Asymptotic scaling of the mass gap in the two-dimensional O(3) nonlinear σ model: A numerical study

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For the two-dimensional standard O(3) nonlinear σ model with the Hamiltonian $H = -\beta \sum_{\langle ij \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$, we report a bulk correlation length (ξ_∞) up to $\beta = 2.7$ (where the corresponding $\xi_\infty \simeq 15\,000$) obtained based on a finite-size-scaling Monte Carlo technique. We find that asymptotic scaling starts to set in around $\beta = 2.3$, and that the analytical prediction on ξ_∞ by Hasenfratz, Maggiore, and Niedermayer agrees better with data as β increases.

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I. INTRODUCTION

Two-dimensional (2D) nonlinear σ models are of special interest in particle physics due to their similarity to four-dimensional (4D) non-Abelian gauge models. For both 4D non-Abelian gauge models [1] and 2D nonlinear σ models [2] perturbation theory predicts that the β functions are negative in the weak coupling region, with their vanishing value at zero coupling; this property is referred to as perturbative asymptotic scaling (AS). The β functions in the 2D nonlinear σ models are negative in the strong coupling region as well, which is known from strong coupling expansion [3]. Determining the sign of the β function in the intermediate region is very important for establishing the validity of QCD: the positive sign of the β function in some intermediate region implies the presence of at least two critical points, which could make QCD nonasymptotic free, and would make it implausible for the quark confining property in the strong coupling regime, for example, to be translated into the continuum limit of the theory.

Usage of various series expansion techniques is limited in the intermediate region [except for $1/N$ expansion], so the behavior of the β function remains by large unknown. The Monte Carlo simulation technique is suitable to the studies in the intermediate region. Indeed, using such effective Monte Carlo algorithms as one cluster or overrelaxation, correlation lengths up to about 200 (in lattice unit) are available for the 2D standard O(3) model, showing that the correlation length scales faster than the prediction of AS [5]. Indirect Monte Carlo methods such as the Monte Carlo renormalization group method and finite size scaling analysis of Monte Carlo data of O(3) [6, 7] and O(4) models [8], however, support AS. For the O(N) models with the larger values of N, AS appears to start to set in with smaller values of correlation length than for the O(3) model [9].

For some other type of the nonlinear σ model such as the RP^2 model, Monte Carlo data reveal that the correlation length scales much faster than the prediction of AS, thus leading some authors to conjecture [10] the possible divergence of the correlation length; i.e., a possible vanishing of the β function in the intermediate region. There have been some rigorous approaches questioning AS also [11]. The current conventional wisdom, nevertheless, holds that there is no critical point in the intermediate region for both O(N) models and RP^{N-1} models with $N \geq 3$.

In this paper bulk correlation lengths (ξ) of the 2D O(3) standard model are computed up to $\xi_\infty \simeq 15\,000$ based on a finite size scaling technique. Our data show that correlation length scales faster than the prediction of AS until $\beta \simeq 2.3$ (where the corresponding $\xi_\infty \simeq 1400$), but the AS tends to set in for $\beta \geq 2.3$ within the statistical errors of the data. The agreement of our ξ_∞ with the analytical prediction by Hasenfratz *et al.* [12] is shown to be improved as β increases.

II. THE MODEL

The Hamiltonian of the standard (nearest neighbor) O($N \geq 3$) nonlinear σ models is defined as

$$H = -\beta \Sigma_{\langle ij \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \tag{1}$$

where β denotes the inverse temperature (coupling), and $\boldsymbol{\sigma}$ takes values in the unit sphere S^{N-1}, and $\langle ij \rangle$ runs over all the links of the nearest sites i and j.

