\ln\frac{s_0}{\mu^2}-\frac{1}{2}\right]\right]\frac{s_0^4}{4}+A_4\langle\alpha_sG^2/\pi\rangle\frac{s_0^2}{2}+A_8\langle\bar{q}q\rangle\langle g_s\bar{q}\sigma Gq\rangle\ , \tag{12b}$$

$$\int_0^{s_0}ds\,s^2\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\left[A+B\left[2\ln\frac{s_0}{\mu^2}-\frac{2}{5}\right]\right]\frac{s_0^5}{5}+A_4\langle\alpha_sG^2/\pi\rangle\frac{s_0^3}{3}-B_6\langle qq\rangle^2\frac{s_0^2}{2}\ , \tag{12c}$$

$$\int_0^{s_0}ds\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle\frac{s_0^2}{2}+\left[C_5+2D_5\left[\ln\frac{s_0}{\mu^2}-1\right]\right]\langle g_s\bar{q}\sigma Gq\rangle s_0+C_7\langle\bar{q}q\rangle\langle\alpha_sG^2/\pi\rangle\ , \tag{12d}$$

$$\int_0^{s_0}ds\,s\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle\frac{s_0^3}{3}+\left[C_5+2D_5\left[\ln\frac{s_0}{\mu^2}-\frac{1}{2}\right]\right]\langle g_s\bar{q}\sigma Gq\rangle\frac{s_0^2}{2}\ , \tag{12e}$$

$$\int_0^{s_0}ds\,s^2\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle\frac{s_0^4}{4}+\left[C_5+2D_5\left[\ln\frac{s_0}{\mu^2}-\frac{1}{3}\right]\right]\langle g_s\bar{q}\sigma Gq\rangle\frac{s_0^3}{3}\ . \tag{12f}$$

We also consider the functions

$$E_1(p^2)=\frac{1}{p^2}[-F_1(p^2)+F_1(0)] \tag{13a}$$

and

$$E_2(p^2)=\frac{1}{p^2}[-F_2(p^2)+F_2(0)] \tag{13b}$$

and take their GW transform. Their zeroth Hermite moments give sum rules

$$\int_0^{s_0} ds\frac{1}{s}\frac{1}{\pi}\,\mathrm{Im}F_1(s)-F_1(0)=\left[A+B\left[2\ln\frac{s_0}{\mu^2}-1\right]\right]\frac{s_0^2}{2}+A_4\langle\alpha_s G^2/\pi\rangle\ln\frac{s_0}{\mu^2}-B_6\langle\bar{q}q\rangle^2\frac{1}{s_0}\ , \tag{14a}$$

$$\int_0^{s_0} ds\frac{1}{s}\frac{1}{\pi}\,\mathrm{Im}F_2(s)-F_2(0)=C_3\langle\bar{q}q\rangle s_0+\left[C_5\ln\frac{s_0}{\mu^2}-D_5\frac{\pi^2}{3}+D_5\left[\ln\frac{s_0}{\mu^2}\right]^2\right]\langle g_s\bar{q}\sigma Gq\rangle\ . \tag{14b}$$

On the RHS of a sum rule, normally only a specific combination of vacuum condensates of a given dimension should appear [10]. However this feature gets lost, first due to radiative corrections of Wilson coefficients and secondly due to the appearance of D_{-n}'s, which do not obey a simple orthogonality relation, in the first few terms of the expansion of GW transforms of spectral functions.

III. THE PHENOMENOLOGICAL MODEL AND THE RENORMALIZATION GROUP IMPROVEMENT OF SUM RULES

We write the phenomenological part of our spectral function by taking the contributions of the ground state and first two excited states [$N(1440)$ and $N(1535)$] only with finite widths; all the states having identical valence quarks and spin:

$$\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\lambda_N^2\delta(s-M^2)+\frac{1}{\pi}\left[\frac{\lambda_{N1}^2M_1\Gamma_1}{(s-M_1^2+\Gamma_1^2/4)^2+M_1^2\Gamma_1^2}+\frac{\lambda_{N2}^2M_2\Gamma_2}{(s-M_2^2+\Gamma_2^2/4)^2+M_2^2\Gamma_2^2}\right]\theta(s-M^2)\ , \tag{15a}$$

$$\frac{1}{\pi}\,\mathrm{Im}F_2(s)=\lambda_N^2M\delta(s-M^2)+\frac{1}{\pi}\left[\frac{\lambda_{N1}^2\frac{\Gamma_1}{2}(s+M_1^2+\Gamma_1^2/4)}{(s-M_1^2+\Gamma_1^2/4)^2+M_1^2\Gamma_1^2}-\frac{\lambda_{N2}^2\frac{\Gamma_2}{2}(s+M_2^2+\Gamma_2^2/4)}{(s-M_2^2+\Gamma_2^2/4)^2+M_2^2\Gamma_2^2}\right]\theta(s-M^2)\ . \tag{15b}$$

Here λ's are couplings of the current (1) with the respective states:

$$\langle 0|\eta(0)|\mathrm{proton},k\rangle=\lambda_N u_N(k)\ ,$$

etc.; M's are their masses and Γ_1 and Γ_2 are decay widths. In the limit $\Gamma_i\to 0$ the excited state contributions also reduce to δ-function form. In order to keep the number of parameters to a minimum, we have assumed that the contribution of excited states to the spectral functions start from the proton mass. The state $N(1535)$ has negative parity and hence its contribution comes with a negative sign in (15b). The parametrization of the spectral function with only the first three states should be sufficient in view of the explicit cutoff scale appearing in the integration of the spectral functions.

We also improve our results (12) and (14) with the help of renormalization group equations. For those coefficients in the OPE for which perturbative corrections are available, we reabsorb QCD perturbative $\ln(s_0/\mu^2)$ contributions in a rescaling of the running coupling constant, as done in Refs. [7,10]. For others, we have used the available results for anomalous dimensions of operators [17]. In the following we have used the abbreviation

$$\frac{5+2t+5t^2}{32(2\pi)^4}=a,\quad \frac{\bar{\alpha}_s(s_0)}{\bar{\alpha}_s(\mu^2)}=R\ .$$

On renormalization group improvement, Eqs. (12) and (14) take the form

$$\int_0^{s_0} ds\frac{1}{\pi}\,\mathrm{Im}F_1(s)=a\left[1+\frac{75}{12}\frac{\bar{\alpha}_s}{\pi}\right]R^{4/9}\frac{s_0^3}{3}+A_4R^{4/9}\langle\alpha_sG^2/\pi\rangle s_0$$
$$+\frac{1}{6}\left[7\left[1-\frac{43}{42}\frac{\bar{\alpha}_s}{\pi}\right]R^{-248/189}-2t\left[1-\frac{1}{6}\frac{\bar{\alpha}_s}{\pi}\right]R^{-8/27}-5t^2\left[1-\frac{29}{30}\frac{\bar{\alpha}_s}{\pi}\right]R^{-184/135}\right]\langle\bar{q}q\rangle^2\ , \tag{16a}$$

$$\int_0^{s_0} ds\, s\frac{1}{\pi}\,\mathrm{Im}F_1(s)=a\left[1+\frac{25}{3}\frac{\bar{\alpha}_s}{\pi}\right]R^{4/9}\frac{s_0^4}{4}+A_4R^{4/9}\langle\alpha_sG^2/\pi\rangle\frac{s_0^2}{2}+A_8\langle\bar{q}q\rangle\langle g_s\bar{q}\sigma Gq\rangle\ , \tag{16b}$$

$$\int_0^{s_0} ds\, s^2 \frac{1}{\pi}\,\mathrm{Im}F_1(s)=a\left[1+\frac{367}{60}\frac{\bar{\alpha}_s}{\pi}\right]R^{4/9}\frac{s_0^5}{5}+A_4R^{4/9}\langle\alpha_s G^2/\pi\rangle\frac{s_0^3}{3}+B_6\langle\bar{q}q\rangle^2\frac{s_0^2}{2}\ , \tag{16c}$$

$$\int_0^{s_0} ds\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle\frac{s_0^2}{2}+\frac{3}{4(2\pi)^2}\left[1+\frac{50}{9}\frac{\bar{\alpha}_s}{\pi}-t^2\left[1+\frac{62}{9}\frac{\bar{\alpha}_s}{\pi}\right]\right]R^{14/27}\langle g_s\bar{q}\sigma Gq\rangle s_0+C_7\langle\bar{q}q\rangle\langle\alpha_s G^2/\pi\rangle\ , \tag{16d}$$

$$\int_0^{s_0} ds\, s\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle\frac{s_0^3}{3}+\frac{3}{4(2\pi)^2}\left[1+\frac{179}{36}\frac{\bar{\alpha}_s}{\pi}-t^2\left[1+\frac{227}{36}\frac{\bar{\alpha}_s}{\pi}\right]\right]R^{14/27}\langle g_s\bar{q}\sigma Gq\rangle\frac{s_0^2}{2}\ , \tag{16e}$$

$$\int_0^{s_0} ds\, s^2\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle\frac{s_0^4}{4}+\frac{3}{4(2\pi)^2}\left[1+\frac{43}{9}\frac{\bar{\alpha}_s}{\pi}-t^2\left[1+\frac{55}{9}\frac{\bar{\alpha}_s}{\pi}\right]\right]R^{14/27}\langle g_s\bar{q}\sigma Gq\rangle\frac{s_0^3}{3}\ , \tag{16f}$$