By the (perturbative) AS of the mass gap, it is meant that the mass gap $m \equiv 1/\xi_\infty$ behaves as

$$m = C_m \Lambda_L \ ,$$
$$\Lambda_L = (2\pi\beta)^{1/(N-2)} e^{-2\pi\beta/(N-2)} [1 + a_1/\beta + O(1/\beta^2)],$$

as $\beta \to \infty$ [2]. The coefficients a_k are nonuniversal; especially, from the lattice three loop perturbative calculation [13] $a_1 \simeq [0.486 + 0.089(N-2)]/[2\pi(N-2)]$. C_m has been exactly computed to be

$$C_m = 2^{5/2} (8/e)^{1/(N-2)} e^{\pi/2(N-2)} / \Gamma(1 + 1/(N-2)) \ , \tag{2}$$

from two separate relations: one is the relation between the scale of the minimal subtraction (MS) regularization scheme and that of the lattice, i.e.,

$$\Lambda_{\text{MS}}/\Lambda_L = 2^{5/2}e^{\pi/[2(N-2)]} \tag{3}$$

obtained based on lattice perturbation theory [14], and the other [12] is

$$m/\Lambda_{\text{MS}} = (8/e)^{1/(N-2)}/\Gamma(1+1/(N-2)) \tag{4}$$

based on Bethe ansatz and Zamolodchikov's S matrix [15].

Lüsher, based on the renormalization group equation, pioneered a finite size scaling study on the 2D O(N) model [16]. Namely, he derived, for $N \geq 3$,

$$\frac{m_L}{\Lambda_{\text{MS}}} = C_N(ze^{\pi/z})^{(N-1)/(N-2)}\left(1+\sum_{k=1}^{\infty} b_k z^k\right), \tag{5}$$

where m_L and z represent, respectively, the mass gap on a $L \times L$ lattice and a dimensionless parameter defined as

$$z \equiv Lm_L = L/\xi_L. \tag{6}$$

The coefficient C_N depends on N only, and b_k can be calculated from perturbation theory.

III. METHOD AND SIMULATIONS

Rewriting Eq. (5) as $m_L = m_\infty q_m(z)$, one immediately notices that Eq. (5) is a particular form of the general finite size scaling formula [17]. Namely, for any multiplicative renormalizable quantity P,

$$P_L(t) = P_\infty(t)q_P(x(t)) \ , \quad x(t) \equiv \xi_L(t)/L \tag{7}$$

where t represents (reduced) temperature.

In particular, Eq. (7) for the correlation length reduces to

$$\xi_L \sim L \tag{8}$$

at a critical point ($t = 0$) [17], and it has been observed numerically that Eq. (8) holds for most models [18] with remarkable accuracy for $L \geq L_{\min}$, where $L_{\min}$ is between 10 and 20, in general with our definition of the correlation length [Eq. (10)]. We stress that Eq. (7) is valid even when x is sufficiently large (i.e., ξ_L/L is sufficiently small) for $L \geq L_{\min}$: otherwise, if Eq. (7) would not be valid for too small a value of L/ξ_∞, one would have never observed the validity of Eq. (8) on a finite lattice, since at a critical point $L/\xi_\infty \to 0$ for any finite L.

Equation (7) implies an efficient method of extracting bulk values (thermodynamic values) in Monte Carlo simulations [19]; the method is significant since measurements on the lattice with $L < \xi_\infty$ are already good enough for extracting accurate bulk values, whereas $L/\xi_\infty \geq 6$ is required for the traditional direct measurements. The crucial observation for this method is that $q_P(x(t))$ in Eq. (7) has no explicit temperature dependence so that some set of $[x(t'), q_P(x(t'))]$ obtained at a certain temperature t', referred to as a *reference* temperature, can be used for the extraction of $P_\infty(t)$ at another temperature t. In order to demonstrate that $q_\xi(x(\beta))$ in the $O(3)$ model indeed has no β dependence, we evaluated $[x(\beta), q_\xi(x(\beta))]$ for several arbitrary points of $x(\beta)$ at some arbitrarily different β, and observe that the data set belonging to different β tend to overlap (Fig. 1).

For a given $x(t)$, therefore, the prime thing for the method is to estimate $q_\xi(x(t))$ by means of interpolation, given some set of $[x(t'), q_\xi(x(t'))]$ available via Monte Carlo simulations at a temperature t' where $\xi_\infty(t')$ is already known. The accuracy and efficiency of this method has already been well demonstrated for the 2D and 3D Ising models [19].

We employed Wolff's one-cluster algorithm [20] for our Monte Carlo simulations, imposing the periodic boundary condition on square lattice. For each β and L, 20–40 different bins were obtained, where each bin is composed of 10 000 measurements each of which is separated by 4–50 consecutive one-cluster updating. We averaged over the bin values estimating the statistical errors by the jackknife method. In most cases, much lower statistics of the data than above were sufficient to extract bulk values with reasonable accuracy. However, when interpolation happens to occur in the region of x where $q_\xi(x)$ is quite steep, sufficiently high statistics of the data were required to obtain accurate results. If an interpolation is made where $q_\xi(x)$ is very small, one may suspect that an estimate of bulk value would be too sensitive to interpolation to get a reasonably accurate result. It turns out, however, that with sufficiently high statistics of Monte Carlo data an estimate can be still reliable even for the values of $q_\xi \simeq 10^{-2}$, because in this region $q_\xi(x)$ becomes almost flat with x (see Fig. 1).

In order to define a correlation length, we consider the Fourier transform of the two-point correlation function

$$G(\mathbf{k}) \equiv \Sigma_{\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}}\langle\sigma_0 \cdot \sigma_{\mathbf{x}}\rangle \ . \tag{9}$$

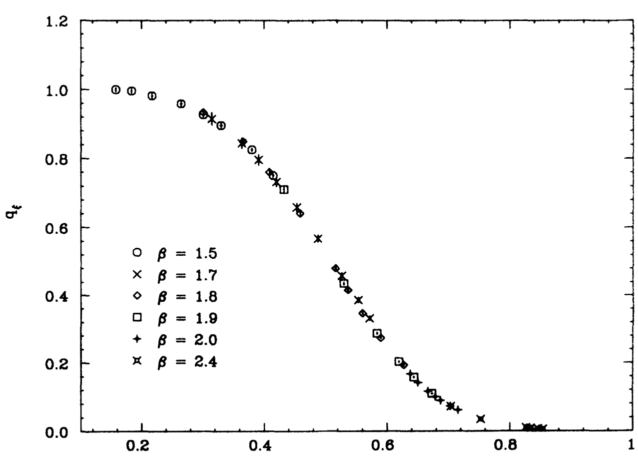

FIG. 1. $[x(\beta), q_\xi(x(\beta))]$ for various values of β, using data in Table II. For each β, the data point with the largest value of x corresponds to $L = 20$. The reader should not be confused with the error bars from the symbols of the data at $\beta = 2.0$; the actual error bars are almost invisible.

TABLE I. Bulk correlation length $[\xi_\infty(\beta)]$ obtained through our method for $2.0 \leq \beta \leq 2.7$. For our evaluation of $\xi_\infty(2.0)$ the reference temperature was $\beta = 1.9$, using $[x(1.9), q_\xi(x(1.9))]$ in Table II with $\xi_\infty(1.9) = 122.0(2.7)$ (see Table III). For our results of $\xi_\infty(\beta)$ in the range $2.05 \leq \beta \leq 2.4$ and $2.5 \leq \beta \leq 2.7$, the reference temperature was $\beta = 2.0$ and 2.4, respectively, using data set in Table II. The interpolation was made in the range of $0.08 \leq q_\xi \leq 0.16$ and $0.07 \leq q_\xi \leq 0.009$ for the reference $\beta = 2.0$ and 2.4 respectively. The data marked with an asterisk are from Ref. [5], and the reference temperature was $\beta = 1.5$ using data in Table II. From the direct Monte Carlo measurement[5] using $L = 1024$, $\xi_\infty(2.0) = 224.3(4.2)$.

β	2.0	2.05	2.05	2.1	2.15	2.2
L	60	34	1024	100	60	140
ξ_L	38.29(0.06)	23.79(0.02)	295.6(5.2)*	40.91(0.04)	66.76(0.08)	92.96(0.03)
ξ_∞	227.8(3.2)	306(4)	315(13)	419(5)	574(8)	766(7)
β	2.3	2.4	2.5	2.6	2.7	
L	180	260	60	100	160	
ξ_L	124.1(0.2)	183.1(0.3)	49.82(0.07)	83.18(0.11)	134.6(0.2)	
ξ_∞	1402(22)	2499(41)	4696(128)	8022(234)	15209(449)	

TABLE II. The values of ξ_L varying L for $1.5 \leq \beta \leq 2.4$, which were used to extract our results in Table I. Results for $\beta = 1.5$ show that ξ_L reaches its bulk value for approximately $L/\xi_L \geq 6$.