$$\int_0^{s_0} ds\frac{1}{s}\frac{1}{\pi}\,\mathrm{Im}F_1(s)=a\left[1+\frac{77}{12}\frac{\bar{\alpha}_s}{\pi}\right]R^{4/9}\frac{s_0^2}{2}+A_4\ln\frac{s_0}{\mu^2}R^{4/9}\langle\alpha_s G^2/\pi\rangle-B_6\langle\bar{q}q\rangle^2\frac{1}{s_0}+F_1(0)\ , \tag{17a}$$

$$\int_0^{s_0} ds\frac{1}{s}\frac{1}{\pi}\,\mathrm{Im}F_2(s)=C_3\langle\bar{q}q\rangle s_0+\frac{3}{4(2\pi)^2}\left[\frac{7\pi^2}{36}(1-t^2)\frac{\bar{\alpha}_s}{\pi}+\left[1+\frac{79}{12}\frac{\bar{\alpha}_s}{\pi}-t^2\left[1+\frac{103}{18}\frac{\bar{\alpha}_s}{\pi}\right]\right]\ln\frac{s_0}{\mu^2}\right]R^{7/27}\langle g_s\bar{q}\sigma Gq\rangle+F_2(0)\ . \tag{17b}$$

The above expressions, evaluated in QCD, provide the RHS of the sum rules. The LHS of the sum rules are obtained from the phenomenological model (15). Using the abbreviations

$$\arctan\frac{s_0-M_1^2+\Gamma_1^2/4}{M_1\Gamma_1}+\arctan\frac{M_1^2-M^2-\Gamma_1^2/4}{M_1\Gamma_1}=Atn_1\ ,$$

etc., and

$$\ln\frac{(s_0-M_1^2+\Gamma_1^2/4)^2+M_1^2\Gamma_1^2}{(M_1^2-M^2-\Gamma_1^2/4)^2+M_1^2\Gamma_1^2}=Lr_1\ ,$$

etc., we obtain, for the LHS,

$$\int_0^{s_0} ds\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\lambda_N^2+\frac{\lambda_{N1}^2}{\pi}Atn_1+\frac{\lambda_{N2}^2}{\pi}Atn_2\ , \tag{18a}$$

$$\int_0^{s_0} ds\, s\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\lambda_N^2M^2+\frac{\lambda_{N1}^2}{\pi}\left[\frac{M_1\Gamma_1}{2}Lr_1+(M_1^2-\Gamma_1^2/4)Atn_1\right]+\frac{\lambda_{N2}^2}{\pi}\left[\frac{M_2\Gamma_2}{2}Lr_2+(M_2^2-\Gamma_2^2/4)Atn_2\right]\ , \tag{18b}$$

$$\begin{aligned}\int_0^{s_0} ds\, s^2\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\lambda_N^2M^4&+\frac{\lambda_{N1}^2}{\pi}M_1\Gamma_1\left[(s_0-M^2)+\frac{(M_1^2-\Gamma_1^2/4)^2-M_1^2\Gamma_1^2}{M_1\Gamma_1}Atn_1+(M_1^2-\Gamma_1^2/4)Lr_1\right]\\&+\frac{\lambda_{N2}^2}{\pi}M_2\Gamma_2\left[(s_0-M^2)+\frac{(M_2^2-\Gamma_2^2/4)^2-M_2^2\Gamma_2^2}{M_2\Gamma_2}Atn_2+(M_2^2-\Gamma_2^2/4)Lr_2\right]\ ,\end{aligned} \tag{18c}$$

$$\int_0^{s_0} ds\frac{1}{\pi}\,\mathrm{Im}F_2(s)=\lambda_N^2M+\frac{\lambda_{N1}^2}{\pi}\left[\frac{\Gamma_1}{4}Lr_1+M_1Atn_1\right]-\frac{\lambda_{N2}^2}{\pi}\left[\frac{\Gamma_2}{4}Lr_2+M_2Atn_2\right]\ , \tag{18d}$$

$$\begin{aligned}\int_0^{s_0} ds\, s\frac{1}{\pi}\,\mathrm{Im}F_2(s)=\lambda_N^2M^3&+\frac{\lambda_{N1}^2}{\pi}\left[(s_0-M^2)\frac{\Gamma_1}{2}+M_1(M_1^2-\tfrac{3}{4}\Gamma_1^2)Atn_1+\frac{\Gamma_1}{4}\left[3M_1^2-\frac{\Gamma_1^2}{4}\right]Lr_1\right]\\&-\frac{\lambda_{N2}^2}{\pi}\left[(s_0-M^2)\frac{\Gamma_2}{2}+M_2(M_2^2-\tfrac{3}{4}\Gamma_2^2)Atn_2+\frac{\Gamma_2}{4}\left[3M_2^2-\frac{\Gamma_2^2}{4}\right]Lr_2\right]\ ,\end{aligned} \tag{18e}$$