β	L	ξ_L	β	L	ξ_L	β	L	ξ_L
1.5	20	8.30(0.02)	1.8	20	12.55(0.02)	2.0	20	14.32(0.01)
	24	9.12(0.03)		30	17.71(0.02)		24	16.88(0.02)
	30	9.90(0.06)		40	22.43(0.03)		30	20.63(0.02)
	34	10.24(0.04)		50	26.86(0.05)		34	23.08(0.02)
	40	10.60(0.06)		60	31.01(0.06)		40	26.68(0.02)
	50	10.86(0.05)		90	41.33(0.04)		50	32.53(0.03)
	60	11.01(0.04)		120	49.08(0.04)		60	38.29(0.04)
	70	11.05(0.03)		150	54.83(0.05)			
	80	11.04(0.03)		200	60.26(0.07)			
1.7	20	11.44(0.02)	1.9	20	13.47(0.02)	2.4	20	17.08(0.02)
	24	13.30(0.03)		30	19.31(0.02)		24	20.30(0.02)
	30	15.82(0.03)		40	24.79(0.02)		30	25.02(0.02)
	40	19.52(0.03)		60	35.06(0.04)		34	28.10(0.03)
	50	22.66(0.02)		100	53.06(0.05)		120	90.35(0.05)
	60	25.21(0.04)		200	86.48(0.12)		260	183.1(0.3)
	70	27.39(0.05)						
	80	29.07(0.03)						
	100	31.49(0.08)						

TABLE III. $\xi_\infty(\beta)$ obtained using different interpolating points for $\beta = 1.7$, 1.8, and 1.9, showing that the results are not dependent on the value of interpolations. $\xi_\infty(\beta)$ obtained through direct Monte Carlo measurements [5] are 34.4(.1), 64.6(.5), and 122.1(1.9) for $\beta = 1.7$, 1.8, and 1.9, respectively.

β	1.7	1.7	1.8	1.8	1.8
L	80	100	60	90	120
ξ_∞	34.3(0.2)	34.5(0.2)	63.5(1.1)	64.2(0.8)	64.8(0.7)
β	1.8	1.8	1.9	1.9	1.9
L	150	200	60	100	200
ξ_∞	64.6(0.3)	65.0(0.3)	121.8(2.6)	121.8(2.0)	122.5(2.4)

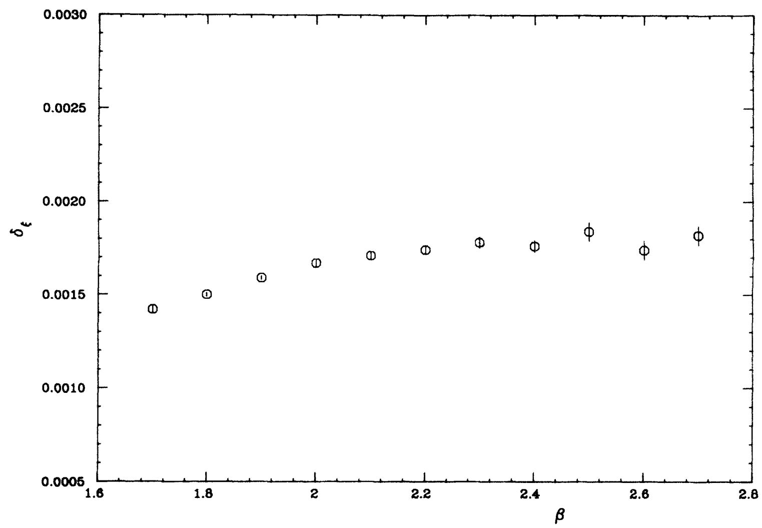

FIG. 2. $\delta_\xi(\beta) \equiv \xi_\infty/[e^{2\pi\beta}(1-0.0915/\beta)/\beta]$ for $1.7 \le \beta \le 2.7$. A constant value of $\delta_\xi(\beta)$ is an indicator of the exact AS from the three loop perturbative calculation.

When $|\mathbf{x}|$ is sufficiently large, $\langle \sigma_0 \cdot \sigma_{\mathbf{x}} \rangle \sim e^{-|\mathbf{x}|/\xi_L}$ holds, so we will have

$$G(\mathbf{k})^{-1} = G(0)^{-1}[1 + k^2\xi_L^2 + O(k^4)]. \tag{10}$$

By choosing $\mathbf{k} = (2\pi/L, 0)$ and by computing $G(0)$ and $G(\mathbf{k})$ through Monte Carlo simulations ξ_L can be evaluated from Eq. (10) when L is so large that $O(k^4)$ can be ignored.

Using $L_{\min} = 20$ seems to be good enough for our definition of the correlation length, and for the complete elimination of the possible correction in Eq. (7) (see Fig. 1).

IV. RESULTS

We present all the ξ_∞ thus evaluated from $\beta = 2.0$ to $\beta = 2.7$ (Table I) by changing β by 0.1 or by 0.05, and the data necessary to evaluate them (Table II). Our results at $\beta = 1.7$, 1.8, and 1.9 are compared with those from direct Monte Carlo measurements [5] (Table III), yielding excellent agreements.

To check AS from the three loop perturbative result, we introduce $\delta_\xi \equiv \xi_\infty/[e^{2\pi\beta}(1 + a_1/\beta)/\beta]$ with $a_1 \simeq -0.0915$, and the results for $\beta = 1.7 \sim 2.7$ are plotted in Fig. 2. Within the statistical error, AS sets in from around $\beta \simeq 2.3$. In Fig. 3, $(\beta, \ln[\xi_\infty(\beta)])$ are plotted, showing a remarkable exponential type critical behavior of $\xi_\infty(\beta)$ in our range of β.

Fitting our data in Table I to, for example, $\xi_\infty(\beta) =$

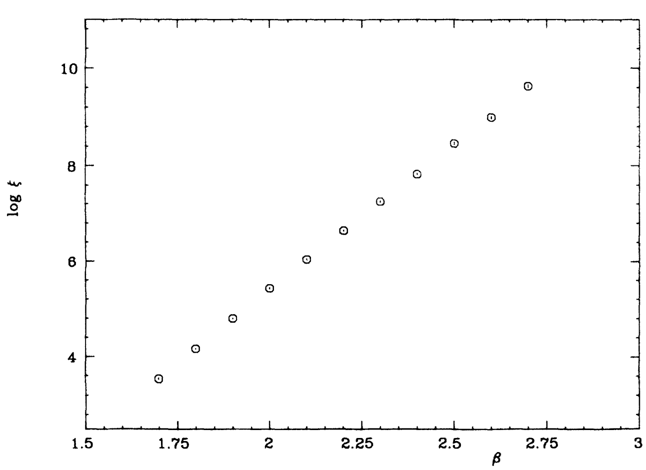

FIG. 3. $\ln(\xi_\infty(\beta))$ for our range of β, showing a qualitative exponential behavior of $\xi_\infty(\beta)$ in the 2D O(3) model.

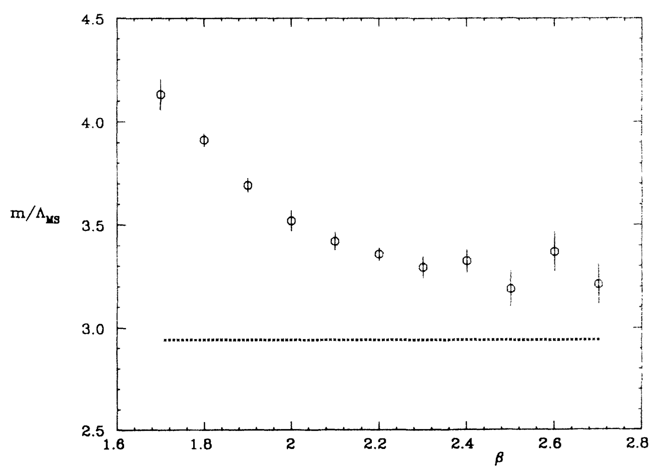

FIG. 4. $m(\beta)/\Lambda_{\rm MS}$ for our range of β, where the dotted line represents the theoretical prediction in Ref. [12], i.e., $m/\Lambda_{\rm MS} = 8/e$. This figure, along with Fig. 2, shows that the prediction in Ref. [12] is incompatible with AS up to at least $\beta = 2.7$. In other words, if $m/\Lambda_{\rm MS}$ happened to be $8/e$ from, say, $\beta = 2.7$, AS would not set in until $\beta = 2.7$. The expected compatibility seems to start to occur from a $\beta > 2.7$.