$$\int_0^{s_0} ds\, s^2\, \mathrm{Im}F_2(s)=\lambda_N^2 M^5+\frac{\lambda_{N1}^2}{\pi}\frac{\Gamma_1}{2}\left[\frac{s_0^2-M^4}{2}+(s_0-M^2)\left[3M_1^2-\frac{\Gamma_1^2}{4}\right]\right.$$
$$\left.+Atn_1\frac{2M_1^6-5M_1^4\Gamma_1^2+\frac{5}{8}M_1^2\Gamma_1^4}{M_1\Gamma_1}+\left[\frac{5}{2}M_1^4-\frac{5}{4}M_1^2\Gamma_1^2+\frac{\Gamma_1^4}{32}\right]Lr_1\right]$$
$$-\frac{\lambda_{N2}^2}{\pi}\frac{\Gamma_2}{2}\left[\frac{s_0^2-M^4}{2}+(s_0-M^2)\left[3M_2^2-\frac{\Gamma_2^2}{4}\right]+Atn_2\frac{2M_2^6-5M_2^4\Gamma_2^2+\frac{5}{8}M_2^2\Gamma_2^4}{M_2\Gamma_2}\right.$$
$$\left.+\left[\frac{5}{2}M_2^4-\frac{5}{4}M_2^2\Gamma_2^2+\frac{\Gamma_2^4}{32}\right]Lr_2\right], \tag{18f}$$

$$\int_0^{s_0} ds\frac{1}{s}\frac{1}{\pi}\,\mathrm{Im}F_1(s)=\frac{\lambda_N^2}{M^2}+\frac{\lambda_{N1}^2/\pi}{(M_1^2-\Gamma_1^2/4)^2+M_1^2\Gamma_1^2}\left[M_1\Gamma_1\ln\frac{s_0}{M^2}-\frac{1}{2}M_1\Gamma_1 Lr_1+(M_1^2-\Gamma_1^2/4)Atn_1\right]$$
$$+\frac{\lambda_{N2}^2/\pi}{(M_2^2-\Gamma_2^2/4)^2+M_2^2\Gamma_2^2}\left[M_2\Gamma_2\ln\frac{s_0}{M^2}-\frac{1}{2}M_2\Gamma_2 Lr_2+(M_2^2-\Gamma_2^2/4)Atn_2\right], \tag{19a}$$

$$\int_0^{s_0} ds\frac{1}{s}\frac{1}{\pi}\,\mathrm{Im}F_2(s)=\frac{\lambda_N^2}{M}+\frac{\lambda_{N1}^2/\pi}{(M_1^2-\Gamma_1^2/4)^2+M_1^2\Gamma_1^2}\left[\frac{\Gamma_1}{2}\left[M_1^2+\frac{\Gamma_1^2}{4}\right]\ln\frac{s_0}{M^2}-\frac{\Gamma_1}{4}\left[M_1^2+\frac{\Gamma_1^2}{4}\right]Lr_1\right.$$
$$\left.+M_1\left[M_1^2+\frac{\Gamma_1^2}{4}\right]Atn_1\right]$$
$$-\frac{\lambda_{N2}^2/\pi}{(M_2^2-\Gamma_2^2/4)^2+M_2^2\Gamma_2^2}\left[\frac{\Gamma_2}{2}\left[M_2^2+\frac{\Gamma_2^2}{4}\right]\ln\frac{s_0}{M^2}-\frac{\Gamma_2}{4}\left[M_2^2+\frac{\Gamma_2^2}{4}\right]Lr_2\right.$$
$$\left.+M_2\left[M_2^2+\frac{\Gamma_2^2}{4}\right]Atn_2\right]. \tag{19b}$$

IV. ANALYSIS OF RESULTS AND DISCUSSION

We have eight equations in the form of sum rules at our disposal and hence we can solve them for only eight unknowns. We take the mass of the proton and the coupling of the current (1) with the proton state to be known; we determine Γ_i, λ_i, and M_i ($i=1,2$), and $F_1(0)$ and $F_2(0)$ in terms of the other known parameters. In principle, we could have determined the proton parameters as well. However, we do not think we can use more and more sum rules by taking various moments of the spectral function for determining more and more unknown parameters and at the same time do justice with the accuracy of results. This is because we have used only a limited number of terms in the OPE as well as in QCD perturbative expansion and also have used the factorization hypothesis for higher-dimensional operators.

First, consider the sum rules obtainable from Eqs. (16) and (18). We first transfer proton contributions to the RHS of the equations. We have used the following values for the constants:

$t=-1,\ \ \bar{\alpha}_s/\pi=0.1,\ \ \mu=0.5$ GeV ,

$\Lambda_{\mathrm{QCD}}=0.15$ GeV, $M=0.939$ GeV ,

$\lambda_N^2=5.13\times10^{-4}$ GeV^6, $\langle\bar{q}q\rangle=-0.012$ GeV^3 ,

$\langle\alpha_s G^2/\pi\rangle=0.036$ GeV^4 ,

$\langle g_s\bar{q}\sigma Gq\rangle=\langle\bar{q}q\rangle\times0.8$ GeV^2, $\langle\bar{q}q\rangle^2=0.0003$ GeV^6 .