$c_1 \exp(2\pi\beta)[1 + c_2/\beta]/\beta$ yields $c_1 \simeq 0.0026$, $c_2 \simeq -0.827$ with $\chi^2/N_{\rm DF} \simeq 2.2$. On the other hand, fitting our data to a power-law critical behavior such as $\xi_\infty(t) \sim t^{-\nu}$ [$t \equiv (\beta_c - \beta)/\beta_c$] yields extremely unstable fitting; β_c tends to be extremely large with a very large value of ν, and the value of $\chi^2/N_{\rm DF}$ decreases extremely slowly with β_c where $\beta_c \geq 100$. Also, it is evident that fitting our data to a Kosterlitz-Thouless-type critical behavior is much worse than that to a power-law critical behavior.

To compare with the analytical result [Eq. (3)], we computed values of m_∞/Λ_{MS} for $1.7 \leq \beta \leq 2.7$, which are plotted in Fig. 4. The values of the correlation length measured in this range of β are uniformly smaller than those from the analytical prediction. Although the convergence rate is very slow, the values of the bulk correlation length seem to converge to the analytical prediction as β grows. Also, note that if the analytical prediction were exact from a β less that 2.7 the AS would set in from a β larger than our prediction.

V. CONCLUSION AND DISCUSSION

Our numerical result of the bulk correlation length demonstrates that the perturbative AS holds at least qualitatively in the region of β considered here, i.e., $1.7 \leq \beta \leq 2.7$. It is very remarkable that the perturbative AS which is supposed to be valid in the limit $\beta \to \infty$, still holds for such *small* values of β, and we take this as strong numerical evidence showing that there is no phase transition separating the weak coupling regime from the strong coupling region. It also suggests that a physical property from a strong coupling expansion such as the quark confining potential between quarks may persist at least qualitatively to the weak coupling region, i.e., to the continuum limit of the asymptotic free theories, being consistent with general expectations. Although it may be hard to rule out the possibility of a power-law critical behavior with, say, $\beta_c = 3000$ [$\chi^2/N_{\rm DF} \simeq 4$], it seems to be highly implausible.

As stressed, one of the crucially important assumptions for our method is that Eq. (7) is valid regardless of the magnitude of the scaling variable L/ξ_∞ for $L \geq L_{\rm min}$. We confirmed this numerically for the O(3) model (Table III), where the bulk values estimated from different L are recorded for $\beta = 1.7$, 1.8, and 1.9. (Also, see Table II for $\beta = 2.0$ and 2.05.)

For interpolations, we always used a rational functional-type interpolation scheme (subroutine RATINT in Ref. [21]) with the degree of interpolation fixed as 4: Given four data points $((x(t'), q_\xi(x)(t'))$ at a reference temperature t', the subroutine finds an optimized rational function. The error from this subroutine results only from whether the given data tend to fit to a rational function or not; for data points sufficiently close to each other it is always possible to find a rational function fit to the data, so the errors turn out to be very small.

The sources of errors in our interpolations are the statistical errors in $x(t')$, $q_\xi(x(t'))$, $x(t)$, and $\xi_\infty(t')$. We are not aware of any interpolation subroutine which takes account of the errors in both $x(t')$ and $q_\xi(x(t'))$, and it is very hard to monitor error propagation due to these errors. Practically, however, one can make the statistical errors in $x(t')$ very small by choosing values of L usually not larger than 40. The significant part of error propagations which can be tracked comes from the statistical error in $x(t)$, and from $\xi_\infty(t')$. Usually when L is small, the most significant error comes from the error in input $\xi_\infty(t')$.

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