Most of the above constants have been used by the ITEP group [3–5]. However, for the four quark condensate and gluon condensate, we use some recent results [12] which were derived using FESR's and hence should be more appropriate for our purpose. In all the equations, λ_{Ni}^2 appear linearly, whereas M_i and Γ_i appear in nonlinear forms. Moreover, nonlinearity in Γ_i's is more pronounced. We have found numerical solutions of the six equations by solving them simultaneously for the six unknowns M_i, Γ_i, and λ_i ($i=1,2$). Initially, we have solved the equations by satisfying them within 5% accuracy while keeping the continuum threshold s_0 a free parameter. Because of the 5% limit on the accuracy, the equations admit solutions within a certain range for each value of the six parameters to be determined at a given s_0. However, the results appear to be relatively stable over the range $s_0=2.3$–2.7 GeV^2. This can be observed from the general trend of the computed values of M_1 and M_2 plotted versus s_0, shown in Figs. 1(a) and 1(b), respectively. Incidentally, the midpoint of this plateau region, $s_0=2.5$ GeV^2, happens to be the onset of the QCD continuum in some of the baryon mass calculations [7]. At this midpoint, we have solved the equations with a better accuracy satisfying them within 2%. We have displayed the results in Table I in set I.

We have studied the stability of our results against the variation of input parameters: $\langle\alpha_s G^2/\pi\rangle$ has been varied from 0.036 GeV^4 to 0.0127 GeV^4 (the latter has been used by the ITEP group); $\langle\bar{q}q\rangle^2$ from 0.0003 GeV^6

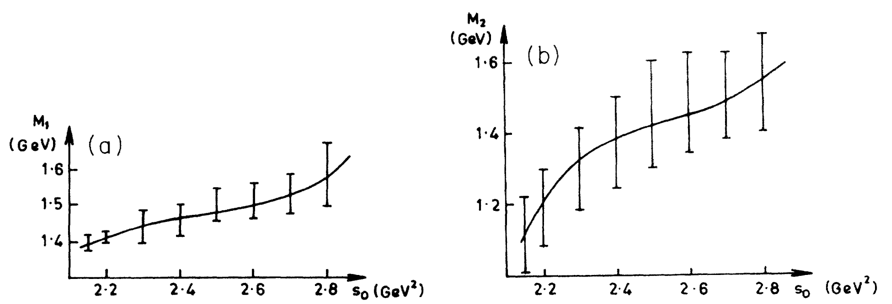

FIG. 1. Error bars in (a) and (b) show the scatter in computed results for M_1 and M_2, respectively (accuracy 5%) versus s_0 (same scale). Solid lines show probable curves that take into account the general trend of error bars.

to 0.000 144 GeV^6 (the latter value corresponds to the vacuum saturation hypothesis); μ from 0.5 GeV to 0.9 GeV; $\bar{\alpha}_s/\pi$ from 0.1 to 0 (i.e., to no perturbative correction to Wilson coefficients; such sum rules along with the vacuum saturation hypothesis have been used by the ITEP group) combined with $\langle \bar{q}q \rangle^2 = 0.000\,144$ GeV^6; and t from -1 to -0.8. The range of solutions obtained and results for the variations of the average values of the solutions from those in set I have been displayed in sets II, III, IV, V, and VI, respectively, of Table I. The solutions were obtained with an accuracy of 3%, 6%, 3.7%, 3%, and 4%, respectively. It was difficult to achieve better accuracy than tried here. In all these cases, barring the case where t has been varied, the masses and the width Γ_1 have been found to vary within 100 MeV. It is interesting to note that when the quark-gluon coupling constant is set to zero (i.e., no perturbative corrections to Wilson coefficients), then the coupling constants of the quark current to the excited states of the proton get considerably reduced. In set VII of Table I, we have displayed the solutions for 20% increment in λ_N^2 while keeping other parameters the same as in set V (i.e., closer to the value obtained for λ_N in Ref. [3], where $\bar{\alpha}_s = 0$ and the vacuum saturation hypothesis have been used). The solutions have been obtained within 7% accuracy and the variation in the average results over their counterparts in set V is mild, except for the Γ_2 which has increased by 120 MeV. However, with $\lambda_N^2 = 6.7 \times 10^{-4}$ GeV^6, the value obtained in Ref. [3], it appears difficult to solve the equations with an accuracy better than 20%; i.e., there seems to be a compatibility problem with other parameters used in the set. The reason for such a discrepancy may be the fact that in Ref. [3], the determination of λ_N^2 has been carried out by saturating the phenomenological side of the sum rule by the ground state plus continuum, whereas we are doing it here with three states plus continuum. The case of no perturbative correction ($\bar{\alpha}_s = 0$) but other parameters the same as in set I is solvable with an accuracy of no better than 8.3%. The results for solutions are qualitatively the same as in the previous cases, except that the value of M_2, now, decreases.

The current with $t = -1$ has been extensively and suc-

TABLE I. Calculated results for various input parameters. In set I, the parameters used are as given in the text. In other sets, the changed parameters are $\langle \alpha_s G^2/\pi \rangle = 1.27 \times 10^{-2}$ GeV^4 in set II; $\langle \bar{q}q \rangle^2 = 1.44 \times 10^{-4}$ GeV^6 in set III; $\mu = 0.9$ GeV in set IV; $\bar{\alpha}_s = 0$ and $\langle \bar{q}q \rangle^2 = 1.44 \times 10^{-4}$ GeV^6 in set V; $t = -0.8$ in set VI; $\lambda_N^2 = 6.16 \times 10^{-4}$ GeV^6, $\bar{\alpha}_s = 0$, and $\langle \bar{q}q \rangle^2 = 1.44 \times 10^{-4}$ GeV^6 in set VII. The accuracies of the results are 2%, 3%, 6%, 3.7%, 3%, 4%, and 7%, respectively. In all the sets, the first column shows the full range of calculated results (masses and widths in GeV and couplings in GeV^6). The second column of set I shows the average results, the second column of sets II through VI shows the change in average results from those of set I, while the second column of set VII shows the change in average results from those of set V. The changes in masses and widths are displayed in MeV units rounded to nearest 10 MeV, while those in couplings are displayed as percentage changes.

Physical quantity	Set I		Set II		Set III	
M_1	1.42–1.44	1.43	1.44–1.46	+20	1.45–1.48	+30
M_2	1.32–1.50	1.41	1.34–1.66	+90	1.40–1.57	+80
Γ_1	0.01–0.12	0.06	0–0.07	−20	0–0.05	−40
Γ_2	0.18–0.74	0.46	0.04–0.67	−100	0–0.18	−360
$10^4 \times \lambda_{N1}^2$	7.70–9.00	8.35	6.93–7.95	−11%	6.70–7.49	−15%
$10^4 \times \lambda_{N2}^2$	3.60–7.20	5.40	2.31–7.44	−10%	2.31–4.00	−29%

Physical quantity	Set IV		Set V		Set VI		Set VII	
M_1	1.42–1.45	+10	1.50–1.52	+80	1.38–1.46	−10	1.54–1.58	+50
M_2	1.38–1.64	+100	1.34–1.66	+90	0.96–1.16	−350	1.35–1.75	+30
Γ_1	0.01–0.04	−30	0.01–0.03	−40	0.04–0.26	+90	0–0.02	−10
Γ_2	0.14–0.51	−130	0.10–0.45	−170	0.05–0.55	−160	0.10–0.70	+120
$10^4 \times \lambda_{N1}^2$	8.59–10.0	+11%	4.10–4.36	−50%	8.21–11.29	+17%	3.33–3.85	−15%
$10^4 \times \lambda_{N2}^2$	5.39–9.23	+33%	1.03–3.08	−65%	2.57–3.59	−43%	1.03–4.11	+25%

cessfully used by various authors [6,9,18]. As argued in Ref. [6], the proton current which is orthogonal to the above choice (i.e., with $t=1$) cannot have nonperturbative contributions. One other form of the current, with $t=-0.2$, which has been used by some authors [7,8], has the property that it minimizes the contributions of excited states. Contrary to these, we want the nonperturbative contributions and the excited states to play a significant role. Furthermore, as pointed out by Ioffe [14], the current with $t=-0.2$ results in a strong increase in the neglected power corrections in the OPE. We do not want this to happen. Deviation from $t=-1$ amounts to an introduction of the above types of currents, and hence is contrary to our purpose.

It is well known [19] that the numerical values of the quark and gluon condensates, in general, depend on μ, the renormalization point. The numerical values of the condensates, used in the text, correspond to the values obtained as a fit for $\mu=0.5$ GeV. Furthermore, a realistic calculation should take into account the perturbative corrections of Wilson coefficients in OPE. In view of the above discussion, we do our final analysis based on the results of sets I, II, and III only of Table I, since the numerical values of four quark condensates and the gluon condensate are still controversial.

The problem of factorization is more severe for higher-dimensional operators and their numerical values more uncertain. We have estimated that the highest dimensional operators that we have used, namely $\langle \bar{q}q\bar{q}g_s\sigma Gq\rangle$ and $\langle \bar{q}g_s^2G^2q\rangle$, contribute less than 5% to the theoretical part of the spectral function. This gives us confidence regarding the convergence of the OPE series. Moreover, in this connection the FESR has an advantage, since it suppresses the contributions of higher-dimensional operators due to the orthogonality of Hermite polynomials and only radiative corrections of their Wilson coefficients contribute. The contributions of higher excited states to the phenomenological part of the sum rules have been subtracted off by an explicit use of a cutoff parameter s_0 and they may contribute only through their finite widths (we assume the latter to be sufficiently small). We summarize our final results of QCD sum rule determination of masses, widths, and coupling constants in Table II.

The results obtained for the negative parity state $[N(1535)]$ have a larger scatter compared to the corresponding results for the positive parity state. This is apparently due to alternate signs with which the quantities corresponding to the negative parity state are appearing in Eqs. (18). The scatter in widths is roughly twice as large as in the respective masses. Realizing the fact that the masses, by the QCD sum rule method, cannot be calculated with an accuracy better than 10% [3], it appears that the QCD sum rule method, at least in the present form, is not very appropriate for the calculation of hadronic widths.

TABLE II. Results of QCD sum rule determination of masses, widths, and coupling constants. The error in the calculated value shows the total uncertainty in the value of the physical quantity as contained in sets I, II, and III of Table I.

Physical parameter	Calculated value (GeV)	Experimental value (GeV)
M_1	1.45 ± 0.03	$1.40-1.48\sim1.44$
M_2	1.49 ± 0.17	$1.52-1.56\sim1.54$
Γ_1	0.06 ± 0.06	$0.12-0.35\sim0.20$
Γ_2	0.37 ± 0.37	$0.10-0.25\sim0.15$
λ_{N1}^2	$(7.85\pm1.15)\times10^{-4}$	
λ_{N2}^2	$(4.88\pm2.57)\times10^{-4}$	

If we parametrize the two-point function (3) at zero momentum by

$$F_1(0)=\frac{\lambda_N^2}{M^2}(1+\delta),\quad F_2(0)=\frac{\lambda_N^2}{M}(1+\delta'),$$

where δ and δ' are contributions of all the excited states, then from Eqs. (17) and (19) for the experimental values of parameters, we obtain

$$\delta=-0.7,\quad \delta'=-0.33 .$$

The smaller value of δ' is due to the fact that the states of positive and negative parity combine with opposite signs.

Finally, in order to get a feel for the Chung-Dosch-Kremer-Schall (CDKS) current [7], we compute $F_1(0)$ and $F_2(0)$ from Eqs. (17) and (19) only, with $t=-0.2$ and keeping other constants as above. This gives $\delta=-0.29$, $\delta'=-0.06$.

On using the values of condensates used by CDKS, $|\delta|$ gets lowered by 40% whereas δ' remains almost unchanged. Thus the use of the CDKS current gives a reduced contribution of excited states to the spectral sum rules. This is expected, since this current has been constructed in a way that ensures minimal coupling of the current to the excited states [7]. In view of this, we do not attempt to use this current for computing masses, etc., of excited states.

Conversely, one could have used the six equations to calculate vacuum condensates if one takes the values of masses and widths from experiments.

In conclusion, using only some phenomenological constants (the vacuum condensates), the proton mass and the coupling constant of the proton with a local quark current as the only inputs and applying the combination of Gauss-Weierstrass transform and finite energy sum rules, we have been able to generate numbers for the values of masses and widths of some of the baryon resonances which are reasonably close to the corresponding experimental numbers.

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APPENDIX

We list below Borel transforms, as defined by Eq. (6), of terms that we have used in the text:

$$\hat{B}\left[\frac{1}{(q^2)^n}\right]=\frac{(-1)^n\sigma^n}{(n-1)!}\ ,$$

$$\hat{B}[(q^2)^n\ln(-q^2)]=-(n!)\sigma^{-n}\ ,$$

$$\hat{B}\{q^2[\ln(-q^2)]^2\}=\frac{2}{\sigma}[\ln\sigma-\psi(1)-1]\ ,$$

$$\hat{B}\{[\ln(-q^2)]^2\}=2[\ln\sigma+\gamma_E]\ ,$$

$$\hat{B}[(q^2)^{-n}\ln(-q^2)]=\frac{(-1)^n\sigma^n}{(n-1)!}[\psi(n)-\ln\sigma]\ ,$$

$$\hat{B}\{(q^2)^{-n}[\ln(-q^2)]^2\}=\frac{(-1)^n\sigma^n}{(n-1)!}[(\ln\sigma)^2-2\psi(n)\ln\sigma+\psi^2(n)-\psi'(n)]\ ,$$

$$\psi(n)=-\gamma_E+\sum_{k=1}^{n-1}\frac{1}{k},\quad \psi(1)=-\gamma_E\ ,$$

$\psi'(n)=\zeta(2,n)$, the generalized zeta function.

Next we list the GW transforms of terms that have been used in the text:

$$\frac{1}{2\tau}\hat{L}\left[\frac{1}{\sigma}\frac{1}{\sigma^{d+1}}e^{-s\sigma}\right]=\frac{1}{\sqrt{\pi}}2^{(d-1)/2}\tau^{d/2}e^{-s^2/8\tau}D_{-d-1}(s/\sqrt{2\tau})\ .$$

Here the parabolic cylinder function $D_{-d-1}(z)$ is defined by

$$D_{-d-1}(z)=\sqrt{2}\frac{(-1)^d}{d!}e^{-z^2/4}\frac{d^d}{(dz)^d}\left[e^{z^2/2}\int_{z/\sqrt{2}}^{\infty}dy\, e^{-y^2}\right],\quad d\geq 0\ ,$$

and

$$e^{z^2/4}D_n(z)=2^{-n/2}H_n(z/\sqrt{2}),\quad n\geq 0\ .$$

$H_n(z)$ is Hermite's polynomial. For log terms, we have

$$\frac{1}{2\tau}\hat{L}\left[\frac{2^{-\beta-1}}{\sigma^{d+1}(\ln\sigma)^{\beta+1}}\frac{1}{\sigma}e^{-s\sigma}\right]=I(s,\tau;d,\beta)\ .$$

For our purpose, we have obtained

$$I(s,\tau;2,-2)=-8\tau I^3\mathrm{Derfc}(s/2\sqrt{\tau})-\frac{4\tau}{\sqrt{\pi}}\frac{s}{2\sqrt{\tau}}+\frac{\tau}{2}-\tau(\ln\tau+\gamma_E)[i^2\mathrm{erfc}(s/2\sqrt{\tau})+i^2\mathrm{erfc}(-s/2\sqrt{\tau})]$$
$$-\tau(\ln\tau+\gamma_E+2\ln 2)[i^2\,\mathrm{erfc}(s/2\sqrt{\tau})-i^2\,\mathrm{erfc}(-s/2\sqrt{\tau})]\ ,$$

$$I(s,\tau;-1,-3)=\frac{1}{\sqrt{\pi\tau}}\left[\frac{1}{2}(\ln 4\tau+\gamma_E)^2-\frac{\pi^2}{4}\right]e^{-s^2/4\tau}$$
$$-\frac{8}{\sqrt{\tau}}e^{-s^2/4\tau}\int_0^{s/2\sqrt{\tau}}dy\, e^{y^2}\int_0^y dx\,\mathrm{Derfc}(x)-\frac{2}{\sqrt{\tau}}[\psi(1/2)-\ln\tau]\mathrm{Derf}\left[\frac{s}{2\sqrt{\tau}}\right]\ ,$$

where

$$i^2f(x)=\int_x^{\infty}dz\int_z^{\infty}f(y)dy\ ,$$

$$I^2f(x)=\int_0^x dz\int_0^z f(y)dy,\quad \text{etc.}\ ,$$

$$\mathrm{erfc}(x)=1-\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}dy\, e^{-y^2}\ ,$$

$$D(x)=e^{-x^2}\int_0^x dy\, e^{y^2}\ ,$$

$$\mathrm{Derf}(x)=e^{-x^2}\int_0^x dy\, e^{y^2}\,\mathrm{erf}(y)\ ,$$

$$\mathrm{Derfc}(x)=e^{-x^2}\int_0^x dy\, e^{y^2}\,\mathrm{erfc}(x)\ .$$

Other functions which are used in the text can be obtained from $I(s,\tau;2,-2)$ and $I(s,\tau;-1,-3)$ by one or more differentiations with respect to s:

$$-\frac{d}{ds}I(s,\tau;n,-m)=I(s,\tau;n-1,-m)\ .$$

For deriving FESR's, we require knowledge of the asymptotic behavior of integrals [10]:

$$\lim_{z\to\infty}\int_0^z dz' D(z')=\frac{1}{2}\ln z+\frac{\gamma_E}{4}+\tfrac{1}{2}\ln 2\ ,$$

$$\lim_{z\to\infty}\int_0^z dz'\,\mathrm{Derf}(z')=\frac{1}{2}\ln z+\frac{\gamma_E}{4}\ ,$$

and others which we have calculated:

$$\lim_{z\to\infty}\int_0^z dz' e^{-z'^2}\int_0^{z'}dy\, e^{y^2}\int_0^y dx\, D(x)=\frac{1}{8}\left[\left[\ln z+\frac{\gamma_E}{2}+\ln 2\right]^2-\frac{\pi^2}{8}\right]\ ,$$

$$\lim_{z\to\infty}\int_0^z dz' e^{-z'^2}\int_0^{z'}dy\, e^{y^2}\int_0^y dx\,\mathrm{Derf}(x)=\frac{1}{8}\left[\left[\ln z+\frac{\gamma_E}{2}\right]^2-\frac{\pi^2}{24}\right]\ .$$

The terms that have been left over, in the above integrals, vanish when $z\to\infty$.

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